International Journal of Pure and Applied Mathematics
Volume 80 No. 5 2012, 647-655
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

LCEM RECIPROCAL GCED MATRICES

Zahid Raza[1 §], Seemal Abdul Waheed[2]

[1,2]Department of Mathematics
National University of Computer and
Emerging Sciences
Lahore Campus, PAKISTAN

Abstract: We have given structure theorem of lcem reciprocal gced matrix on a set $S = \{x_1, x_2, \ldots, x_n\}$ where $x_1 < x_2 < \ldots < x_n$ and the gced and lcem exists for every $x_i, x_j \in S$. We have calculated the determinant and inverse of lcem reciprocal gced matrix on an exponential divisor closed set.

AMS Subject Classification: 15A36, 15A60, 15B36
Key Words: Lcem reciprocal gced matrix, exponential divisor and exponential closed set

1. Introduction

In 1876, H.J. Smith [20] proved that the determinant of a GCD matrix on $S = \{1, 2, \ldots, n\}$ is equal to $\varphi(1)\varphi(2)\ldots\varphi(n)$ where φ is Euler's totient function. The result holds if S is a factor closed set. Structure theorems for Reciprocal GCD matrices and LCM matrices were introduced by S.J. Beslin [2]. Structures of Power GCD matrix, Power LCM matrix, Reciprocal LCM matrix, GCD Reciprocal LCM matrix, GCUD Reciprocal LCUM matrices,GCED Reciprocal GCED matrices, GCED Reciprocal LCEM matrices, LCEM Reciprocal GCED matrices have been determined [1], [3], [15], [18], [19] and [25]. Structure, determinant and inverses of Fermat GCD matrices, Mersenne GCUD matrices and Reciprocal Mersenne GCUD matrices have been determined by S. Büyükköse and D. Tasci in [5], [6] and [7]. Research has also been extended to divisibility properties of the above mentioned matrices and their applications [4], [8], [9], [10], [11], [12], [13], [14], [16], [17], [22], [23], [24], [27], [28] and [29].

Received: June 5, 2012

© 2012 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

We recall that an integer $d = \prod_{i=1}^{t} {p_i}^{a_i}$ is said to be an exponential divisor of $m = \prod_{i=1}^{t} {p_i}^{b_i}$, if $a_i|b_i$ for every $1 \leq i \leq t$ and is denoted by $d|_e m$. This notion was introduced by M. V. Subarrao [21]. Note that unlike divisor and unitary divisor, 1 is not an exponential divisor for every $m > 1$. By convention $1|_e 1$. The smallest exponential divisor of $m > 1$ is its square free kernel $\kappa(m) = \prod_{i=1}^{r} p^i$ [26].

Two integers n and m have common exponential divisor if and only if they have the same prime factors. Two integers $m = \prod_{i=1}^{r} {p_i}^{b_i}$ and $n = \prod_{i=1}^{r} {p_i}^{c_i}$ are exponentially co-prime if $(b_i, c_i) = 1$ for every $1 \leq i \leq r$. We denote the lcem of two integers m and n by $[m, n]_e$. By convention $[1, 1]_{(e)} = 1$ and $[1, m]_{(e)}$ does not exist for every $m > 1$.

Lcem of two integers $m = \prod_{i=1}^{k} {p_i}^{a_i}$ and $n = \prod_{i=1}^{k} {p_i}^{b_i}$ is given as

$$[m, n]_e = \prod_{i=1}^{k} p_i^{[a_i, b_i]},$$

where $[a_i, b_i]$ is the usual lcm for all $i = 1, 2, \ldots, k$.

A set $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is said to be an exponential divisor closed set if the exponential divisors of every element of S belongs to S. For example $\{12, 18, 36\}$ is not an exponential divisor closed set. But, $\{6, 12, 18, 36\}$ is an exponential divisor closed set.

