A CLASS OF EIGENVALUE PROBLEMS
FOR THE $(p, q)$-LAPLACIAN IN $\mathbb{R}^N$

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Abstract: This paper concerns the study of a nonlinear eigenvalue problem for the $(p, q)$–Laplacian with a positive weight

$$-\Delta_p u - \Delta_q u = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N.$$ 

Using the Mountain-Pass Theorem, we show the existence of a continuous set of positive eigenvalues.

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1. Introduction

In recent years much attention was given to the study of stationary solutions of the reaction-diffusion equation

$$u_t = div \left( (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \right) + c(x, u) \quad (1.1)$$

that appears in physics and related sciences such as biophysics, plasma physics and chemical reaction design.
When \( q = p \) and \( c(x, u) = \lambda g(x)|u|^{p-2}u \), stationary problem associated to 
1.1 becomes an eigenvalue \( p \)-Laplacian problem of the form

\[
-\Delta_p u = \lambda g(x)|u|^{p-2}u
\]

which has been widely studied both in bounded domains and \( \mathbb{R}^N \). By means
Ljusternik-Schnirelmann theory, it was established the existence of a non
decreasing positive sequence of eigenvalues \( 0 < \lambda_1 < \ldots < \lambda_n < \ldots \) (see [4] and
[8]). A characterization of the first eigenvalue was given by

\[
\lambda_1 = \inf_{u \in W^{1,p} \setminus \{0\}} \frac{\|u\|_{1,p}^p}{\int g|u|^p\,dx}
\]

It was also shown (see [1]) that \( \lambda_1 \) is simple, principal and isolated.

For the case \( q \neq p \), few studies appeared for special cases of
\( c(x, u) \). For example in [6], the author gives an existence result of a non
trivial solution of the problem

\[
-\text{div} \left( (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \right) = m|u|^{p-2}u + n|u|^{q-2}u + f(x, u)
\]

on \( \mathbb{R}^N \), under suitable conditions on the coefficients and the exponents. In
[9], a result of the existence of an infinitely many weak solutions of a similar
problem with a concave-convex nonlinearity in bounded domain is given. In [5],
the authors established a multiplicity existence result for the \((p, q)\) -Laplacian
problem with critical exponent on a bounded domain.

In this paper, we are interested in finding eigenvalues of the problem

\[
-\text{div} (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u = \lambda g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N.
\]

under the hypotheses:

\[
1 < q < p < q^*
\]

and

\[
0 \leq g \in L^{(\frac{q^*}{p^*})'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)
\]

where \( \alpha = \frac{p(1-t)}{1-\frac{p}{q^*}} \), for some \( t \in (0, 1) \) (so \( 0 < \alpha < p \)).

We will look for weak solutions of 1.4 in the framework of the reflexive
Banach space \( W = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) \), defined as the completion of \( C_0^\infty(\mathbb{R}^N) \)
with respect to the norm \( \|u\| = \|u\|_{1,p} + \|u\|_{1,q} \).

As we will see, the energy functional associated to problem 1.4 is given by

\[
I(u) = \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|u\|_{1,q}^q - \frac{\lambda}{p} \int_{\mathbb{R}^N} g(x)|u|^p\,dx.
\]
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\[ \langle I'(u), v \rangle = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2} \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} g(x)|u|^{p-2} uv dx, \]
for all \( v \in W \).

Notice that critical points of the functional \( I \) are precisely weak solutions of 1.4. To get critical points of \( I \) we will apply the Mountain-Pass Theorem. Establishing the Palais-Smale condition is the most difficulty in studying such problems. First, there is a lack of compactness for the Sobolev embedding. So the boundedness of the Palais-Smale sequence is not evident. Another difficulty, as it was mentioned in [5, 9] becomes from the fact that the Banach space framework does not ensure that

\[ |\nabla u_n|^{p-2} \nabla u_n \rightharpoonup |\nabla u|^{p-2} \nabla u \text{ in } L^{p-1}_p(\mathbb{R}^N) \]
for a Palais-Smale sequence \( \{u_n\} \). That is why in both works the authors used the Concentration-Compactness principal.

In our case, the use of the weight \( g \) in an appropriate Lebesgue space overcomes these difficulties.

The principal result of this paper is the following theorem.

**Theorem 1.** If \( p, q, \) and \( g \) fulfill 1.5 and 1.6, then there exits \( \lambda^* > 0 \) such that any \( \lambda > \lambda^* \) is an eigenvalue of problem 1.4.

In what follows, the letter \( C \) will be indiscriminately used to denote various constants when the exact values are irrelevant. The symbol \( \int \) will denote \( \int_{\mathbb{R}^N} \).

