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SCHEME TERMS

Martin Dowd
1613, Wintergreen Pl.
Costa Mesa, CA 92626, USA

Abstract: In earlier papers the author has used schemes to establish some bounds in small large cardinal theory. Here the methods are improved, yielding improved bounds.

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1. Introduction

The author has produced a series of papers on schemes; see [5] for an overview. This paper adds to the series. Notation as in [5] will be used. Section 2 provides a justification of axiom G. Section 3 gives an overview of chains in the Galvin-Hajnal order, and in the stationary reflection order. In Section 4 scheme terms are defined. Section 5 gives a construction from scheme terms of function in the Galvin-Hajnal order. Section 6 gives a construction from scheme terms of stationary subsets of κ. Section 7 proves enforceability of the sets constructed in Section 6, giving an improved bound on the Mahlo rank of s weakly compact cardinals. Section 8 gives a discussion of new axioms, and proposes a specific axiom which seems justified in view of the results of this paper. Section 9 provides some observations on conjecture 2 of [4].

2. Axiom G

Recall from [5] the definitions of M_Σ and A_Σ. Axiom G is the statement, "$\forall\Sigma A_\Sigma$". A straightforward attempt to justify axiom G by induction on Σ

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encounters the difficulty that, if M_Σ is stationary in V, it can't be concluded that there is a κ where M_Σ is stationary, because Σ turns in to a set if V is collected and is meaningless in the enlarged V.

The difficulty suggests its solution. A "sufficiently large" universe V is postulated, and "current universes" collected within V. V provides a "stock" of schemes. The current universe equals V_μ where μ is the current value of Ord; M_Σ equals $M_\Sigma \cap \mu$, and $\mathrm{LimM}_\Sigma(X)$ equals $\mathrm{Lim}(X) \cap M_\Sigma \cap \mu$. With these adjustments, the inductive justification of cases 1 to 4 of axiom A_Σ is similar to the justification of the axioms I1 to I4 given in [4].

In justifying A_Σ it may be assumed that the current V can be taken (after collecting) to be in M_Σ. In case 1, collecting the current V yields an element of M_Σ; iterating, M_Σ is unbounded in a larger V. In case 2, collecting the current V yields an element of LimM_Σ; iterating, M_Σ is unbounded in a larger V. In case 3, it may be assumed that $\mu \notin T_\Sigma$; then Σ may be replaced by $\Sigma \downarrow \mu$ (see [5] for T_Σ and $\downarrow$). Collecting V, $\mu \in \mathrm{LimM}_\Sigma(X_\xi)$ for each ξ, so $\mu \in X_\xi$ for each ξ, so $\mu \in \cap_\xi$; iterating, M_Σ is unbounded in a larger V. Case 4 is similar to case 3.

The preceding argument justifies A_Σ inductively if $\mathrm{cf}(\sigma)$ is a successor ordinal. If the rank of Σ is a limit ordinal less than Ord^+, it may be assumed that $\mu \notin T_\Sigma$. If V is M_{Σ_ξ} for all ξ then V is M_Σ, so M_Σ is unbounded. The argument is similar when the rank of Σ is Ord^+.

The exercise of proving that the existence of greatly Mahlo cardinals is equivalent to the existence of V_κ satisfying axiom G, and of justifying axiom G, may seem like an academic one, and indeed the author takes the viewpoint in [2] that it is reasonable to assume that the Mahlo operation can be iterated in collecting the universe. But a greater understanding of the behavior of small ("generalized Mahlo") cardinals, and of their relation to foundational issues, has been achieved; and the supposition that the Mahlo operation may be iterated more thoroughly justified.

3. Chains

Adopting notation from earlier papers, Let Inac denote the inaccessible cardinals, and for $\kappa \in \mathrm{Inac}$ let In_κ denote $\mathrm{Inac} \cap \kappa$, and let $\mathrm{In}^\emptyset_\kappa$ denote $\mathrm{In}_\kappa \cup \{\emptyset\}$.

For a filter F on a cardinal κ and $f, g : \kappa \mapsto \kappa$ write $f <_F g$ if $\{\lambda : f(\lambda) < g(\lambda)\} \in F$. Similarly $f \leq_F g$ if $\{\lambda : f(\lambda) \leq g(\lambda)\} \in F$ Basic facts concerning these relation may be found in [6]. ρ_F will be used to denote the rank function of the order $<_F$. For $\kappa \in \mathrm{Inac}$ let F_C denote the club filter; $<_{F_C}$ is often called the Galvin-Hajnal order.

For a nonempty subset $S \subseteq \kappa$, write $F \restriction S$ for $\{X \cap S : X \in F\}$; this is readily seen to be a filter. If κ is Mahlo, F is the club filter, and S is In_κ, the resulting order will be denoted $<_*$, and the rank function ρ_*. This is an alternative to $<_{F_C}$ which is convenient for some uses. For example, the canonical functions f_Σ for schemes Σ may be defined in $<_{F_C}$, and considered in $<_*$ by restricting to In_κ. The function f_{κ^+} is conveniently defined in $<_*$ as $\lambda \mapsto \lambda^+$; it can be defined in $<_{F_C}$ as $\alpha \mapsto |\alpha|^+$.

Recalls (see, e.g., [5]) that for F a normal κ-complete filter on κ, the following operations are of defined. Given an ordinal $\eta < \kappa$ and a sequence f_ξ for $\xi < \eta$ let $\sup_{\xi<\eta} f_\xi$ be the function f where $f(\gamma) = \sup_{\xi<\eta} f_\xi(\gamma)$. Given a sequence f_ξ for $\xi < \kappa$ let $\mathrm{dsup}_{\xi<\kappa}\mathrm{f}_\xi$ be the function f where $f(\gamma) = \sup_{\xi<\gamma} f_\xi(\gamma)$. It is readily seen that $\sup_{\xi<\eta} f_\xi$ is the supremum in the order $\leq_F$ of the f_ξ; and similarly for Finally, note that for κ Mahlo (in fact under weaker hypotheses) $F_C \restriction \mathrm{Inac}$ is normal.

In what follows, classes $\boldsymbol{\alpha}$ in V_κ will be used as codes for ordinals $\alpha < \kappa^{++}$. Functions $f_{\boldsymbol{\alpha}}$ will be defined, so that if $\beta < \alpha$ then $f_{\boldsymbol{\beta}} <_* f_{\boldsymbol{\alpha}}$. It is readily seen that it suffices that $f_{\boldsymbol{\beta}} \leq_* f_{\boldsymbol{\alpha}}$ if $\beta < \alpha$, and if $\alpha = \beta+1$ then $f_{\boldsymbol{\beta}} <_* f_{\boldsymbol{\alpha}}$. Choosing a code for each ordinal $\alpha \leq \sigma$ for an ordinal σ yields a (strictly increasing) chain $\langle f_{\boldsymbol{\alpha}} : \alpha \leq \sigma \rangle$ in $<_*$.

As in earlier papers, for example [5], for $\kappa \in \mathrm{Inac}$, for $X, Y \subseteq \kappa$ say that $X \subseteq_{\mathrm{t}} Y$ if $X - Y$ is thin. For $X \subseteq \mathrm{In}_\kappa^\emptyset$ let $\mathrm{H}(X) = \{\lambda \in X \colon X \cap \lambda$ is a stationary subset of $\lambda\}$. For $X, Y \subseteq \mathrm{In}_\kappa^\emptyset$ say that $X <_R Y$ if $Y \subseteq_{\mathrm{t}} \mathrm{H}(X)$ and let ρ_R denote the rank function.

Defining sets $S_{\boldsymbol{\alpha}}$ for codes $\boldsymbol{\alpha}$ is also of interest, where $S_{\boldsymbol{\alpha}}$ is a stationary subset of $\mathrm{In}_\kappa^\emptyset$. The desired property of these sets is $S_{\boldsymbol{\beta}} <_R S_{\boldsymbol{\alpha}}$ for $\beta < \alpha$. It suffices that $S_{\boldsymbol{\beta}} \supseteq_{\mathrm{t}} S_{\boldsymbol{\alpha}}$ if $\beta < \alpha$, and $S_{\boldsymbol{\beta}} <_R S_{\boldsymbol{\alpha}}$ if $\alpha = \beta + 1$.