Similarly, a set $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is said to be lcem closed if $[x_i, x_j]_{(e)} \in S$ for every $x_i, x_j \in S$. Note that $\{6, 12, 18, 36\}$ is also an lcem closed set.

The exponential convolution of two arithmetic functions f and g is given as

$$(f \odot g)(n) = \sum_{k_1 l_1 = m_1} \cdots \sum_{k_r l_r = m_r} f({p_1}^{k_1} {p_2}^{k_2} \ldots {p_r}^{k_r}) g({p_1}^{l_1} {p_2}^{l_2} \ldots {p_r}^{l_r})$$

where $n = {p_1}^{m_1} {p_2}^{m_2} \ldots {p_r}^{m_r}$.

The inverse with respect to $\odot$ of the constant function 1 is called the exponential analogue of Möbius function and is denoted by $\mu^{(e)}$.

2. Structure of LCEM Reciprocal GCED Matrix

Consider the set $S = \{x_1, x_2, \ldots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$. The $n \times n$ matrix $(S)_{(e)} = (s_{ij})_{(e)}$ having $s_{ij} = \frac{[x_i, x_j]_{(e)}}{(x_i, x_j)_{(e)}}$ as its ij^{th} entry is called lcem reciprocal gced matrix on S, where $(x_i, x_j)_{(e)}$ is the greatest common exponential divisor of x_i and x_j and $[x_i, x_j]_{(e)}$ is the greatest common exponential multiple of x_i and x_j. LCEM reciprocal gced matrices are symmetric.

We define an arithmetic function $f(n)$ as follows:

$$f(n) = \sum_{a_1 b_1 = c_1} \sum_{a_2 b_2 = c_2} \cdots \sum_{a_r b_r = c_r} \frac{1}{{p_1}^{2a_1} {p_2}^{2a_2} \cdots {p_r}^{2a_r}} \mu^{(e)}({p_1}^{b_1} {p_2}^{b_2} \cdots p_r^{b_r}) \quad (1)$$

where $n = {p_1}^{c_1} {p_2}^{c_2} \cdots {p_r}^{c_r}$.

Theorem 1. *Let $R = \{y_1, y_2, \ldots, y_m\}$ be an exponential closure of the set $S = \{x_1, x_2, \ldots, x_n\}$ where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$.*

Define the $n \times m$ matrix $C = (c_{ij})$ by

$$c_{ij} = \begin{cases} x_i, & y_j |_e x_i \\ 0, & \text{otherwise} \end{cases}$$

and the $m \times m$ diagonal matrix by

$$\Psi = diag(f(x_1), f(x_2), \ldots, f(x_m))$$

Then,

$$(S)_{(e)} = C \Psi C^t.$$

Proof. The ij^{th} entry of $C\Psi C^t$ is equal to

$$\begin{aligned} (C\Psi C^t)_{ij} &= \sum_{k=1}^{n} c_{ik} f(y_k) c_{jk} = \sum_{y_k |_e x_i, y_k |_e x_j} x_i x_j f(y_k) \\ &= \sum_{y_k |_e (x_i, x_j)_{(e)}} x_i x_j f(y_k) \end{aligned}$$

where the function f is defined in Equation 1.

By Möbius Inversion formula, we have,

$$\sum_{d |_e n} f(d) = \frac{1}{n^2}$$

Finally, we get,

$$(C\Psi C^t)_{ij} = \frac{x_i x_j}{(x_i, x_j)_{(e)}^2} = \frac{[x_i, x_j]_{(e)}}{(x_i, x_j)_{(e)}}.$$

Theorem 2. *Let $R = \{y_1, y_2, \ldots, y_m\}$ be an exponential closure of the set $S = \{x_1, x_2, \ldots, x_n\}$ where $y_1 < y_2 < y_3 < \cdots < y_m$ and $x_1 < x_2 < x_3 < \cdots < x_n$. Then*

$$\det(S)_{(e)} = \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} (\det C_{(k_1,k_2,\ldots,k_n)})^2 f(x_{k_1}) f(x_{k_2}) \ldots f(x_{k_n})$$

where $C_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of C consisting of the ${k_1}^{th}$, ${k_2}^{th}, \ldots, {k_n}^{th}$ columns of C.

