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ALGEBRAIC CHARACTERIZATION OF STRONGLY CONNECTED GRAPHS

Zhivko Nedev
International University VNU-HCM
Thu Duc District
HCM City, VIETNAM

Abstract: We introduce the notion of *uniform rank*. A matrix **C** has an uniform rank r if its (normal) rank(C) equals r and each of its minors of size r is non-zero.

1. Introduction

All digraphs in this paper are loop-free and simple, i.e. multiple edges are not allowed. Let $\mathbf{G} = (V, E)$ be a digraph with a vertex set $V = \{1, 2, \dots, n\}$ and an edge set E. For $1 \leq i \leq n$, we denote with $\mathbf{out}(i)$ the outgoing degree of the vertex i; that is, $\mathbf{out}(i)$ is the number of edges in E that begin at i. Then the laplacian matrix for **G**, **C**, is constructed as usual:

$$\mathbf{C}_{i,j} = \begin{cases} \mathbf{out}(i) & \text{if } i = j, \\ -1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

A well known result by W. T. Tutte from [5], is that the number of oriented spanning subtrees of **G** with root a vertex v equals the determinant taken from the Laplacian with its row and column v deleted. Since a digraph is strongly connected iff for any one (root) vertex v, there exists both an out-oriented spanning tree and an in-oriented one, we can test (using linear algebra) a digraph for strong connectivity with two applications of Tutte's formula. In

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this paper, we show that testing if a digraph is strong connected can be done 2 times faster: with a single run of Gaussian Elimination on the laplacian $\mathbf{C}$.

The author discovered these ideas while researching a family of discrete number-theoretic games called Magnus-Derek Game and Vector Game that were described for the first time in [1, 2]. And some of the techniques in this paper appeared initially in two previous papers by Nedev and Quas [4, 3]. More specifically, in [4], the proof of Lemma 9 is based on strongly connected digraphs with constant out-degree 2. Furthermore, Lemma 1 and Lemma 2 from this paper use techniques similar to the techniques used in Lemma 1 and Lemma 4 from [3].

Definition. We say that a matrix has an uniform rank r if its (normal) rank equals r and if each of its minors of size $r \times r$ is non-zero. (Note that some matrices do not have uniform rank.)

Lemma 1. *Let* $\mathbf{C}$ *be a matrix with* $\operatorname{rank}(\mathbf{C}) = r$. *Then the rank of* $\mathbf{C}$ *is uniform if and only if these two conditions are both true: every* r *row vectors and every* r *column vectors in* $\mathbf{C}$ *are linearly independent.*

Proof. By definition, if $\mathbf{C}$ has an uniform rank r, each of its $r \times r$ minors is non-zero. Thus by choosing an appropriate minor, we can show that every r row vectors and every r column vectors in $\mathbf{C}$ are linearly independent.

Now let every r row vectors and every r column vectors in $\mathbf{C}$ be linearly independent. Suppose the rank of $\mathbf{C}$ is not uniform. Then there exists a $r \times r$ submatrix $\mathbf{C}'$ with $\det(\mathbf{C}') = 0$. Without loss of generality, let $\mathbf{C}'$ be the submatrix in the top left corner (where the first r rows and first r columns of $\mathbf{C}$ intersect). That is

$$\mathbf{C}' = (\mathbf{C}_{i,j})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}}$$

We now consider the submatrix $\mathbf{C}^*$ consisting of the first r columns of $\mathbf{C}$

$$\mathbf{C}^* = (\mathbf{C}_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq r}}$$

By assumption the column rank of $\mathbf{C}^*$ is r.

We now view $\mathbf{C}^*$ as n row vectors each of size r, and note the following:

1. Since $\operatorname{rank}(\mathbf{C}) = r$ and the first r rows of $\mathbf{C}$ are linearly independent, each of the (last) $n - r$ rows in $\mathbf{C}$ is a linear combination of the first r rows. Thus the same is true of the last $n - r$ rows of $\mathbf{C}^*$.