Given functions $f_{\boldsymbol{\alpha}}$ such that $f_{\boldsymbol{\beta}} <_* f_{\boldsymbol{\alpha}}$ if $\beta < \alpha$, consider the sets $S_{\boldsymbol{\alpha}} = \{\lambda \in \mathrm{In}_\kappa : \rho_R(\lambda) \geq f_{\boldsymbol{\alpha}}(\lambda)\}$. Clearly $S_{\boldsymbol{\beta}} \supseteq_{\mathrm{t}} S_{\boldsymbol{\alpha}}$ if $\beta < \alpha$. However whether, for example, $\rho_R(\lambda) > \lambda^+$ implies that $\{\mu \in \mathrm{Inac} : \mu < \lambda$ and $\rho(\mu) \geq \mu^+\}$ is stationary, is a question which the author has been unable to settle. Indeed, if $\rho_R(\kappa) > \kappa^+$ where κ is the smallest greatly Mahlo cardinal, then the question has a negative answer. In view of this, the sets $S_{\boldsymbol{\alpha}}$ must be defined directly, in particular so that $S_{\boldsymbol{\alpha}} \equiv_{\mathrm{t}} \mathrm{H}(S_{\boldsymbol{\beta}})$ when $\alpha = \beta + 1$.

4. Scheme Terms

Similarly to definitions in [10], a scheme term over $\kappa \in$ Inac is defined to be a finitary term whose interior nodes are ordinal valued functions of ordinal arguments, and whose leaves are schemes over κ. A scheme term system is given by an adequately effective enumeration of a countable collection of integer names for the functions, and rules for singling out which terms are in normal form. In [10], successively stronger term-based ordinal notation systems are considered, namely, Cantor normal form (CNF), binary Veblen functions (BV), finitary Veblen functions, and Klammersymbols. In this paper, only CNF and BV will be considered; consideration of stronger systems, and of scheme term systems in general, is left to further research.

CNF has already been considered in [5]. As is well-known (see [8] for example) if $\beta > 1$ is an ordinal then any nonzero ordinal α can be written uniquely as $\beta^{\eta_k} \cdot \sigma_k + \cdots + \beta^{\eta_1} \cdot \sigma_1$, where $k \geq 1$, $\sigma_i < \beta$, and $\eta_k > \cdots > \eta_1$. In the CNF scheme term system over κ where $\kappa \in$ Inac, $\beta = \kappa^+$. There is a function f_k for each k and a constant 0. An NF term is of the form $f(\boldsymbol{\eta}, \boldsymbol{\sigma}_k, \ldots, \boldsymbol{\eta}_1, \boldsymbol{\sigma}_1)$ where the $\boldsymbol{\eta}_i$ are NF terms with $\eta_k > \cdots > \eta_1$, and $\boldsymbol{\sigma}_i$ is a scheme. Note that 0 can occur only as the root, or as some η_1.

As in [5], let $\odot$ denote the ordinal exponentiation function, and define $\uparrow$ by the recursion $\alpha \uparrow 0 = 1$, $\alpha \uparrow (\beta + 1) = \alpha \odot (\alpha \uparrow \beta)$, and $\alpha \uparrow \beta = \sup_{\beta' < \beta} \alpha \uparrow \beta'$ for $\beta \in$ Lim. It is easily seen that the closure ordinal of the scheme term system over κ with just the Cantor normal form functions is $\kappa^+ \uparrow \omega$; this ordinal will be denoted $\epsilon_{0\kappa}$.

As for ω, the function $\alpha \mapsto \kappa^{+\alpha}$ can serve as the base of a "Veblen hierarchy", which for the purposes of this paper will be said to be over κ. The hierarchy is given by the binary function ϕ where $\phi(0, \alpha) = \kappa^\alpha$; and for $\beta > 0$ $\phi(\beta, \alpha)$ is the αth element in the enumeration of the values which are fixed points of each function $\alpha \mapsto \phi(\beta', \alpha)$ for $\beta' < \beta$. The function ϕ over κ will also be denoted as ϕ_κ.

In a NF term in any system, the value at a node must be greater than the value at its sons (this restriction is redundant for the system CNF). The system BV over κ is obtained from CNF by adding the function ϕ, and imposing the foregoing restriction, and the restriction that 0 not occur as the first argument of ϕ. The closure ordinal of BV will be denoted as $\Gamma_{0\kappa}$. Using arguments as in [9] the following are readily shown. For every ordinal $\alpha < \Gamma_{0\kappa}$ there is a term $\boldsymbol{\alpha}$ for α; this term is unique up to replacing schemes by schemes of the same rank.

Writing the Cantor normal form for α as $\kappa^{+\eta_k} \cdot \sigma_k + \cdots + \kappa^{+\eta_1} \cdot \sigma_1$, $\boldsymbol{\alpha}$ may be determined by recursion by breaking the root into the following cases. In cases 2 to 6, $k = 1$ and α is written as $\kappa^{+\eta} \cdot \sigma$

Case 0: $\alpha = 0$
Case 1: $k > 1$
Case 2: $\eta = 0$ (so $\boldsymbol{\alpha}$ is a scheme)
Case 3: $\sigma > 1$, $\eta > 0$
Case 4: $\sigma = 1$, $\eta = 1$ (so $\alpha = \kappa^+$)
Case 5: $\sigma = 1$, $\eta => 1$, $\eta < \kappa^{+\eta}$
Case 6: $\sigma = 1$, $\eta = \kappa^{+\eta}$. In this case, $\boldsymbol{\alpha} = \phi(\boldsymbol{\beta}, \boldsymbol{\gamma})$ where $\beta, \gamma < \alpha$.

Recall from [5] the definitions for a scheme $\boldsymbol{\sigma}$ of $\boldsymbol{\sigma}_{\leq \tau}$ for $\tau \leq \sigma$, $T_{\boldsymbol{\sigma}}$, and $\boldsymbol{\sigma} \downarrow \lambda$ for $\lambda \in \mathrm{In}_\kappa$; and their basic properties. For a term $\boldsymbol{\alpha}$ let $T_{\boldsymbol{\alpha}}$ be the union over the $\boldsymbol{\sigma}$ occurring at leaves of $\boldsymbol{\alpha}$ of the $T_{\boldsymbol{\sigma}}$. Let $\boldsymbol{\alpha} \downarrow \lambda$ be $\boldsymbol{\alpha}$, with each leaf $\boldsymbol{\sigma}$ replaced by $\boldsymbol{\sigma} \downarrow \lambda$. Given $\boldsymbol{\alpha}$, the notation $\alpha \downarrow \lambda$ will be used to denote the ordinal specified by $\boldsymbol{\alpha} \downarrow \lambda$; note that in this computation, κ^+ is replaced by λ^+ and ϕ by ϕ_λ.

When giving proofs by induction on schemes, in case 1 of the recursive definition (q.v. see [5]) $\boldsymbol{\tau}$ will be used to denote $\boldsymbol{\sigma}_{\leq \tau}$, and in cases 2 and 3 $f_{\boldsymbol{\sigma}_\xi}$ will be used to denote $\boldsymbol{\sigma}_{\leq \sigma_\xi}$. Also, familiarity with lemma 3 of [5] will be assumed.

Lemma 1. *Suppose $\boldsymbol{\alpha}$ is a term, $\lambda \in \mathrm{In}_\kappa$, and $\lambda \notin T_{\boldsymbol{\alpha}}$. If α is a successor ordinal then $\alpha \downarrow \lambda$ is a successor ordinal. If cf(α) $= \theta$ where $\theta < \kappa$ then $\theta < \lambda$ and cf($\alpha \downarrow \lambda$) $= \theta$. If cf(α) $= \kappa$ then cf($\alpha \downarrow \lambda$) $= \lambda$.*

Proof. This follows readily by induction on $\boldsymbol{\sigma}$ and then $\boldsymbol{\alpha}$. The case $\alpha = \phi(\beta, \gamma)$ may be broken into the following cases; cf(α) is given for each case.

$\beta = 1$, $\gamma = 0$. κ^+.
$\beta = 1$, $\gamma = \gamma' + 1$. ω.
$\beta = \beta' + 1$, $\gamma = 0$. ω.
$\beta = \beta' + 1$, $\gamma = \gamma' + 1$. ω.
$\beta \in \mathrm{Lim}$, $\gamma = 0$. cf(β).
$\beta \in \mathrm{Lim}$, $\gamma = \gamma' + 1$. cf(β).
$\gamma =\in \mathrm{Lim}$. cf(γ). □

5. Functions from Terms

Suppose κ is a Mahlo cardinal. For each BV term $\boldsymbol{\alpha}$, a function $f_{\boldsymbol{\alpha}} : \mathrm{In}_\kappa \mapsto \kappa$ will be defined. First, the definition of $f_{\boldsymbol{\sigma}}$ for a scheme $\boldsymbol{\sigma}$ will be reviewed. The cases are as follows.

Case 0: If $\sigma = 0$ then $f_0 = 0$ (the zero function).

Case 1: If $\sigma = \tau + 1$ then $f_{\boldsymbol{\sigma}} = f_{\boldsymbol{\tau}} + 1$.