Proof. By Theorem 1, we have, $(S)_{(e)} = (C\Psi^{\frac{1}{2}})(C\psi^{\frac{1}{2}})^t$. Thus we can write $E = C\Psi^{\frac{1}{2}}$ which leads us to $(S)_{(e)} = EE^t$ and by applying Cauchy-Binet formula, we get

$$\begin{aligned}\det(S)_{(e)} &= \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} \det E_{(k_1,k_2,\ldots,k_n)} \det {E^t}_{(k_1,k_2,\ldots,k_n)} \\ &= \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} (\det E_{(k_1,k_2,\ldots,k_n)})^2\end{aligned}$$

where $E_{(k_1,k_2,\ldots,k_n)}$ is the submatrix of E consisting of the ${k_1}^{th}, {k_2}^{th}, \ldots, {k_n}^{th}$ columns of E.

$$\det E_{(k_1,k_2,\ldots,k_n)} = \sqrt{g(x_{k_1}) g(x_{k_2}) \ldots g(x_{k_n})} \det C_{(k_1,k_2,\ldots,k_n)}.$$

Hence,

$$\det(S)_{(e)} = \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} (\det C_{(k_1,k_2,\ldots,k_n)})^2 g(x_{k_1}) g(x_{k_2}) \ldots g(x_{k_n}).$$

Corollary 3. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite ordered set of distinct positive integers. If $S = R$, then the determinant of lcem reciprocal gced matrix $(S)_{(e)}$ defined on S is*

$$\det(S)_{(e)} = \prod_{k=1}^{n} x_k^2 f(x_k).$$

Proof. C is a lower triangular matrix with diagonal $(x_1, x_2, \ldots, x_n)$. This implies that $\det C = \prod_{k=1}^{n} x_k$. The determinant of a diagonal matrix is equal to the product of its diagonal entries so the determinant of Ψ is equalt to $\prod_{k=1}^{n} f(x_k)$ which leads us to the desired result.

Corollary 4. *If $(S)_{(e)}$ is an $n \times n$ lcem reciprocal gced matrix on a set $S = \{x_1, x_2, \ldots, x_n\}$, then the trace is given as*

$$tr(S)_{(e)} = n.$$

Lemma 5. *Let $(S)_{(e)} = (s_{ij})_{(e)}$ is an $n \times n$ lcem reciprocal gced matrix defined on an exponential divisor closed set S. Consider $n \times n$ matrix $C = (c_{ij})$ as defined in Theorem 1. Then, the $n \times n$ matrix $W = (w_{ij})$ defined by*

$$w_{ij} = \begin{cases} \frac{1}{x_j}\mu^{(e)}(\frac{x_i}{x_j}), & x_j|_e x_i \\ 0, & otherwise \end{cases}$$

is the inverse of C.

Proof. The ij^{th} entry of CW is given by

$$\begin{aligned} (CW)_{ij} &= \sum_{k=1}^{n} c_{ik} w_{kj} = \sum_{x_k|_e x_i, x_j|_e x_k} \frac{x_i}{x_j}\mu^{(e)}(\frac{x_k}{x_j}) \\ &= \frac{x_i}{x_j} \sum_{x_d|_e \frac{x_i}{x_j}} \mu^{(e)}(x_d) = \begin{cases} 1, & if \;\; x_i = x_j \\ 0, & otherwise \end{cases} \end{aligned}$$

Theorem 6. *Let $(S)_{(e)}$ be $n \times n$ lcem reciprocal gced matrix on an exponential divisor closed set.*