2. Since $\det(\mathbf{C}') = 0$, the first r rows of $\mathbf{C}^*$ are linearly dependent.

Therefore, the row rank of $\mathbf{C}^*$ is at most $r-1$. Since the row rank equals the column rank, this is a contradiction. Therefore, every minor of $\mathbf{C}$ of size $r \times r$ is non-zero. □

The lemma gives an equivalent definition for matrix with an uniform rank.

Definition. We say that a matrix has an uniform rank r if its (normal) rank equals r and if every r row vectors and every r column vectors in $\mathbf{C}$ are linearly independent.

In the next two sections, we will prove that a digraph with n vertices is strongly connected if and only if its laplacian has an uniform rank $n-1$ (the number of its vertices minus one). Although the proof can be done using the Tutte's formula from [5], for greater insight into the problem, we develop a new proof based only on the notions of linear independence and graph strong connectivity.

2. Necessary Condition

Suppose $\mathbf{G}$ is strongly connected, i.e. it contains a directed path from u to v for every pair of vertices u, v. Then its laplacian $\mathbf{C}$ has the following two properties:

Property 1) In the i-th row of $\mathbf{C}$, the entry on the main diagonal is an integer, say k_i with $1 \leq k_i \leq n-1$, exactly k_i other row entries are -1, and the remaining entries are zero. Thus in each row, the entries sum to zero.

Property 2) For each non-trivial proper subset T of $\{1, 2, \ldots, n\}$, there exist $i \in T$ and $j \in \{1, 2, \ldots, n\} \setminus T$ such that $\mathbf{C}_{i,j} = -1$.

The first property is by construction; the second is an equivalent definition for $\mathbf{G}$ to be strongly connected. We denote by $\mathcal{A}(n)$ the family of $n \times n$ matrices satisfying the above two properties.

Lemma 2. *Let the matrix $\mathbf{C} \in \mathcal{A}(n)$ and let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ be the row vectors of $\mathbf{C}$. Then any proper subset of these row vectors is linearly independent.*

Proof. Suppose otherwise. Let t, with $t < n$, be an integer such that there exist t linearly dependent row vectors of $\mathbf{C}$. Clearly $t \geq 2$. Without loss of generality, let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_t$ be linearly dependent. Then, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_t$, not all zero, such that

$$\sum_{i=1}^{t} \alpha_i \vec{r}_i = \vec{0} \tag{1}$$

If all α_i are nonpositive, we multiply equation (1) by -1; so assume that at least one $\alpha_i > 0$. Without loss of generality, let $\alpha_i > 0$ for $i = 1, \ldots, s$, with $1 \leq s \leq t$ and $\alpha_i \leq 0$ for $i = s+1, \ldots, t$. Thus, the first s scalars are positive and the rest, if any, are negative.

We consider the first s scalar equations from the vector equation (1). For $\forall j$, $1 \leq j \leq s$

$$\sum_{i=1}^{t} \alpha_i \mathbf{C}_{i,j} = 0 \tag{2}$$

Since, in addition, $\mathbf{C}_{i,j} \leq 0$ for $i > s, 1 \leq j \leq s$, we have $\alpha_i \mathbf{C}_{i,j} \geq 0$ for $s+1 \leq i \leq t$ and $1 \leq j \leq s$. Thus, we obtain the following s inequalities

$$\sum_{i=s+1}^{t} \alpha_i \mathbf{C}_{i,j} \geq 0, \qquad \forall\, 1 \leq j \leq s \tag{3}$$

From (2) and (3), it follows that

$$\sum_{i=1}^{s} \alpha_i \mathbf{C}_{i,j} \leq 0, \qquad \forall\, 1 \leq j \leq s$$

Summing the above s inequalities, we get

$$\sum_{j=1}^{s} \Big(\sum_{i=1}^{s} \alpha_i \mathbf{C}_{i,j} \Big) \leq 0 \Longleftrightarrow \sum_{i=1}^{s} \Big(\sum_{j=1}^{s} \alpha_i \mathbf{C}_{i,j} \Big) \leq 0 \tag{4}$$