Case 2: If $\eta < \kappa$ is a limit ordinal and σ_ξ for $\xi < \eta$ is the ascending chain with limit σ then $f_{\boldsymbol{\sigma}} = \sup_{\xi<\eta} f_{\boldsymbol{\sigma}_\xi}$.

Case 3: If σ_ξ for $\xi < \kappa$ is the ascending chain with limit σ then $f_{\boldsymbol{\sigma}} = \mathrm{dsup}_{\xi<\kappa} \mathrm{f}_{\boldsymbol{\sigma}_\xi}$.

Lemma 2. *If $\sigma' \le \sigma$ then $f_{\boldsymbol{\sigma}'} \le_* f_{\boldsymbol{\sigma}}$.*

Proof. This follows by properties of canonical functions, but a direct proof is of interest; the proof is by induction on σ. In case 0, $\sigma' = 0$ also and the claim is immediate. In case 1, if $\sigma' \le \tau$ then inductively $f_{\sigma'} \le_* f_\tau$, and clearly $f_\tau <_* f_\sigma$. If $\sigma' = \sigma$ then inductively $f_{\sigma'_{\le\tau}} \equiv_* f_{\sigma_{\le\tau}}$, and $f_{\sigma'} \equiv_* f_\sigma$ follows. In case 2, if $\sigma' < \sigma$ then $\sigma' < \sigma_\xi$ for some ξ where $\langle \sigma_\xi : \xi < \eta \rangle$ is the sequence ascending to σ. Inductively, $f_{\boldsymbol{\sigma}'} \le_* f_{\boldsymbol{\sigma}_{\le\sigma_\xi}}$, and clearly $f_{\boldsymbol{\sigma}_{\le\sigma_\xi}} \le_* f_{\boldsymbol{\sigma}}$. If $\sigma' = \sigma$ let $\langle \sigma'_\xi : \xi < \eta' \rangle$ be the sequence ascending to σ'. By what was just proved, $f_{\boldsymbol{\sigma}'_\xi} \le_* f_\sigma$ for all $\xi < \eta'$, whence $f_{\boldsymbol{\sigma}'} \le_* f_\sigma$. The argument for case 3 is similar to that for case 2. $\square$

Lemma 3. *If $\sigma = \tau + 1$ then $f_{\boldsymbol{\sigma}} \equiv_* f_{\boldsymbol{\tau}} + 1$.*

Proof. Let $\tau_1 = \sigma_{\le\tau}$ and use lemma 2. $\square$

For a BV term $\boldsymbol{\alpha}$, $f_{\boldsymbol{\alpha}} : \mathrm{In}_\kappa \mapsto \kappa$ is defined by recursion on $\boldsymbol{\alpha}$, per case as given in the previous section, as follows.

0. $f_0 = 0$
1. $f_{\kappa^{+\eta_k}\cdot\boldsymbol{\sigma}_k} + \cdots + f_{\kappa^{+\eta_0}\cdot\boldsymbol{\sigma}_0}$
2. $f_{\boldsymbol{\sigma}}$ where $\boldsymbol{\sigma}$ is the scheme $\boldsymbol{\alpha}$.
3. $f_{\kappa^{+\eta}} \cdot f_{\boldsymbol{\sigma}}$
4. $f_{\boldsymbol{\alpha}}(\lambda) = \lambda^+$
5. $f_{\boldsymbol{\alpha}}(\lambda) = \lambda^{+f_{\boldsymbol{\eta}}(\lambda)}$
6. $f_{\boldsymbol{\alpha}}(\lambda) = \phi_\lambda(f_{\boldsymbol{\beta}}(\lambda), f_{\boldsymbol{\gamma}}(\lambda))$

Theorem 4. *Suppose $\boldsymbol{\alpha}$ is a term, $\lambda \in \mathrm{In}_\kappa$, and $\lambda \notin T_{\boldsymbol{\alpha}}$.*

a. $f_\alpha \restriction \lambda = f_{\alpha\downarrow\lambda}$.

b. $f_\alpha(\lambda) = \alpha \downarrow \lambda$.

Proof. For schemes this is lemma 5 of [4] and lemma 5 of [5]; for CNF it is lemma 17 of [5]. Case 6 of the recursion will be given. For part a, for $\mu < \lambda < \kappa$ $f_{\boldsymbol{\alpha}\downarrow\lambda}(\mu) = f_{\phi_\lambda}(\boldsymbol{\beta}\downarrow\lambda, \boldsymbol{\gamma}\downarrow\lambda)(\mu) = \phi_\mu(f_{\boldsymbol{\beta}\downarrow\lambda}(\mu), f_{\boldsymbol{\gamma}\downarrow\lambda}(\mu)) = \phi_\mu(f_{\boldsymbol{\beta}}(\mu), f_{\boldsymbol{\gamma}}(\mu)) = f_{\phi(\boldsymbol{\beta},\boldsymbol{\gamma})}(\mu) = f_{\boldsymbol{\alpha}}(\mu)$. For part b, $f_{\boldsymbol{\alpha}}(\lambda) = f_{\phi(\boldsymbol{\beta},\boldsymbol{\gamma})}(\lambda) = \phi_\lambda(f_{\boldsymbol{\beta}}(\lambda), f_{\boldsymbol{\gamma}}(\lambda)) = \phi_\lambda(\beta\downarrow\lambda, \gamma\downarrow\lambda) = \phi(\beta,\gamma)\downarrow\lambda = \alpha)\downarrow\lambda$. $\square$

Theorem 5. *If $\alpha' = \alpha$ then $f_{\boldsymbol{\alpha}'} \equiv_* f_{\boldsymbol{\alpha}}$, and if $\alpha' < \alpha$ then $f_{\boldsymbol{\alpha}'} <_* f_{\boldsymbol{\alpha}}$.*

Proof. Both statements are proved by induction on $\boldsymbol{\alpha}$. If $\alpha' = \alpha$ then case 2 of the induction follows by lemma 2, and in the other cases the ordinals of the subterms $\boldsymbol{\tau}$ are all equal, so inductively the $f_{\boldsymbol{\tau}}$ are equal mod $\equiv_*$, so $f_{\boldsymbol{\alpha}'} \equiv_* f_{\boldsymbol{\alpha}}$.

Suppose $\alpha' < \alpha$. Write α in CNF as usual, and write α' as $\kappa^{+\eta'_{k'}} \cdot \sigma'_{k'} + \cdots + \kappa^{+\eta'_1} \cdot \sigma'_1$. The cases of the induction are as follows.

Case 0 is vacuous.

In case 1, by properties of CNF and induction, leading equal terms may be removed from $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}$. The rest of the argument is similar to that for case 3.

Case 2 follows by lemmas 2 and 3.

In case 3, if $\alpha' \le \kappa^{+\eta} \cdot \sigma_1$ for some $\sigma_1 < \sigma$, let $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{\le \sigma_1}$. Then inductively $f_{\boldsymbol{\alpha}'} <_* f_{\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_1}$, and from the definition of $f_{\boldsymbol{\alpha}}$ and properties of canonical functions $f_{\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_1} \le_* f_{\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}}$. Otherwise, the first term of the CNF for α' is $\kappa^{+\eta} \cdot \sigma'$ where $\sigma = \sigma' + 1$. Letting $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{\le \sigma'}$, $f_{\boldsymbol{\alpha}'} <_* f_{\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_1} + f_{\kappa^{+\boldsymbol{\eta}}}$. The right side equals $f_{\kappa^{+\boldsymbol{\eta}}} \cdot (f_{\boldsymbol{\sigma}_1} + 1) = f_{\kappa^{+\boldsymbol{\eta}}} \cdot f_{\boldsymbol{\sigma}} = f_{\boldsymbol{\alpha}}$.

In case 4, $\boldsymbol{\alpha}'$ is a scheme and the claim follows because for any scheme $\boldsymbol{\sigma}$ and $\lambda \in \mathrm{In}_\kappa$, $f_{\boldsymbol{\sigma}}(\lambda) < \lambda^+$.

In case 5, if $\alpha' \le \kappa^{+\eta_1}$ for some $\eta_1 < \eta$, let $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_{\le \eta_1}$. Then inductively $f_{\boldsymbol{\alpha}'} \le_* f_{\kappa^{+\boldsymbol{\eta}_1}}$, and from the definition of $f_{\boldsymbol{\alpha}}$ $f_{\kappa^{+\boldsymbol{\eta}_1}} <_* f_{\kappa^{+\boldsymbol{\eta}}}$. Otherwise, the first term of the CNF for α' is $\kappa^{+\eta'} \cdot \sigma'$ where $\eta = \eta' + 1$; let $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_{\le \eta'}$. Similarly to case 3, $f_{\boldsymbol{\alpha}'} <_* f_{\kappa^{+\boldsymbol{\eta}'}} \cdot f_{\kappa^+}$. Letting $f \odot g$ denote the pointwise operation, the right side equals $f_{\kappa^+} \odot f_{\boldsymbol{\eta}_1} \cdot f_{\kappa^+} = f_{\kappa^+} \odot (f_{\boldsymbol{\eta}_1} + 1) = f_{\kappa^+} \odot f_{\boldsymbol{\eta}} = f_{\boldsymbol{\alpha}}$.