### 2. Proof of the Results

We say that \( \lambda \in \mathbb{R} \) is an eigenvalue of Problem 1.4 if there exists \( u \in W, u \neq 0 \) such that

\[ \int |\nabla u|^{p-2} \nabla u. \nabla v dx + \int |\nabla u|^{q-2} \nabla u. \nabla v dx = \lambda \int g(x)|u|^{p-2} uv dx, \]
\[ \forall v \in W. \quad (2.1) \]

This is equivalent to

\[ I'(u) = 0 \text{ in } W'. \]

Standard argument shows that \( I \in C^1(W, \mathbb{R}) \). Since the functional \( I \) is not bounded from below, we will look for local minimizers by means of the Mountain-Pass Theorem [7].
We begin this section by establishing some results required in the proof of Theorem 1.

**Lemma 2.** For all \( u \in W \) we have

\[
\int g|u|^p \, dx \leq C \|g\|_\infty \|g\|_{\frac{p(1-t)}{t}} \|u\|_{1,p} \|u\|_{1,q}^{(1-t)}. \tag{2.2}
\]

**Proof.** For any \( u \in W \), we have by the Hölder inequality

\[
\int g|u|^p \, dx \leq A_1 A_2.
\]

Where \( A_1 = \left( \int g|u|^\alpha \, dx \right)^{\frac{p(1-t)}{\alpha}} \) and \( A_2 = \left( \int g|u|^{p^*} \, dx \right)^{\frac{pt}{p^*}} \). By Sobolev injection it yields

\[
A_2 \leq C \|g\|_\infty \|u\|_{1,p}^{pt}.
\]

To estimate \( A_1 \) we apply the Hölder inequality to get

\[
A_1^{\frac{\alpha}{p(1-t)}} \leq \left( \int g^{\frac{p^*}{\alpha}} \, dx \right)^{\frac{1}{p^*}} \left( \int |u|^{q^*} \, dx \right)^{\frac{\alpha}{q^*}}.
\]

By Sobolev embedding we can conclude that

\[
A_1 \leq C \|g\|_{\frac{p^*}{\alpha}} \|u\|_{1,q}^{(1-t)}
\]

and Lemma 2 follows.

**Lemma 3.** The functional \( I \) satisfies the Palais-Smale condition \((PS)_c\) for any \( c \in \mathbb{R} \).

**Proof.** Let \( (u_n) \subset W \) be a Palais-Smale sequence at a level \( c \in \mathbb{R} \). This means that

\[
I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } W'.
\]

We will show that \( u_n \) is a bounded sequence. Since \( I(u_n) \) is a real convergent sequence then there exists \( M > 0 \) such that

\[
I(u_n) = \frac{1}{p} \|u_n\|_{1,p}^p + \frac{1}{q} \|u_n\|_{1,q}^q - \frac{\lambda}{p} \int g|u_n|^p \, dx < M
\]

\[
\frac{1}{p} \left( \|u_n\|_{1,p}^p + \|u_n\|_{1,q}^q \right) \leq \frac{1}{p} \|u_n\|_{1,p}^p + \frac{1}{q} \|u_n\|_{1,q}^q \leq M + \frac{\lambda}{p} \int g|u_n|^p \, dx.
\]
We entail that
\[
\langle I'(u_n), u_n \rangle = \|u_n\|^p_{1,p} + \|u_n\|^q_{1,q} - \lambda \int g|u_n|^p dx \leq pM.
\]
This means that \(\langle I'(u_n), u_n \rangle\) is bounded. We conclude that \(\frac{1}{q} - \frac{1}{p}\|u_n\|^q_{1,q} = I(u_n) - \frac{1}{p}\langle I'(u_n), u_n \rangle\) is a bounded real sequence i.e. \((u_n)\) is bounded in \(W^{1,q}(\mathbb{R}^N)\).

Put
\[
J(u_n) = \int |\nabla u_n|^p dx - \lambda \int g|u_n|^p dx.
\]
By the above considerations, \(J(u_n)\) is a real convergent sequence. So, there exists \(C > 0\) such that
\[
\int |\nabla u_n|^p dx \leq C + \lambda \int g|u_n|^p dx.
\]
By Lemma 2 and the boundedness of \(u_n\) in \(W^{1,q}(\mathbb{R}^N)\) we can find a positive constant, still denoted \(C\), such that
\[
\|u_n\|^p_{1,p} \leq C(1 + \|u_n\|^{pt}_{1,p}). \tag{2.3}
\]

If \(u_n\) is not bounded in \(W^{1,p}(\mathbb{R}^N)\), we can suppose that \(\|u_n\|_{1,p} \to \infty\). By relation 2.3, we have that
\[
\|u_n\|^{p(1-t)}_{1,p} \leq C(1 + \|u_n\|^{-pt}_{1,p}).
\]
Since \(0 < t < 1\) we conclude that \(\|u_n\|_{1,p}\) is bounded which is a contradiction.