Then, its inverse matrix $(A)_{(e)} = (a_{ij})_{(e)}$ is given as

$$(a_{ij})_{(e)} = \frac{1}{x_i x_j} \sum_{x_i|_{(e)} x_d, x_j|_{(e)} x_d} \frac{\mu^{(e)}(\frac{x_d}{x_i})\mu^{(e)}(\frac{x_d}{x_j})}{g(x_d)}.$$

Proof. Since $(S)_{(e)} = (C\Psi C^t)$ and by *Lemma* 5, we have $C^{-1} = W$, then

$$(S)_{(e)}^{-1} = (C\Psi C^t)^{-1} = W^t \Psi^{-1} W$$

where ij^{th} entry of $(S)_{(e)}^{-1}$ is given as

$$(a_{ij})_{(e)} = \sum_{x_i|_{(e)} x_d, x_j|_{(e)} x_d} \frac{\frac{1}{x_i x_j}\mu^{(e)}(\frac{x_d}{x_i})\mu^{(e)}(\frac{x_d}{x_j})}{g(x_d)}.$$

Hence, the required result.

3. Numerical Results

Example 1. Let $S = \{12, 18, 36\}$. The lcem reciprocal gced matrix $(S)_{(e)}$ on S is given as:

$$(S)_{(e)} = \begin{pmatrix} 1 & 6 & 3 \\ 6 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

Note that $S = \{12, 18, 36\}$ is not an exponential divisor closed set. Its exponential closure is $R = \{6, 12, 18, 36\}$. The 3×4 matrix $(C)_{(e)}$ is

$$C = \begin{pmatrix} 12 & 12 & 0 & 0 \\ 18 & 0 & 18 & 0 \\ 36 & 36 & 36 & 36 \end{pmatrix}$$

$f(6) = \frac{1}{2^2 3^2}\mu^{(e)}(2 \cdot 3) = \frac{1}{36}$,
$f(12) = \frac{1}{2^2 3^2}\mu^{(e)}(2^2 \cdot 3) + \frac{1}{2^4 3^2}\mu^{(e)}(2 \cdot 3) = \frac{-1}{48}$,
$f(18) = \frac{1}{2^2 3^2}\mu^{(e)}(2 \cdot 3^2) + \frac{1}{2^2 3^4}\mu^{(e)}(2 \cdot 3) = \frac{-2}{81}$,
$f(36) = \frac{1}{2^2 3^2}\mu^{(e)}(2^2 3^2) + \frac{1}{2^4 3^2}\mu^{(e)}(2 \cdot 3^2) + \frac{1}{3^4 2^2}\mu^{(e)}(2^2 \cdot 3) + \frac{1}{2^4 3^4}\mu^{(e)}(2 \cdot 3) = \frac{1}{54}$.

By applying the formula given in *Theorem* 2 , the determinant of lcem reciprocal gced matrix is

$$\det(S)_{(e)} = \begin{vmatrix} 12 & 12 & 0 \\ 18 & 0 & 18 \\ 36 & 36 & 36 \end{vmatrix}^2 f(6)f(12)f(18) + \begin{vmatrix} 12 & 12 & 0 \\ 18 & 0 & 0 \\ 36 & 36 & 36 \end{vmatrix}^2 f(6)f(12)f(36) +$$

$$\begin{vmatrix} 12 & 0 & 0 \\ 18 & 18 & 0 \\ 36 & 36 & 36 \end{vmatrix}^2 f(6)f(18)f(36) + \begin{vmatrix} 12 & 0 & 0 \\ 0 & 18 & 0 \\ 36 & 36 & 36 \end{vmatrix}^2 f(12)f(18)f(36)$$

Hence, the determinant is given as: $\det(S)_{(e)} = (7776)^2 \left[\frac{36-27-32+24}{2519424}\right] = 24$

Example 2. Let $S = \{2, 4, 16\}$. The set is exponential divisor closed, so we apply the corollary to Theorem 2 directly to calculate the determinant. The lcem reciprocal gced defined on S is

$$(S)_{(e)} = \begin{bmatrix} 1 & 6 & 3 \\ 6 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