The proof of case 6 will be broken up into subcases. First note that a simple induction shows that $f_{\boldsymbol{\alpha}}(\lambda) < \Gamma_{0\lambda}$ for any $\boldsymbol{\alpha}$ and λ. It follows that $f_{\boldsymbol{\beta}}, f_{\boldsymbol{\gamma}} < f_{\phi(\boldsymbol{\beta},\boldsymbol{\gamma})}$ for any $\boldsymbol{\beta}, \boldsymbol{\gamma}$ (i.e., the inequality holds for all λ).

$\beta = 1, \gamma = 0$. Then $\alpha' < \kappa^+ \uparrow n$ for some n, so $f_{\boldsymbol{\alpha}'} <_* f_{\kappa^+ \uparrow n} <_* f_{\kappa^+ \uparrow \omega}$.

$\beta = 1$ and $\alpha' < \phi(1, \gamma_1)$ for some $\gamma_1 < \gamma$. Then $f_{\boldsymbol{\alpha}'} <_* f_{\phi(1,\boldsymbol{\gamma}_1)} <_* f_{\phi(1,\boldsymbol{\gamma})}$.

$\beta = 1$, $\gamma = \gamma_1 + 1$, and $\phi(1, \gamma_1) <\le \alpha'$. Let $\theta_0 = \phi(1, \gamma_1)$ and for $n < \omega$ let $\theta_{n+1} = \kappa^+ \odot \theta_n$. Then $\alpha' < \theta_n$ for some n, and $f_{\boldsymbol{\alpha}'} <_* f_{\boldsymbol{\theta}_n} <_* f_{\boldsymbol{\alpha}}$.

$\beta > 1$. $\alpha = \phi(1, \xi)$ where $\xi \in \mathrm{Lim}$, so $\alpha' < \phi(1, \zeta)$ for some $\zeta < \xi$; and $\phi(1, \zeta) = \phi(\beta_1, \gamma_1)$ for a unique β_1, γ_1 with $\beta_1, \gamma_1 < \phi(\beta_1, \gamma_1)$. Choose terms $\boldsymbol{\beta}_1, \boldsymbol{\gamma}_1$ for β_1, γ_1. Then $f_{\boldsymbol{\alpha}'} <_* f_{\phi(\boldsymbol{\beta}_1,\boldsymbol{\gamma}_1)} <_* f_{\phi(\boldsymbol{\beta},\boldsymbol{\gamma})}$.

This concludes the proof of theorem 5. □

Theorem 6. *If $\alpha = \beta + 1$ then $f_{\boldsymbol{\alpha}} \equiv_* f_{\boldsymbol{\beta}} + 1$.*

Proof. If $\beta = 0$ the claim is immediate. If η_1 in the CNF for β equals 1 the claim follows using lemma 3. If $\eta_1 > 1$ $\boldsymbol{\alpha}$ has the additional term "1" to β and the claim follows easily. □

Theorem 7. *Suppose $\theta < \kappa$ and $\boldsymbol{\alpha}_\xi$ for $\xi < \theta$ is a sequence of terms; and $\boldsymbol{\alpha}$ is a term. Suppose α_ξ is ascending and $\alpha = \sup_\xi \alpha_\xi$. Then $f_{\boldsymbol{\alpha}} \equiv_* \sup_\xi f_{\boldsymbol{\alpha}_\xi}$. Suppose the sequence of terms is $\boldsymbol{\alpha}_\xi$ for $\xi < \kappa$. Then $f_{\boldsymbol{\alpha}} \equiv_* \mathrm{dsup}_\xi \mathrm{f}_{\boldsymbol{\alpha}_\xi}$.*

Proof. The proof is by induction on $\boldsymbol{\alpha}$. In each case $\boldsymbol{\alpha}_\xi$ is replaced by a sequence $\boldsymbol{\alpha}'_\xi$, and the claim proved either using the induction hypothesis or directly.
Case 0 is impossible.

In case 1, $\boldsymbol{\alpha}'_\xi$ differs from $\boldsymbol{\alpha}$ only in the last term; the claim follows using the induction hypothesis.

In case 2, let $\boldsymbol{\sigma}'_\xi$ be the sequence given by the last node of $\boldsymbol{\sigma}$.

In case 3, $\boldsymbol{\alpha}'_\xi$ equals $\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}'_\xi$ for some σ'_ξ; the claim follows inductively.

Case 4 is impossible.

In case 5, η must be a limit ordinal, and $\boldsymbol{\alpha}'_\xi$ equals $\kappa^{+\boldsymbol{\eta}_\xi}$ for some $\boldsymbol{\eta}'_\xi$; the claim follows inductively.

The proof of case 6 is divided into subcases. Some direct verifications involve verifying an equation per λ, possibly using lemmas 1 and 4.

$\beta = 1$, $\gamma = 0$. $\boldsymbol{\alpha}'_n$ for $n < \omega$ equals $\kappa^+ \uparrow n$; the claim follows directly.

$\beta = 1$, $\gamma = \gamma' + 1$. $\boldsymbol{\alpha}'_0 = \phi(1, \boldsymbol{\gamma}') + 1$, and for $n < \omega$ $\boldsymbol{\alpha}'_{n+1} = \kappa^+ \odot \boldsymbol{\alpha}'_n$; the claim follows directly.

$\beta = \beta' + 1$, $\gamma = 0$. $\boldsymbol{\alpha}'_0 = \phi(\boldsymbol{\beta}', 0)$, and for $n < \omega$ $\boldsymbol{\alpha}'_{n+1} = \phi(\boldsymbol{\beta}', \boldsymbol{\alpha}'_\xi)$; the claim follows directly.

$\beta = \beta' + 1$, $\gamma = \gamma' + 1$. $\boldsymbol{\alpha}'_0 = \phi(\boldsymbol{\beta}', \boldsymbol{\gamma}' + 1)$, and for $n < \omega$ $\boldsymbol{\alpha}'_{n+1} = \phi(\boldsymbol{\beta}', \boldsymbol{\alpha}'_\xi)$; the claim follows directly.

$\beta \in \mathrm{Lim}$, $\gamma = 0$. $\boldsymbol{\alpha}'_\xi = \phi(\boldsymbol{\beta}_\xi, 0)$ for some $\boldsymbol{\beta}_\xi$; the claim follows inductively.

$\beta \in \mathrm{Lim}$, $\gamma = \gamma' + 1$. Let $\boldsymbol{\beta}_\xi$ be terms with β_ξ an ascending sequence with supremum β. Let $\boldsymbol{\alpha}'_0 = \phi(\boldsymbol{\beta}, \boldsymbol{\gamma}')$, let $\boldsymbol{\alpha}'_{\xi'+1} = \phi(\boldsymbol{\beta}', \boldsymbol{\alpha}'_\xi)$. and for $\xi \in \mathrm{Lim}$ let $\boldsymbol{\alpha}'_\xi$ be some term with $\alpha'_\xi = \sup_{\xi' < \xi} \alpha'_{\xi'}$. The claim follows in the case $\mathrm{cf}(\beta) = \mathrm{cf}(\alpha) < \kappa$, and in the case $\mathrm{cf}(\beta) = \kappa$ also.

$\gamma = \in \mathrm{Lim}$. $\boldsymbol{\alpha}'_\xi = \phi(\boldsymbol{\beta}, \boldsymbol{\gamma}_\xi)$ for some $\boldsymbol{\gamma}_\xi$; the claim follows inductively.

This concludes the proof of theorem 7. □

Theorem 8. *If U is a normal ultrafilter on κ then f_σ represents σ in V^κ/U.*

Proof. For CNF this is theorem 16 of [5]. The argument for case 6 of the recursion is as follows. Let $Phi(\lambda, \beta, \gamma) = \phi_\lambda(\beta, \gamma)$, and let Φ act on functions pointwise. Then $f_{\boldsymbol{\alpha}} = \Phi(f_{\kappa^+}, f_{\boldsymbol{\beta}}, f_{\boldsymbol{\gamma}})$. The claim follows by Los' theorem. □

6. Sets from Terms

To construct sets $S_{\boldsymbol{\alpha}}$ from terms, functionals $\mathrm{H}_n^{\boldsymbol{\alpha}}$ of height n will be constructed, simultaneously for all n by recursion on $\boldsymbol{\alpha}$. $S_{\boldsymbol{\alpha}} = H_1^{\boldsymbol{\alpha}}(\mathrm{In}_\kappa)$. Various ingredients of the construction have already been given in [3]. Relevant facts will be summarized below; see [3] for further details.