Consequently, \(u_n\) is bounded in \(W\) and then there exists \(u \in W\) such that \(u_n \rightharpoonup u\).

It is easy to show that
\[
\lim_{n \to +\infty} \langle I'(u_n) - I'(u), u_n - u \rangle = 0. \tag{2.4}
\]

Next, we will show that
\[
\lim_{n \to +\infty} \int g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx = 0 \text{ for all } v \in W. \tag{2.5}
\]
We have by Hölder inequality and assumption 1.6 that for any \(R > 0\)
\[
| \int_{B_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)v dx |
\]
\[ \leq \|g\|_\infty \left( \int_{B_R} |u_n|^{p-2}u_n - |u|^{p-2}u|^{|p'|} \right)^{\frac{1}{|p'|}} \left( \int_{B_R} |v|^{p'} \right)^{\frac{1}{p'}}. \]

Here \( B_R \) denotes the ball of radius \( R \) in \( \mathbb{R}^N \) centred at the origin and we will denote \( B'_R = \mathbb{R}^N \setminus B_R \).

So by Sobolev embedding it yields

\[ |\int_{B_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx| \]
\[ \leq C\|g\|_\infty \|v\| \left( \int_{B_R} |u_n|^{p-2}u_n - |u|^{p-2}u|^{|p'|} \right)^{\frac{1}{|p'|}}. \]

Since \( u_n \) is weakly convergent to \( u \) in \( W^{1,p}(\mathbb{R}^N) \) then \( (\chi_{B_R}u_n) \) is also weakly convergent to \( (\chi_{B_R}u) \) in \( W^{1,p}(B_R) \). We can deduce that \( (\chi_{B_R}u_n) \) converges strongly to \( (\chi_{B_R}u) \) in \( L^{(p')'(p-1)}(B_R) \) since \( (p')'(p-1) < p^* \). Then there exists a subsequence, still denoted \( (\chi_{B_R}u_n) \), and \( h \in L^{(p')'(p-1)}(B_R) \) such that \( \chi_{B_R}u_n \to \chi_{B_R}u \) a.e. in \( B_R \) as \( n \to \infty \) and for all \( n \), \( |\chi_{B_R}u_n| \leq h \) a.e. in \( B_R \). It follows that \( \chi_{B_R}|u_n|^{p-2}u_n \to \chi_{B_R}|u|^{p-2}u \) a.e. in \( B_R \) and \( \chi_{B_R}|u_n|^{p-1} \to h^{p-1} \) a.e. in \( B_R \). By the Lebesgue Theorem there exists another subsequence, still denoted \( (\chi_{B_R}u_n) \), such that \( \chi_{B_R}|u_n|^{p-2}u_n \to \chi_{B_R}|u|^{p-2}u \) strongly in \( L^{(p')'}(B_R) \).

In another hand we have by the Hölder inequality that for any \( v \in W \)

\[ |\int_{B'_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx| \leq \left( \int_{B'_R} g\|u_n|^{p-2}u_n - |u|^{p-2}u|^{|\alpha_p^*|} \right)^{\frac{(p-1)q^*}{|p|}} \cdot \left( \int_{B'_R} g|v|^{|\alpha_p^*$\frac{|q^*-p|}{|p-1|q^*}|} \right)^{\frac{1}{|p|}}. \]

Applying again the Hölder inequality we get

\[ \int_{B'_R} g|v|^{|\alpha_p^*$\frac{|q^*-p|}{|p-1|q^*}|} dx \leq \|g\|_{L^{(\frac{\alpha q^*}{p^*})'}(B'_R)} \left( \int_{B'_R} g\frac{p^{q^*-p\alpha} q^{p^*}}{|q^*-p|} |v|^{p^*} dx \right)^{\frac{\alpha q^*}{p^*} |q^*-p|} \]
\[ \leq C\|g\|_{L^{(\frac{\alpha q^*}{p^*})'}(B'_R)} \|v\|_{1,p}^{\frac{q^*}{q^*+p\alpha}} \|v\|_{1,p}^{\frac{q^*}{q^*+p\alpha}}. \]

Boundedness of \( (u_n) \) together with the above assertions yield

\[ |\int_{B'_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx| \leq C\|g\|_{L^{(\frac{\alpha q^*}{p^*})'}(B'_R)} \|v\|. \]
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where \( A = \frac{\left( \frac{q^*}{n} \right)'}{\left( \frac{\alpha p^-}{(p-1)q^*} \right)'} \). It follows that

\[
\int_{B_R} g(|u_n|^{p-2}u_n - |u|^{p-2}u)dx \to 0
\]

when \( R \to \infty \), since \( g \in L^1(\frac{q^*}{n})(\mathbb{R}^N) \).

