ON CONNECTIONS BETWEEN DOMINATING SETS AND TRANSVERSALS IN SIMPLE HYPERGRAPHS

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Abstract: This article focuses on characterizing the hypergraphs in which dominating sets are transversals.

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1. Introduction

The cardinality (or, size) of a finite nonempty set $V$ is denoted by $|V|$. The set of all subsets (including the empty set $\phi$) of $V$ is denoted by $2^V$ which is called the \textit{power set} \cite{7} of $V$. The set of all nonempty subsets of $V$ is denoted by $2^V*$; that is, $2^V* = 2^V - \{\phi\}$.

Let $E$ be a family of nonempty subsets of $V$. If $\bigcup_{X \in E} X = V$, we say $E$ \textit{fills out} $V$. A \textit{hypergraph} \cite{2} on $V$ is a pair (or, couple) $H = (V, E)$ where $V$ is a nonempty finite set and $E$ is a family of nonempty subsets of $V$ that fills out $V$. The set $V$ is called the \textit{vertex set} of $H$ and each member of $E$ is called a \textit{hyperedge} of $H$. If the members of $E$ are all distinct (that is, no two members are equal as subsets of $V$; or, $E \subseteq 2^V*$) then $H$ is called \textit{simple}. If no member of $E$ is a subset (proper or otherwise) of another, then $H$ is called a Sperner

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hypergraph. Some authors (instances: [2] and [3]) take Sperner hypergraphs to be simple and vice versa but there is distinction [6] between the two: Sperner hypergraphs are necessarily simple but not conversely. See 1.1 that follows.

**Example 1.1.** Let \( H = (V, E) \) where \( V = \{1, 2, 3, 4, 5\} \), \( E = \{X_1, X_2, X_3\} \) with \( X_1 = \{2, 3\} \), \( X_2 = \{2\} \) and \( X_3 = \{1, 3, 4, 5\} \). \( H \) is simple because the three hyperedges are all distinct as subsets of \( V \). But \( H \) is not Sperner because \( X_2 \subset X_1 \).

All the hypergraphs in the coming discussion are assumed simple unless there is some unambiguous indication to the contrary. The motivation for this research work comes principally from [4] that discusses transversal number and dominating number in simple hypergraphs in substantial detail.

### 2. Transversals and Dominating Sets

Let \( H = (V, E) \) be a simple hypergraph and \( T \in 2^V \). Then \( T \) is called a transversal [2] in \( H \) iff \( T \cap A \neq \emptyset \) for each \( A \in E \) - which criterion is also rephrased as: \( T \) intersects every hyperedge of \( H \). If \( T \) is a transversal and \( T \neq V \) then \( T \) is called a proper transversal in \( H \).

Two vertices \( x \) and \( y \) in \( V \) are said to be adjacent if there is a hyperedge that contains \( x \) and \( y \); that is: \( x, y \in A \) for some \( A \in E \). Evidently every vertex is adjacent to itself. Let \( D \in 2^V \). Then \( D \) is called a dominating set [1] in \( H \) iff: (i) \( D \neq V \), and (ii) each \( x \in V - D \) is adjacent to some \( y \in D \).

**Proposition 2.1.** If \( T \) is a proper transversal in \( H = (V, E) \), then \( T \) is a dominating set in \( H \).

**Proof.** Let \( x \in V - T \). Let \( A \in E \) be such that \( x \in A \). Then \( T \cap A \neq \emptyset \) since \( T \) is a transversal. Let \( y \in T \cap A \). At once we have \( x \neq y \) and \( x \) is adjacent to \( y \), whence \( T \) is a dominating set.

**Example 2.2.** Not every dominating set in \( H \) is a transversal, though. Consider \( H = (V, E) \) where \( V = \{1, 2, 3, 4, 5\} \), \( E = \{X_1, X_2, X_3, X_4\} \) with \( X_1 = \{1, 2\} \), \( X_2 = \{2, 3, 4\} \), \( X_3 = \{4, 5\} \) and \( X_4 = \{3, 5\} \). Let \( D = \{1, 5\} \). Each element of \( V - D = \{2, 3, 4\} \) is adjacent to either 1 or 5, and so \( D \) is a dominating set in \( H \). But \( D \) is not a transversal because \( D \cap X_2 = \emptyset \).