For $\kappa \in \mathrm{Inac}$ and $n \geq 1$ the set $\mathcal{L}_n$ of local functionals of height n over κ is defined recursively. If $F_n \in \mathcal{L}_n$ then $F_n : \mathcal{L}_{n-1} \mapsto \mathcal{L}_{n-1}$, where for convenience $\mathcal{L}_0$ denotes $\mathrm{Pow}(\kappa)$. If necessary $\mathcal{L}_n$ may be denoted as $\mathcal{L}_n^\kappa$, and for $\lambda \in \mathrm{In}_\kappa$ there is a restriction operator $\downharpoonleft$ from $\mathcal{L}_n$ to $\mathcal{L}_n^\lambda$. $\mathcal{L}_n$ is closed under the composition operation $\circ$. The operations $\cap$ and $\triangle$ may be defined on $\mathcal{L}_n$ "recursively pointwise". The operation $\downharpoonleft$ commutes with $\circ$, $\cap$ and $\triangle$. For a scheme $\boldsymbol{\sigma}$ and a functional $F_n \in \mathcal{L}_n$ the functional $F_n^{\boldsymbol{\sigma}}$ is readily defined, and is an element of $\mathcal{L}_n$.

The operation $F_n \mapsto F_n^*$ is defined by the requirement that for $\lambda \in \mathrm{In}_\kappa$, $\lambda \in F_n^*(F_{n-1}) \cdots (F_1)(X)$ iff $\lambda \in F_{n-1} \cdots (F_1)(X)$ and $F_n^{\boldsymbol{\sigma}}(F_{n-1}) \cdots (F_1)(X \cap \lambda)$ is stationary for all schemes $\boldsymbol{\sigma}$ over λ (where of course in the latter expression F_j is $F_j \downharpoonleft \lambda$). $F_n^* \in \mathcal{L}_n$, and $F_n^* \downharpoonleft \lambda = (F_n \downharpoonleft \lambda)^*$. For $n \geq 2$ let H_n be the map $F_{n-1} \mapsto F_{n-1}^*$ (let H_1 denote H). $\mathrm{H}_n \in \mathcal{L}_n$ for $n \geq 1$, and each $\mathcal{L}_n$ contains the identity function, which will be denoted Id.

For $F_n, G_n \in \mathcal{L}_n$, say that $F_n \subseteq G_n$ iff
$\forall F_{n-1} \cdots \forall F_1 \forall X (F_n(F_{n-1}) \cdots (F_1)(X) \subseteq G_n(F_{n-1}) \cdots (F_1)(X))$
The relation $F_n \subseteq_{\mathrm{t}} G_n$ is similarly defined. Basic properties of these relations are left to the reader.

Initially the construction will be given for CNF terms. For each term $\boldsymbol{\alpha}$, functionals $\mathrm{H}_{n\boldsymbol{\alpha}} \in \mathcal{L}_{\boldsymbol{\alpha}}$ are defined by recursion on $\boldsymbol{\alpha}$. The cases of the recursion are as follows.

Case 0: Id
Case 1: $\mathrm{H}_{n\kappa+\boldsymbol{\eta}_1\cdot\boldsymbol{\sigma}_1} \circ \cdots \circ \mathrm{H}_{n\kappa+\boldsymbol{\eta}_k\cdot\boldsymbol{\sigma}_k}$
Case 2: $\mathrm{H}_n^{\boldsymbol{\sigma}}$
Case 3: $\mathrm{H}_{n\kappa+\boldsymbol{\eta}}^{\boldsymbol{\sigma}}$
Case 4: $\mathrm{H}_{n+1}(\mathrm{H}_n)$
Case 5: $\mathrm{H}_{n+1,\boldsymbol{\eta}}(\mathrm{H}_n)$

Lemma 9. *1. If $F_{n-1} \subseteq_t G_{n-1}$ then $H_n(F_{n-1}) \subseteq_t H_n(G_{n-1})$.*
2. $H_n(F_{n-1}) \subseteq F_{n-1}$.
3. If $F_n \subseteq_t G_n$ then $F_n(F_{n-1}) \subseteq_t G_n(F_{n-1})$.

Proof. These all follows directly from the definitions. ☐

Lemma 10. *If $\sigma' \leq \sigma$ then $H_{n\boldsymbol{\sigma}'} \supseteq_{\mathrm{t}} H_{n\boldsymbol{\sigma}}$.*

Proof. A direct proof by induction on σ will be given. In case 0, $\sigma' = 0$ also and the claim is immediate. In case 1, if $\sigma' \leq \tau$ then inductively $\mathrm{H}_{n\sigma'} \supseteq_{\mathrm{t}} \mathrm{H}_{n\tau}$, and by lemma 9 $\mathrm{H}_\tau \supseteq \mathrm{H}_\sigma$. If $\sigma' = \sigma$ then inductively $\mathrm{H}_{n\sigma'_{\leq\tau}} \equiv_{\mathrm{t}} \mathrm{H}_{n\sigma_{\leq\tau}}$, and $\mathrm{H}_{n\sigma'} \equiv_{\mathrm{t}} \mathrm{H}_{n\sigma}$ follows. In case 2, if $\sigma' < \sigma$ then $\sigma' < \sigma_\xi$ for some ξ where $\langle \sigma_\xi : \xi < \eta \rangle$ is the sequence ascending to σ. Inductively, $\mathrm{H}_{n\boldsymbol{\sigma}'} \supseteq_{\mathrm{t}} \mathrm{H}_{n\boldsymbol{\sigma}_{\leq\sigma_\xi}}$, and clearly $\mathrm{H}_{n\boldsymbol{\sigma}_{\leq\sigma_\xi}} \supseteq \mathrm{H}_{n\boldsymbol{\sigma}}$. If $\sigma' = \sigma$ let $\langle \sigma'_\xi : \xi < \eta' \rangle$ be the sequence ascending to σ'. By what was just proved, $\mathrm{H}_{\boldsymbol{\sigma}'_{n\xi}} \supseteq_{\mathrm{t}} \mathrm{H}_{n\sigma}$ for all $\xi < \eta'$, whence $\mathrm{H}_{n\boldsymbol{\sigma}'} \supseteq_{\mathrm{t}} \mathrm{H}_{n\sigma}$. The argument for case 3 is similar to that for case 2. □

Lemma 11. *If $\sigma = \sigma' + 1$ then $H_{n\boldsymbol{\sigma}} \equiv_{\mathrm{t}} H_n \circ H_{n\boldsymbol{\sigma}'}$.*

Proof. By lemma 10 $\mathrm{H}_{n\boldsymbol{\sigma}'} \equiv_{\mathrm{t}} \mathrm{H}_{n\boldsymbol{\tau}}$ □

Lemma 12. *For $n \geq 1$ and a term $\boldsymbol{\alpha}$ let $F_n = H_{n\boldsymbol{\alpha}}$. Then*

$$F_n(F_{n-1}) \cdots (F_1)(X) \subseteq F_{n-1} \cdots (F_1)(X) \cup \{0\}.$$

Proof. For ease of notation, in expressions $F_n(F_{n-1}) \cdots (X)$, $n = 1$ is allowed; $F_{n-1} = X$ and "$\cdots (X)$" is the null string. The claim is first proved for a scheme $\boldsymbol{\sigma}$, by induction on $\boldsymbol{\sigma}$, with cases as follows.