Consequently, \( \lim_{n \to +\infty} \int g(|u_n|^{p-2}u_n - |u|^{p-2}u)dx = 0 \) for all \( v \in W \).

From relations 2.4 and 2.5 we get

\[
\lim_{n \to +\infty} \int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx + \\
+ \int (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u)(\nabla u_n - \nabla u)dx = 0.
\]

By the Hölder inequality we have

\[
\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)dx \geq \int |\nabla u_n|^p dx + |\nabla u|^p dx - \\
- \left( \int |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \left( \int |\nabla u|^p dx \right)^{\frac{1}{p}} - \left( \int |\nabla u_n|^p dx \right)^{\frac{1}{p}} \left( \int |\nabla u|^p dx \right)^{\frac{p-1}{p}} \\
= (\|u_n\|_{1,p}^{p-1} - \|u\|_{1,p}^{p-1})(\|u_n\|_{1,p} - \|u\|_{1,p}) \geq 0.
\]

By the same argument it yields

\[
\int (|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u)(\nabla u_n - \nabla u)dx \geq (\|u_n\|_{1,q}^{q-1} - \|u\|_{1,q}^{q-1})(\|u_n\|_{1,q} - \|u\|_{1,q}) \geq 0.
\]

It follows that \( \lim_{n \to +\infty} \|u_n\|_{1,p} = \|u\|_{1,p} \) and \( \lim_{n \to +\infty} \|u_n\|_{1,q} = \|u\|_{1,q} \). This together with the weak convergence of \( u_n \) to \( u \) in \( W \) implies that \( u_n \) is strongly convergent to \( u \) in \( W \) and the proof is complete. \( \square \)

Next, we will show that the functional \( I \) given by 1.7 satisfies the Mountain pass geometry.

**Lemma 4.**

1. There exist \( \rho, \beta > 0 \) such that \( I(u) \geq \beta \) on \( \|u\| = \rho \).

2. There exists \( u_0 \in W \) with \( \|u\| > \rho \) and \( I(u_0) < 0 \).
Proof. (1) Let \( u \in W \), we put \( \rho = \| u \| = \rho_1 + \rho_2 \) were \( \| u \|_{1,p} = \rho_1 \) and \( \| u \|_{1,q} = \rho_2 \). By relation 2.2 it yields

\[
I(u) \geq \frac{1}{p} \rho_1^p - \frac{\lambda}{p} C \| g \|_\infty \| g \|_{(\frac{q}{\alpha})'}^{p(1-t)} \rho_1^p \rho_2^{q(1-t)}
\]

\[
\geq \frac{1}{p} \rho_1^p \left( \rho_1^{p(1-t)} - \lambda C \| g \|_\infty \| g \|_{(\frac{q}{\alpha})'}^{p(1-t)} \rho_2^{q(1-t)} \right)\]

We can choose \( \rho_2 = \varepsilon \) and \( \rho_1 = \left( 1 + \lambda C \| g \|_\infty \| g \|_{(\frac{q}{\alpha})'}^{p(1-t)} \varepsilon^{q(1-t)} \right)^{\frac{1}{p(1-t)}} \) for a sufficiently small \( \varepsilon > 0 \). Consequently

\[
I(u) \geq \frac{1}{p} \left( 1 + \lambda C \| g \|_\infty \| g \|_{(\frac{q}{\alpha})'}^{p(1-t)} \varepsilon^{q(1-t)} \right)^{\frac{1}{(1-t)}} > 0
\]

for any \( u \in W \) such that \( \| u \| = \left( 1 + \lambda C \| g \|_\infty \| g \|_{(\frac{q}{\alpha})'}^{p(1-t)} \varepsilon^{q(1-t)} \right)^{\frac{1}{p(1-t)}} + \varepsilon \).