Recall that each $\alpha_i > 0$ for $1 \leq i \leq s$. By property 1 of $\mathbf{C}$, for each $i, 1 \leq i \leq s$, the sum $\sum_{j=1}^{n} \alpha_i \mathbf{C}_{i,j} = 0$. And in this sum, only the term $\alpha_i \mathbf{C}_{i,i}$ is positive. Therefore for each $i, 1 \leq i \leq s$, the sum $\sum_{j=1}^{s} \alpha_i \mathbf{C}_{i,j} \geq 0$ because it still contains the only positive term. Furthermore, by property 2 of $\mathbf{C}$, there must exist $k \in \{1, \ldots, s\}$ and $l \in \{s+1, \ldots, n\}$ such that $\mathbf{C}_{k,l} = -1$. Thus, the sum $\sum_{j=1}^{s} \alpha_k \mathbf{C}_{k,j} > 0$ because $\sum_{j=1}^{n} \alpha_k \mathbf{C}_{k,j} = 0$. It follows that

$$\sum_{i=1}^{s} \Big(\sum_{j=1}^{s} \alpha_i \mathbf{C}_{i,j} \Big) > 0$$

which contradicts inequality (4). Therefore, any proper subset of row vectors from $\mathbf{C}$ is linearly independent. □

Lemma 3. *Let $\mathbf{C} \in \mathcal{A}(n)$. Then its rank is $n-1$.*

Proof. By property 1, the column vectors of $\mathbf{C}$ sums to $\vec{0}$, so its rank is less than n. But by lemma 2, every $n-1$ row vectors of $\mathbf{C}$ are linearly independent. Therefore the rank of $\mathbf{C}$ is $n-1$. □

Lemma 4. *Let $\mathbf{C}$ be a $n \times n$ matrix with column vectors $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ and rank $n-1$, and let there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, all non-zero, such that*

$$\sum_{i=1}^{n} \alpha_i \vec{c}_i = \vec{0} \tag{5}$$

Then any $n-1$ column vectors from $\mathbf{C}$ are linearly independent.

Proof. Suppose otherwise. Since $\text{rank}(\mathbf{C}) = n-1$, there are $n-1$ linearly independent column vectors. Without loss of generality, let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-1}$ be linearly independent.

By assumption, there exists a set of $n-1$ linearly dependent column vectors. Then $\vec{c}_n$ must be a member of this set. Without loss of generality, let one set of linearly dependent vectors be $\vec{c}_2, \vec{c}_3, \ldots, \vec{c}_n$. Then there exist $\beta_2, \beta_3, \ldots, \beta_n$, not all zero, such that

$$\sum_{i=2}^{n} \beta_i \vec{c}_i = \vec{0} \tag{6}$$

where $\beta_n \neq 0$.

From (5) and (6) we obtain

$$\vec{c}_n = -\alpha'_1 \vec{c}_1 - \alpha'_2 \vec{c}_2 - \cdots - \alpha'_{n-1} \vec{c}_{n-1} \qquad \text{where } \alpha'_i = \frac{\alpha_i}{\alpha_n}$$

$$\vec{c}_n = -\beta'_2 \vec{c}_2 - \cdots - \beta'_{n-1} \vec{c}_{n-1} \qquad \text{where } \beta'_i = \frac{\beta_i}{\beta_n}$$

Thus

$$\alpha'_1 \vec{c}_1 + (\alpha'_2 - \beta'_2)\vec{c}_2 + \cdots + (\alpha'_{n-1} - \beta'_{n-1})\vec{c}_{n-1} = \vec{0}$$

with $\alpha'_1 \neq 0$. But $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-1}$ are linearly independent, so this is a contradiction. Therefore, every set of $n-1$ column vectors of $\mathbf{C}$ is linearly independent.