Case 0 is trivial. For case 1, $\mathrm{H}_n^{\boldsymbol{\sigma}}(F_{n-1}) \cdots (X) = \mathrm{H}_n(\mathrm{H}_n^{\boldsymbol{\tau}}(F_{n-1})) \cdots (X) \subseteq \mathrm{H}_n^{\boldsymbol{\tau}}(F_{n-1})) \cdots (X) \subseteq F_{n-1} \cdots (X) \cup \{0\}$, where the first inclusion follows by definition of H_n and the second follows inductively. Case 2 follows because

$$\mathrm{H}_n^{\boldsymbol{\sigma}}(F_{n-1}) \cdots (X) \subseteq \mathrm{H}_n^{\boldsymbol{\sigma}_0}(F_{n-1}) \cdots (X).$$

Case 3 follows because

$$\mathrm{H}_n^{\boldsymbol{\sigma}}(F_{n-1}) \cdots (X) \subseteq \mathrm{H}_n^{\boldsymbol{\sigma}_0}(F_{n-1}) \cdots (X) \cup \{0\}.$$

The cases for an arbitrary term $\boldsymbol{\alpha}$ are as follows. Case 0 is trivial. Suppose $F_n(F_{n-1}) \cdots (X) \subseteq F_{n-1} \cdots (X) \cup \{0\}$ and $G_n(F_{n-1}) \cdots (X) \subseteq F_{n-1} \cdots (X) \cup \{0\}$. Then if $\lambda \in (F_n \circ G_n)(F_{n-1}) \cdots (X)$ then $\lambda \in G_n(F_{n-1}) \cdots (X) \cup \{0\}$ so $\lambda \in F_{n-1} \cdots (X) \cup \{0\}$. Case 1 follows. Case 2 has already been proved. For case 3, inductively $\mathrm{H}_{n\kappa^{+\eta}}(F_{n-1}) \cdots (X) \subseteq F_{n-1} \cdots (X) \cup \{0\}$. The same induction as given above for H_n shows that $\mathrm{H}^{\boldsymbol{\sigma}}_{n\kappa^{+\eta}}(F_{n-1}) \cdots (X) \subseteq F_{n-1} \cdots (X) \cup \{0\}$, except that in case 1, $\mathrm{H}_{n\kappa^{+\eta}}(\mathrm{H}^{\boldsymbol{\tau}}_{n\kappa^{+\eta}}(F_{n-1})) \cdots (X) \subseteq \mathrm{H}^{\boldsymbol{\tau}}_{n\kappa^{+\eta}}(F_{n-1}) \cdots (X) \cup \{0\}$ follows inductively. For case 4, $\mathrm{H}_n^*(F_{n-1}) \cdots (X) \subseteq F_{n-1} \cdots (X)$ by definition of F_n^*. For case 5, $\mathrm{H}_{n+1,\boldsymbol{\eta}}(\mathrm{H}_n)(F_{n-1}) \cdots (X) \cup \{0\} \subseteq \mathrm{H}_n(F_{n-1}) \cdots (X) \cup \{0\} \subseteq F_{n-1} \cdots (X) \cup \{0\}$, where the first inclusion follows inductively.

This concludes the proof of lemma 12. □

Lemma 13. *For $n \geq 1$ and a term $\boldsymbol{\alpha}$ let $F_n = H_{n\boldsymbol{\alpha}}$. Suppose that $\lambda \in F_{n-1} \cdots (F_1)(X)$ and $F_n(F_{n-1}) \cdots (F_1)(X) \cap \lambda$ is stationary. Then $\lambda \in F_n(F_{n-1}) \cdots (F_1)(X)$.*

Proof. The claim is first proved for a scheme $\boldsymbol{\sigma}$, by induction on $\boldsymbol{\sigma}$, with cases as follows.

Case 0 is trivial.

For case 1, from the hypothesis that $\mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X) \cap \lambda$ is stationary and lemma 12, $\mathrm{H}_{n\boldsymbol{\tau}}(F_{n-1}) \cdots (X) \cap \lambda$ is stationary. Inductively, $\lambda \in \mathrm{H}_{n\boldsymbol{\tau}}(F_{n-1}) \cdots (X)$. By the definition of H_n, $\lambda \in \mathrm{H}_n(\mathrm{H}_{n\boldsymbol{\tau}}(F_{n-1})) \cdots (X) = \mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X)$.

For case 2, from the hypothesis that $\mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X) \cap \lambda$ is stationary, $\mathrm{H}_{n\boldsymbol{\sigma}_\xi}(F_{n-1}) \cdots (X) \cap \lambda$ is stationary for all ξ. Inductively, $\lambda \in \mathrm{H}_{n\boldsymbol{\sigma}_\xi}(F_{n-1}) \cdots (X)$ for all ξ. It follows that $\lambda \in \mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X)$.

For case 3, from the hypothesis that $\mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X) \cap \lambda$ is stationary, $\mathrm{H}_{n\boldsymbol{\sigma}_\xi}(F_{n-1}) \cdots (X) \cap \lambda$ is stationary for all $\xi < \lambda$. (To see this, note that $(\triangle_{\xi<\kappa} X_\xi) \cap \lambda = \triangle_{\xi<\lambda}(X_\xi \cap \lambda)$. Inductively, $\lambda \in \mathrm{H}_{n\boldsymbol{\sigma}_\xi}(F_{n-1}) \cdots (X)$ for all $\xi < \lambda$. It follows that $\lambda \in \mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X)$.

The cases for an arbitrary term $\boldsymbol{\alpha}$ are as follows.

Case 0 is trivial.

For case 1, let $Y_i = (\mathrm{H}_{n\kappa^{+\eta_i} \cdot \boldsymbol{\sigma}_i} \circ \cdots \circ \mathrm{H}_{n\kappa^{+\eta_k} \cdot \boldsymbol{\sigma}_k})(F_{n-1} \cdots (X)$. Using lemma 12, successively $Y_1 \cap \lambda$, $Y_2 \cap \lambda$, $\ldots Y_k \cap \lambda$ are stationary. Using the induction hypothesis, successively $\lambda \in Y_k$, $\lambda \in Y_{k-1}$, $\ldots \lambda \in Y_1$.

Case 2 has already been proved.

For case 3, the same induction as given above for H_n shows that $\lambda \in \mathrm{H}_{n\boldsymbol{\sigma}}(F_{n-1}) \cdots (X)$, except that in case 1, $\lambda \in \mathrm{H}_{n\kappa^{+\eta}}(\mathrm{H}^{\boldsymbol{\tau}}_{n\kappa^{+\eta}}(F_{n-1})) \cdots (X)$ follows inductively.

For case 4, by hypothesis $\mathrm{H}_{n+1}(\mathrm{H}_n)(F_{n-1}) \cdots (X) \cap \lambda$ is stationary. By lemma 12 $\mathrm{H}_n(F_{n-1}) \cdots (X) \cap \lambda$ is stationary. Inductively $\lambda \in \mathrm{H}_n(F_{n-1}) \cdots (X)$. The claim follows.

Case 5 is similar to case 4.

This concludes the proof of theorem 13. □

Theorem 14. *If $\alpha' \leq \alpha$ then $H_{n\boldsymbol{\alpha}'} \supseteq_{\mathrm{t}} H_{n\boldsymbol{\alpha}}$.*

Proof. If $\alpha' = \alpha$ then cases 0 and 2 of the induction follows by lemma 10, and in the other cases the ordinals of the subterms $\boldsymbol{\tau}$ are all equal, so inductively the $\mathrm{H}_{n\boldsymbol{\tau}}$ are equal mod $\equiv_{\mathrm{t}}$, so $\mathrm{H}_{n\boldsymbol{\alpha}'} \equiv_{\mathrm{t}} \mathrm{H}_{n\boldsymbol{\alpha}}$.

Thus, $\alpha' < \alpha$ may be assumed. Write α in CNF as usual, and write α' as $\kappa^{+\eta'_{k'}} \cdot \sigma'_{k'} + \cdots + \kappa^{+\eta'_1} \cdot \sigma'_1$. The cases of the induction are as follows.

Case 0 is vacuous.

In case 1, leading equal terms may be removed from $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}$. The rest of the argument is similar to that for case 3.

Case 2 follows by lemma 10.

In case 3, if $\alpha' \leq \kappa^{+\eta} \cdot \sigma_1$ for some $\sigma_1 < \sigma$, let $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{\leq \sigma_1}$. Then inductively $\mathrm{H}_{n\boldsymbol{\alpha}'} \supseteq_{\mathrm{t}} \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_1}$, and $\mathrm{H}_{n\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_1} \supseteq_{\mathrm{t}} \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}}$. Otherwise, the first term of the CNF for α' is $\kappa^{+\eta} \cdot \sigma'$ where $\sigma = \sigma' + 1$. Letting $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_{\leq \sigma'}$, $\mathrm{H}_{n\boldsymbol{\alpha}'} \subseteq_{\mathrm{t}} \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}}} \circ \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}} \cdot \boldsymbol{\sigma}_1}$. The right side equals $\mathrm{H}^{\boldsymbol{\sigma}}_{n\kappa^{+\boldsymbol{\eta}}}$, which equals $\mathrm{H}_{n\boldsymbol{\alpha}}$.

Case 4 follows because $\mathrm{H}^{\boldsymbol{\sigma}}_n \supseteq_{\mathrm{t}} \mathrm{H}^*_n$ for any scheme σ. This is shown in [3].