The determinant is

$$\det(S)_{(e)} = \prod_{k=1}^{3} x_k^2 f(x_k) = 45,$$

where, $f(2) = \frac{1}{2^2}\mu^{(e)}(2) = \frac{1}{4}$,
$f(4) = \frac{1}{2^2}\mu^{(e)}(2^2) + \frac{1}{2^4}\mu^{(e)}(2) = \frac{-3}{16}$ and
$f(16) = \frac{1}{2^2}\mu^{(e)}(2^4) + \frac{1}{2^4}\mu^{(e)}(2^2) + \frac{1}{2^8}\mu^{(e)}(2) = \frac{-15}{256}$.

Example 3. Let $S = \{2, 4, 16\}$. The 3×3 lcem reciprocal gced matrix $(S)_{(e)}$ defined on S has already been defined in *Example* 2. The entries of the inverse matrix are calculated by using *Theorem* 6.

$$a_{11} = \frac{1}{4}\frac{\mu^{(e)}(2)\mu^{(e)}(2)}{(\frac{1}{4})} + \frac{1}{4}\frac{\mu^{(e)}(2^2)\mu^{(e)}(2^2)}{(\frac{-3}{16})} + \frac{1}{4}\frac{\mu^{(e)}(2^4)\mu^{(e)}(2^4)}{(\frac{-15}{256})} = \frac{-1}{3},$$
$$a_{12} = \frac{1}{8}\frac{\mu^{(e)}(2^2)\mu^{(e)}(2)}{(\frac{-3}{16})} + \frac{1}{8}\frac{\mu^{(e)}(2^4)\mu^{(e)}(2^2)}{(\frac{-15}{256})} = \frac{2}{3},$$
$$a_{13} = \frac{1}{32}\frac{\mu^{(e)}(2^4)\mu^{(e)}(2)}{(\frac{-15}{256})} = 0,$$
$$a_{22} = \frac{1}{16}\frac{\mu^{(e)}(2)\mu^{(e)}(2)}{(\frac{-3}{16})} + \frac{1}{16}\frac{\mu^{(e)}(2^2)\mu^{(e)}(2^2)}{(\frac{-15}{256})} = \frac{-7}{5},$$
$$a_{23} = \frac{1}{64}\frac{\mu^{(e)}(2^2)\mu^{(e)}(2)}{(\frac{-15}{256})} = \frac{4}{15},$$
$$a_{33} = \frac{1}{256}\frac{\mu^{(e)}(2)\mu^{(e)}(2)}{(\frac{-15}{256})} = \frac{-1}{15}.$$

$$(S)^{-1}_{(e)} = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{-7}{5} & \frac{4}{15} \\ 0 & \frac{4}{15} & \frac{-1}{15} \end{bmatrix}$$

References

[1] E. Altinisk, D. Tasci, On a generalization of LCM matrix, *Common. fac. Sci. Univ. Ank. Series A1*, **51** (2) (2002), 37-46.

[2] S.J. Beslin, S. Ligh, Greatest common divisor matrices, *Linear Algebra Appl.*, **118** (1989), 69-76.

[3] S. Beslin, Reciprocal GCD matrices and LCM matrices, *Fibonacci Quart.*, **29** (1991), 271-274.

[4] G. Bhowmik, S. Hong, Divisibilty properties of power GCD matrices and power LCM matrices, *Linear Algebra Appl.*, **428** (4) (2008), 1001-1008.

[5] S. Büyükköse, Fermat GCD matrices, *Int. J. Contemp-Math. Science*, **3** (35) (2008), 1745-1753.

[6] S. Büyükköse, D. Tasci, On the Mersenne GCUD matrices, *International Math Journal*, **3** (1) (2003), 101-105.