Note that the same lemma is valid if we replace the column vectors with the row vectors of $\mathbf{C}$. □

Since for every digraph, the columns of its laplacian sum to zero, we have

Corollary 5. *If a matrix* $\mathbf{C} \in \mathcal{A}(n)$*, then any* $n-1$ *of its column vectors are linearly independent.*

From all the lemmas to this point, we have

Corollary 6. *Let the matrix* $\mathbf{C} \in \mathcal{A}(n)$*. Then its rank is* $n-1$ *and each of its minors of size* $(n-1) \times (n-1)$ *is non-zero. So any matrix* $\mathbf{C} \in \mathcal{A}(n)$ *has an uniform rank* $n-1$*.*

3. Sufficient Condition

Lemma 7. *Let* $\mathbf{G} = (V, E)$ *be a digraph with* $V = \{1, 2, \ldots, n\}$ *and let its laplacian matrix* $\mathbf{C}$ *have an uniform rank* $n-1$*. Then* $\mathbf{G}$ *is strongly connected.*

Proof. Suppose $\mathbf{G}$ is not strongly connected. Then $\mathbf{G}$ has a strongly connected component, say $\mathbf{S}$, from which there is no edge going out to any other component. Let $\mathbf{S}$ have k vertices; $k < n$ since $\mathbf{G}$ is not strongly connected.

Without loss of generality, we rename the vertices of $\mathbf{G}$ so that the vertices of $\mathbf{S}$ become $\{1, 2, \ldots, k\}$. But then for every i: $1 \leq i \leq k$, end every j: $(k+1) \leq j \leq n$, the matrix entry $C_{i,j} = 0$. Since the component $\mathbf{S}$ is strongly connected, the first $k-1$ rows of $\mathbf{C}$ are linearly independent, but more importantly its k-th row is a linear combination of the first $k-1$.

It follows that any $(n-1) \times (n-1)$ minor of $\mathbf{C}$, that contains the first k rows of $\mathbf{C}$ is zero. This is a contradiction with $\mathbf{C}$ having an uniform rank $n-1$. Therefore $\mathbf{G}$ is strongly connected. □

Combining the results from both sections, we reach

Theorem 8. *A digraph with* n *vertices is strongly connected if and only if its laplacian has an uniform rank* $n-1$*.*

4. Testing if a Digraph is Strongly Connected

Lemma 9. *Let* $\mathbf{G}$ *be a digraph with* n *vertices and laplacian* $\mathbf{C}$*, and let* $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ *be the row vectors of* $\mathbf{C}$*. Then* $\mathbf{C}$ *has an uniform rank* $n-1$ *or equivalently* $\mathbf{G}$ *is strongly connected if and only if these two conditions are both true:*

1. *the normal rank of* $\mathbf{C}$ *is* $n-1$

2. *there exist scalars* $\alpha_1, \alpha_2, \ldots, \alpha_n$*, all non-zero, such that* $\sum_{i=1}^{n} \alpha_i \vec{r}_i = \vec{0}$.

Proof. Let the laplacian $\mathbf{C}$ have an uniform rank $n-1$. Then its normal rank is $n-1$, and so there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that $\sum_{i=1}^{n} \alpha_i \vec{r}_i = \vec{0}$. If any of the scalars is zero, then there is proper subset of $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ that are linearly dependent and the normal rank of $\mathbf{C}$ is not uniform. Contradiction!

Let now $\text{rank}(\mathbf{C}) = n-1$ and all non-zero α s exist, such that $\sum_{i=1}^{n} \alpha_i \vec{r}_i = \vec{0}$. We want to prove that $\mathbf{C}$ has an uniform rank $n-1$. Since $\mathbf{C}$ is the laplacian, the sum of all its columns is zero. Then by lemma 4, every $n-1$ column vectors of $\mathbf{C}$ are linearly independent. Applying lemma 4 again, we get that also every $n-1$ row vectors of $\mathbf{C}$ are linearly independent. By lemma 1, the rank of $\mathbf{C}$ is uniform and equals $n-1$. □

Now to test if the graph's laplacian $\mathbf{C} \in \mathcal{A}(n)$, we can apply Gaussian Elimination(GE) on the transpose of $\mathbf{C}$. In this way, we solve the homogenous equation $\mathbf{C}^T \vec{x} = \vec{0}$ for $\vec{x} = (x_1, x_2, \ldots, x_n)$. By the above lemma, if we get $\text{rank}(\mathbf{C}) = n-1$ and a solution $\vec{x}$ with all x_i non-zero, it follows that the digraph is strongly connected. Otherwise, the digraph has more than one strongly connected component.

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