For case 5, if $\alpha' \leq \kappa^{+\eta_1}$ for some $\eta_1 < \eta$, let $\boldsymbol{\eta}_1$ be a term with ordinal η_1. Then inductively $\mathrm{H}_{n\boldsymbol{\alpha}'} \supseteq_{\mathrm{t}} \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}}}$, and $\mathrm{H}_{n+1,\boldsymbol{\eta}_1} \supseteq_{\mathrm{t}} \mathrm{H}_{n+1,\boldsymbol{\eta}}$, whence $\mathrm{H}_{n\kappa^{+\boldsymbol{\eta}_1}} \supseteq_{\mathrm{t}} \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}}}$. Otherwise, the first term of the CNF for α' is $\kappa^{+\eta'} \cdot \sigma'$ where $\eta = \eta' + 1$; let $\boldsymbol{\eta}_1$ be a term with ordinal η_1 and let $\boldsymbol{\sigma}$ be $\boldsymbol{\sigma}'$ with one position added. Suppose

(1) $\quad \mathrm{H}^{\boldsymbol{\sigma}'}_{n\kappa^{+\boldsymbol{\eta}'}} \supseteq_{\mathrm{t}} \mathrm{H}^*_{n\kappa^{+\boldsymbol{\eta}'}}$

for any scheme σ. Then $\mathrm{H}_{n\boldsymbol{\alpha}'} \supseteq_{\mathrm{t}} \mathrm{H}^{\boldsymbol{\sigma}}_{n\kappa^{+\boldsymbol{\eta}'}} \supseteq_{\mathrm{t}} \mathrm{H}^*_{n\kappa^{+\boldsymbol{\eta}'}} = \mathrm{H}_{n+1}(\mathrm{H}_{n+1,\boldsymbol{\eta}'}(\mathrm{H}(\mathrm{H}_n))) = \mathrm{H}_{n+1,\boldsymbol{\eta}}(\mathrm{H}(\mathrm{H}_n)) = \mathrm{H}_{n\kappa^{+\boldsymbol{\eta}}} = \mathrm{H}_{\boldsymbol{\alpha}}$.

For the proof of (1), for ease of notation let F_n denote $\mathrm{H}_{n\kappa^{+\boldsymbol{\eta}}}$. Suppose $\lambda \in F^*_n(F_{n-1}) \cdots (X)$; note that $\lambda \in F_{n-1} \cdots (X)$ follows. Given $\boldsymbol{\sigma}$, suppose $\lambda \notin T_{\boldsymbol{\sigma}}$. By definition of F^*_n, $F^{\boldsymbol{\sigma}\downharpoonleft\lambda}_n(F_{n-1} \downharpoonleft \lambda) \cdots (X \downharpoonleft \lambda)$ By lemma 3 of [5], $(F^{\boldsymbol{\sigma}}_n \downharpoonleft \lambda)(F_{n-1} \downharpoonleft \lambda) \cdots (X \downharpoonleft \lambda) = F^{\boldsymbol{\sigma}}_n(F_{n-1}) \cdots (X) \downharpoonleft \lambda$ is stationary. The claim follows by lemma 13.

This concludes the proof of theorem 14. $\square$

It follows that if $\mathrm{H}_{1\boldsymbol{\alpha}}(\mathrm{In}_\kappa)$ is stationary then $\rho_R(\mathrm{H}_{1\boldsymbol{\alpha}}(\mathrm{In}_\kappa) \geq \alpha$. The proof given above is simplified from that given in [5], indeed an omission is repaired. In addition, it permits attempting to proceed further by considering BV terms.

Some remarks on extending to BV will be given. Let BV_1 be the subsystem of BV where the first argument of ϕ must be 1, and let BV_{1c} be the subsystem where as well the second argument must be a CNF term. It is easily seen that the closure ordinal of BV_1 is $\phi(2,0)$, and that of BV_{1c} is $\phi(1,\epsilon_0)$.

To extend the definition of $\mathrm{H}_{1\boldsymbol{\alpha}}$ to BV_{1c}, some additional functionals are defined. As usual, the first subscript denotes the level in the $\mathcal{L}_n$ hierarchy. Functionals $\vec{F} : \times_{n=1}^{<omega} \mathcal{L}_n \mapsto \times_{n=1}^{<omega} \mathcal{L}_n$; expressions $\vec{F}(\vec{X})$, $F_n(\vec{X})$, etc., will be used; the component of $\vec{F}$ mapping $\vec{X}$ to $\mathcal{L}_n$ is denoted F_{n+1}. The operations of $\circ$, $\cap$, and $\triangle$ are defined pointwise, and $\vec{F}^{\boldsymbol{\sigma}}$ for a scheme $\boldsymbol{\sigma}$ is defined by the usual recursion.

P^0_n equals $\cap_{i<\omega} \mathrm{H}_{n,\kappa^+\uparrow i}$.

Suppose $F_n \in \mathcal{L}_n$. Let $G_{n0} = \mathrm{H}_n \circ F_1$, $G_{n,i+1} = G_{n+1,i}(\mathrm{H}_n)$, and $G_{n\omega} = \cap_{i<\omega} G_{ni}$. Let $\mathrm{P}_{n+1}(\vec{F}) = G_{n\omega}$. P_2 is the "next fixed point" functional; the intent is that $\mathrm{P}_2(\vec{\mathrm{H}}_{\boldsymbol{\gamma}}) = \mathrm{H}_{1,\phi(1,\boldsymbol{\gamma})}$.

$\vec{\mathrm{P}}_{\boldsymbol{\alpha}}$ is defined by recursion on $\boldsymbol{\alpha}$, with cases as follows.

Case 0: $\mathrm{P}_{n0} = \mathrm{Id}$

Case 1: $\vec{\mathrm{P}}_{\kappa+\boldsymbol{\eta}_1\cdot\boldsymbol{\sigma}_1} \circ \cdots \circ \vec{\mathrm{P}}_{\kappa+\boldsymbol{\eta}_k\cdot\boldsymbol{\sigma}_k}$

Case 2: $\vec{\mathrm{P}}^{\boldsymbol{\sigma}}$

Case 3: $\vec{\mathrm{P}}^{\boldsymbol{\sigma}}_{\kappa+\boldsymbol{\eta}}$

Case 4: $\mathrm{P}_{n\kappa^+} = \mathrm{H}_{n+1}(\mathrm{P}_n)$

Case 5: $\mathrm{P}_{n\kappa^{+\boldsymbol{\eta}}} = \mathrm{H}_{n+1,\boldsymbol{\eta}}(\mathrm{P}_n)$

The following case is added to the definition of $\mathrm{H}_{n\boldsymbol{\alpha}}$.

Case 6: $\mathrm{H}_{n,\phi(1,\boldsymbol{\gamma})} = \mathrm{P}_{n+1,\gamma}(\mathrm{P}^0_n)$

This definition improves on [5], where $\mathrm{H}_{1\boldsymbol{\alpha}}$ is defined for $\alpha < \phi(1,\kappa^+)$.

Theorem 15. *If $\alpha' \le \alpha$ then $H_{n\boldsymbol{\alpha}'} \supseteq_t H_{n\boldsymbol{\alpha}}$.*

Proof. Only an outline will be given. Let $X \subseteq_0 Y$ denote $X \subseteq Y \cup \{0\}$, and recursively for the relation on $\mathcal{L}_n$. It may be verified that $\mathrm{P}^0_n \subseteq \mathrm{Id}$, and $P_{n+1}(\vec{F}) \subseteq F_n$ for $n > 1$. It follows inductively that $P^{\boldsymbol{\gamma}}_{n+1}(\vec{F}) \subseteq_0 F_n$ for $n \ge 1$ and CNF γ. Case 6 of lemma 12 then follows, proving the lemma for BV_{1c} terms. Say that F_n is S-closed if $\lambda \in F_n(F_{n-1})\cdots(F_1)(X)$ whenever $\lambda \in F_{n-1}\cdots(F_1)(X)$ and $F_n(F_{n-1})\cdots(F_1)(X) \cap \lambda$ is stationary. One verifies that P^0_n is S-closed, if F_n is S-closed for all n then $P_{n+1}(\vec{F})$ is S-closed for all n, and the same holds for $P^{\boldsymbol{\gamma}}_{n+1}$ for CNF γ. Case 6 of lemma 13 then follows, proving the lemma for BV_{1c} Next,since $\mathrm{P}_{n+1}(\mathrm{H}_{n\phi(1,\gamma)})/subseteq\mathrm{H}_{n\phi(1,\gamma)}$, it follows by induction on γ as in theorem 14 that if $\gamma' \le \gamma$ then $\mathrm{H}_{n\phi(1,\gamma')})/supseteq\mathrm{H}_{n\phi(1,\gamma)}$. Now, suppose $\boldsymbol{\alpha} = \phi(1,\boldsymbol{\gamma})$ and $\alpha' < \alpha$. If $\alpha' < \phi(1,\boldsymbol{\gamma}_1)$ for some $\gamma_1 < \gamma$ then $\mathrm{H}_{n\boldsymbol{\alpha}'} \supseteq_{\mathrm{t}} \mathrm{H}_{n\phi(1\boldsymbol{\gamma}_1)} \supseteq_{\mathrm{t}} \mathrm{H}_{n\phi(1\boldsymbol{\gamma})}$. Otherwise $\gamma = \gamma_1 + 1$ for some γ_1. In this case, choosing γ_1 and letting G_{ni} be as in the definition of P, for some i, $\mathrm{H}_{n\boldsymbol{\alpha}'} \supseteq_{\mathrm{t}} G_{ni} \supseteq_{\mathrm{t}} \mathrm{H}_{n\boldsymbol{\alpha}}$. ☐

Improvements to theorem 15 are left to further research.