[7] S. Büyükköse, D. Tasci, On the Reciprocal Mersenne GCUD matrices, *International Math Journal*, **3** (6) (2003), 641-645.

[8] W. Feng, S. Hong, J. Zhao, Divisibilty of power LCM matrices by power GCD matrices on gcd-closed sets, *Discrete Mathematics*, **309** (2009), 2627-2639.

[9] C. He, Divisibilty of determinant of power matrices on GCD-closed sets, *Acta Math. Sinica (Series A)*, **49** (2006), 647-650.

[10] C. He, J. Zhao, More on divisibilty of determinants of LCM matrices on GCD closed sets, *Sotheast Asian Bull. Math.*, **29** (2005), 887-893.

[11] S. Hong, Divisibility properties of power GCD matrices and power LCM matrices, *Linear Algebra Appl.*, **428** (2008), 1001-1008.

[12] S. Hong, Divisibility of determinants of least common multiple matrices on gcd closed sets, *Southeast Asian Bull. Math.*, **27** (2003), 615-621.

[13] P. Haukkanen, I. Korkee, Notes on the divisibility of LCM and GCD matrices, *International J. Math. and Math. Science*, **6** (2005), 925-935.

[14] P. Haukkanen, S. Kirkland, A topological approach to divisibility of arithmetic functions and GCD matrices, *Linear and Multilinear Algebra*, **1** 9(2011)iFirst.

[15] A. Nalli, D. Tasci, The GCD-Reciprocal LCM matrices on gcd closed sets, *Mathematical and Computational Applications*, **9** (2004), 101-106.

[16] B. V. Rajarama Bhat, On greatest common divisor matrices and their applications, *Linear Algebra Appl.*, **158** (1991), 77-97.

[17] Z. Raza, S. A. Waheed, Divisibility properties of GCED and LCEM matrices (submitted).

[18] Z. Raza, S. A. Waheed, GCED and Reciprocal GCED matrices (submitted).

[19] Z. Raza, S. A. Waheed, GCED Reciprocal LCEM matrices (submitted).

[20] H. J. Smith, On the value of a certain arithmetical determinant, *Proc. London Math Soc.*, **7** (1875/1876), 208-212.

[21] M. V. Subarrao, On some arithmetical convolutions, *The theory of arithmetic functions, Lecture notes in Mathematics No. 251*, Springer (1972).

[22] Q. Tan, Z. Lin, L. Liu, Divisibilty among power GCD matrices and among power LCM matrices on finitely many quasi coprime divisor chains, *Applied Mathematics and Computation*, **59** (9) (2011), 969-983.

[23] Q. Tan, F. Zhang, Divisibilty among power GCD matrices and among power LCM matrices on two coprime divisor chains, *Linear and Multilinear Algebra*, **58** (5) (2010), 659-671.

[24] Q. Tan, Z. Lin, L. Liu, Divisibilty among power GCD matrices and among power LCM matrices on two coprime divisor chains II, *Linear and Multilinear Algebra*, **59** (9) (2011), 969-983.

[25] N. Tuglu, D. Tasci, On the LCUM-Reciprocal GCUD matrices, *Competes Rendus de L' academie Bulgare des Sciences*, **55** (2002).

[26] L. *Tóth*, On certain arithmetical functions involving exponential divisors, ArXiv: math/0610274v2.

[27] J. Zhao, A characterization for the gcd closed set S with $|S| = 4$ such that (S^e) divides $[S^e]$, *Sichuan Univ. Nat. Sci. Ed.*, **45** (2008), 475-477.

[28] J. Zhao, S. Hong, Q. Liao, K.P. Shum, On the divisibility of power LCM matrices by power GCD matrices, *Czechoslovak Math. J.*, **57** (2007), 115-125.

[29] W. Zhao, J. Zhao, A characterization for the gcd closed set S with $|S| = 5$ such that (S^e) divides $[S^e]$, *Southeast Asian Bull. Math*, **33** (2009), 1023-1028.