(2) We denote by \( \varphi \) the normalized eigenfunction associated to the first eigenvalue \( \lambda_1 \) of the \( p \)-Laplacian with weight \( g \), namely

\[-\text{div}(\nabla |\varphi|^{p-2} \nabla \varphi) = \lambda_1 g |\varphi|^{p-2} \varphi \text{ in } \mathbb{R}^N\]

and

\[
\int |\nabla \varphi|^p dx = 1.
\]

Hence,

\[
I(\tau \varphi) = \frac{\tau^p}{p} + \frac{\tau^q}{q} \int |\nabla \varphi|^q dx - \frac{\lambda \tau^p}{p} \int g |\varphi|^p dx, \tau > 0.
\]

Since \( \int g |\varphi|^p dx = \frac{1}{\lambda_1} \) we get

\[
I(\tau \varphi) = \frac{\tau^p}{p} (1 - \frac{\lambda}{\lambda_1}) + \frac{\tau^q}{q} \int |\nabla \varphi|^q dx.
\]

We claim that any eigenvalue of problem 1.4 satisfies \( \lambda > \lambda_1 \). So \( I(\tau \varphi) \rightarrow -\infty \) when \( \tau \rightarrow +\infty \). Consequently, there exists \( \tau_0 > 0 \) such that \( I(\tau_0 \varphi) < 0 \) and we put \( u_0 = \tau_0 \varphi \).
We return now to the claim that any eigenvalue $\lambda$ of problem 1.4 satisfies $\lambda > \lambda_1$. For this, we introduce the quantity

$$\lambda^* = \inf_{\substack{u \in W, \ u \neq 0}} \frac{\int |\nabla u|^p dx + \int |\nabla u|^q dx}{\int g|u|^p dx}.$$ 

For any $u \in W$ we have

$$\frac{\int |\nabla u|^p dx + \int |\nabla u|^q dx}{\int g|u|^p dx} \geq \inf_{\substack{u \in W, \ u \neq 0}} \frac{\int |\nabla u|^p dx}{\int g|u|^p dx} \geq \inf_{u \in W^{1,p}(\mathbb{R}^N)} \frac{\int |\nabla u|^p dx}{\int g|u|^p dx} = \lambda_1.$$ 

So it follows that $\lambda^*$ is a positive real number.

We suppose that there exists an eigenvalue $\lambda$ of problem 1.4 such that $\lambda < \lambda^*$. So there exists $v \in W, \ v \neq 0$ that verifies

$$\int |\nabla v|^p dx + \int |\nabla v|^q dx = \lambda \int g|v|^p dx.$$ 

Then we get

$$\lambda^* > \lambda = \frac{\int |\nabla v|^p dx + \int |\nabla v|^q dx}{\int g|v|^p dx} \geq \inf_{\substack{u \in W, \ u \neq 0}} \frac{\int |\nabla u|^p dx + \int |\nabla u|^q dx}{\int g|u|^p dx} = \lambda^*$$ 

which is a contradiction. So, there is no eigenvalue less than $\lambda^*$ and it is clear that $\lambda_1 < \lambda^*$. In addition, $\lambda^*$ cannot be an eigenvalue of problem 1.4. Indeed, let $u_n \in W$ a minimizing sequence of $\lambda^*$. Similar arguments used in [3] show that $u_n$ converges strongly to a nontrivial function $u \in W$ that satisfies

$$p \int |\nabla u|^{p-2} \nabla u. \nabla wdx + q \int |\nabla u|^{q-2} \nabla u. \nabla wdx = \lambda^* p \int g|u|^{p-2} wdx$$ 

for all $w \in W$. This fact together with definition 2.1 implies that

$$\int |\nabla u|^{q-2} \nabla u. \nabla wdx = 0 \text{ for all } w \in W.$$ 

Hence, $u \equiv 0$ which is a contradiction.

**Proof of Theorem 1.** Define the minimax class

$$B = \{ \psi \in C([0, 1], W), \psi(0) = 0, \psi(1) = u_0 \}$$

and the corresponding minimax level

$$c = \inf_{\psi \in B} \max_{\tau \in [0, 1]} I(\psi(\tau)).$$
By the previous lemmas it follows that the assumptions of the Mountain-Pass Theorem are fulfilled. Therefore for any $\lambda > \lambda^*$, $c$ is a critical value of $I$ associated to a critical point $u_\lambda \in W$. Namely, $I'(u_\lambda) = 0$ and $I(u_\lambda) = c$. By Lemma 4(1) we have necessarily $c \geq \frac{1}{p}(1 + \lambda C \|g\|_\infty \|g\|_{p\frac{p-1}{2}} \varepsilon_{q(1-t)}^{1-t} (q^{1-t})^{\frac{1-p}{p-1}} > 0$. Hence, $u_\lambda$ cannot be trivial since $I(0) = 0$. Hence, Theorem 1 is proved. 

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References