Every hypergraph has a transversal - the vertex set is always one. But there are hypergraphs without dominating sets. When can a hypergraph have a dominating set?
Proposition 2.3. $H = (V, E)$ has a dominating set if and only if $|X| \geq 2$ for some $X \in E$.

Proof. Assume $H$ has a dominating set, say $D$. Let $x \in V - D$ be given. Were $|X| = 1$ for every $X \in E$ then $\{x\}$ is the only hyperedge containing $x$, and so $x$ is not adjacent to any member of $D$, contradicting the dominating nature of $D$.

Conversely, suppose $|X| \geq 2$ for some $X \in E$. Let $a, b \in X$ and $a \neq b$. Let $D = V - \{a\}$. Then (i) $D \neq V$, (ii) $b \in D$, (iii) $V - D = \{a\}$ and (iv) $a$ is adjacent to $b$, whence $D$ is a dominating set in $H$.

We are looking to characterize the hypergraphs with the property that every dominating set is a transversal. Section 3 deals with this, culminating in Proposition 3.4.

3. Dominating Sets in Trim Hypergraphs

This section is devoted to trim hypergraphs. Let $H = (V, E)$. A hyperedge $X$ in $H$ is called redundant in $H$ (or, redundant in $E$) if there exists $S \subseteq E - \{X\}$ such that $S$ covers $X$; that is, $X \subseteq \bigcup_{Y \in S} Y$. If $H = (V, E)$ has no redundant hyperedges then we call $E$ a minimal hyperedge cover for $H$ and we call $H$ a trim hypergraph [5]. For a vertex $x \in V$, the number of hyperedges that contain $x$ is defined to be the degree of $x$ in $H$, and this number is denoted by $dx(H)$ or $dx$.

Proposition 3.1. A hyperedge $X$ is redundant in $H$ if and only if no vertex of $X$ is of degree 1. In other words, $H = (V, E)$ is trim if and only if each hyperedge has a vertex of degree 1.

The proof of 3.1 is discussed in [5].

Proposition 3.2. If $H$ is trim then every dominating set in $H$ is a transversal.

Proof. Let $D$ be a given dominating set in the trim hypergraph $H$, and let $X$ be a given hyperedge in $H$. Then $dz = 1$ for some $z \in X$. If $z \in D$ then the conclusion follows at once. If $z \notin D$, then $z$ is adjacent to some $y \in D$. Then $y \in X$, in view of $dz = 1$, whence $y \in D \cap X$. 

**Proposition 3.3.** If \( H \) is a Sperner hypergraph and if every dominating set in \( H \) is a transversal, then \( H \) is trim.

**Proof.** Suppose \( H \) is not trim, and so let \( Y \) be a redundant hyperedge in \( H \). Clearly \( V - Y \) is nonempty. Let \( D = V - Y \). Given \( y \in Y \), there is a hyperedge \( X (\neq Y) \) such that \( y \in X \). Since \( H \) is Sperner, there is \( z \in X \) with \( z \neq y \) and \( z \in V - Y \). Then \( y \) is adjacent to \( z \), and so \( D \) is a dominating set in \( H \). But then \( D \) fails to be a transversal because \( D \cap Y = \emptyset \).

**Proposition 3.4.** Let \( H \) be Sperner. Then every dominating set in \( H \) is a transversal if and only if \( H \) is trim. (3.4 is a consequence of 3.2 and 3.3.)

**4. Summing Up**

(i) In a simple hypergraph (not necessarily Sperner), each proper transversal is a dominating set (2.1), though not conversely (2.2), and

(ii) in a Sperner hypergraph, each dominating set is a transversal if and only if the hypergraph is trim (3.4).

Thus, it is precisely in the class of trim hypergraphs that every dominating set is a transversal. This is of theoretical interest at this point, and possibilities of applications are being studied.

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**References**