7. Enforceability

Enforceability of $\mathrm{H}^{\boldsymbol{\sigma}}(\mathrm{In}_\kappa)$ for a scheme $\boldsymbol{\sigma}$ was proved in [1]. Recalling some facts from [2] and [5], subsets of V_κ are called classes, and various standard methods can be used to code sequences of classes as a single class, etc. A scheme can be coded as a class; the predicate "X is a scheme" is Δ^1_0. x, y, etc. will be used to denote points of the well-order.

Assuming κ is 1-inaccessible, an element $F_n \in \mathcal{L}_n$d for $n > 0$ will be coded as the class in V_κ, $\{\langle \lambda, F_n \downharpoonleft \lambda \rangle : \lambda \in \mathrm{In}_\kappa\}$. With this coding, the predicate $G_n = F_{n+1}(F_n)$ is Δ^1_0.

Let H_{nx} denote the value of $\mathrm{H}^{\boldsymbol{\sigma}}_n$ at stage x of the iteration. The class of triples $\langle x, \lambda, \mathrm{H}_{nx} \downharpoonleft \lambda \rangle$ may serve as a witness that $Y = \mathrm{H}^{\boldsymbol{\sigma}}_n$. The predicate "$W$ is the witness to $Y = \mathrm{H}^{\boldsymbol{\sigma}}_n$" is Δ^1_0.

A term $\boldsymbol{\alpha}$ may be given as a class coding the sequence of classes $\langle t, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k \rangle$ where T is a hereditarily finite set coding the tree of the term, and $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k$ are (codes for) the schemes at the leaves in order.

Theorem 16. *There is a Π^1_1 parameter-free sentence Φ, which holds in V_κ for a 1-inaccessible κ iff "for all BV_{1c} terms $\boldsymbol{\alpha}$, $H_{1\boldsymbol{\alpha}}(In_\kappa)$ is stationary".*

Proof Sketch. The claim is first shown for CNF terms. A witness consists of a witness W_t for t, and witnesses W_i for each W_i. W_t is the class of triples $\langle \nu, \lambda, \vec{\mathrm{H}}_\nu \downharpoonleft \lambda \rangle$. By considering the cases other than case 2, it may be seen that the predicate "W_t witnesses that $\vec{Y} = \vec{\mathrm{H}}_{\boldsymbol{\alpha}}$" is Δ^1_0. It has already been noted that "W_i witnesses $\vec{\mathrm{H}}_{\nu_i}(\vec{\mathrm{H}})$ is Δ^1_0. The predicate "W witnesses that $\vec{Y} = \vec{\mathrm{H}}_{\boldsymbol{\alpha}}(\vec{\mathrm{H}})$" can be used in a well known way to construct Φ. To extend the argument to BV_{1c} terms, add a witness for each case 6 node, and a Π^1_0 subformula stating that "W witnesses $\vec{y} = \vec{\mathrm{P}}_{\boldsymbol{\gamma}}$; finally adjust the Φ to incorporate these. □

As in [5], Let $\mathcal{E}_0$ be the collection of Π^1_1-enforceable subsets of κ, and for an integer $n > 0$, inductively let $\mathcal{E}_n$ be the elements $F_n \in \mathcal{L}_n$ such that $F[\mathcal{E}_{n-1}] \subseteq \mathcal{E}_{n-1}$. $\mathrm{In}_\kappa \in \mathcal{E}_0$, and $\mathcal{E}_0$ is closed under $\cap$ and $\triangle$ (see [2] for some details). For $n > 0$, $\mathrm{H}_n \in \mathcal{E}_n$, and $\mathcal{E}_n$ is closed under $\circ$, $\cap$, and $\triangle$ (see [5] for some details).

Theorem 17. *If κ is weakly compact then $\models_{V_\kappa} \Phi$. Thus, $\rho_R(\kappa) \geq \phi(1, \epsilon_{0\kappa})$, and $\{\lambda \in In_\kappa : \rho_R(\lambda) \geq \phi_\lambda(1, \epsilon_{0\lambda})\}$ is a Π^1_1-enforceable subset of κ.*

Proof Sketch. In fact, $\mathrm{H}_{n\boldsymbol{\alpha}} \in \mathcal{E}_n$, as may be by induction. The cases for CNF are as follows. Case 0 is immediate. Case 1 follows by closure under composition. Case 2 follows by induction on $\boldsymbol{\sigma}$; case 3 does also. Case 4 is immediate. Case 5 follows by induction on $\boldsymbol{\alpha}$. For BV_1 terms, it is easy to check that $\mathrm{P}^0_n \in \mathcal{E}_n$; and if $F_n \in \mathcal{E}_n$ for all n then $\mathrm{P}_n(\vec{F}) \in \mathcal{E}_n$. By induction on $\boldsymbol{\gamma}$, $\mathrm{P}_{n\gamma} \in \mathcal{E}_n$ for CNF γ. The theorem follows readily. □

The following theorem is needed below.

Theorem 18. *Suppose $\theta < \kappa$ and $\boldsymbol{\alpha}_\xi$ for $\xi < \theta$ is a sequence of BV_{1c} terms; and $\boldsymbol{\alpha}$ is a term. Suppose α_ξ is ascending and $\alpha = \sup_\xi \alpha_\xi$. Then*

$H_{1\boldsymbol{\alpha}} \equiv_t \cap\xi H_{1\boldsymbol{\alpha}_\xi}$. Suppose the sequence of terms is $\boldsymbol{\alpha}_\xi$ for $\xi < \kappa$. Then $H1\boldsymbol{\alpha} \equiv_t Bt_\xi H1\boldsymbol{\alpha}_\xi$.

Proof. The proof for CNF terms is similar to that of 7 and is Left to the reader. For BV_{1c} terms, the proof for case 6 is also similar, except only $\beta = 1$ need be considered; this is left to the reader also. □

8. New Axioms

The results of this paper suggest a two-step method for proposing new axioms for set theory.

Step 1. Construct a sequences of subsets of κ, such that if they are all stationary then they form a strictly ascending chain in the order $<_R$.

Step 2. Postulate that the subsets are all stationary.

Step 1 is a work in progress, which as of this paper has reached rank $\phi(1, \epsilon_0)$. Even this admittedly small a value has required refining methods developed by the author in a series of papers.

By theorem 28 of [5], axiom G may be stated as, $\mathrm{H}_{1\boldsymbol{\sigma}}(\mathrm{Inac})$ is a stationary class for all class schemes $\boldsymbol{\sigma}$. Let M_{κ^+} be another name for axiom G, as it states that Ord is Ord^+-Mahlo.

The author proposes that the axiom $M_{\phi(1,\epsilon_0)}$ be added to BGC. Class terms are readily defined. The axiom states that $\mathrm{H}_{1\boldsymbol{\alpha}}(\mathrm{Inac})$ is a stationary class for all class BV_1 terms $\boldsymbol{\alpha}$.

Axiom $M_{\phi(1,\epsilon_0)}$ seems a mild enough generalization of axiom M_{κ^+} that it is a reasonable candidate. A more detailed discussion is clearly of interest, though. As will be seen, the axiom can be deduced from more specific ones which are easier to justify, along the lines of axiom G.

To specify these axioms, let $\mathcal{S}_0$ be the classes $\mathrm{H}_{1\boldsymbol{\alpha}}(\mathrm{Inac})$ which are stationary. and for $n > 0$, inductively let $\mathcal{S}_n$ be those H_n such that $\mathrm{H}_{n\boldsymbol{\alpha}}[\mathcal{S}_{n-1}] \subseteq \mathcal{S}_{n-1}$. The following theorem will be useful.

The axioms for CNF terms as follows.

1. Inac $\in \mathcal{S}_0$. This has already been justified.
2. If $\mathrm{H}_{1\boldsymbol{\alpha}}(\mathrm{Inac})$ is nonempty then it is stationary. This axiom is redundant, but useful for justifying subsequent axioms. It is a statement of a principle which is a consequence of the principle of collecting the universe. It is a generalization of axiom G, and the justification of axiom G can be generalized to justify it. Note that axiom 1 can be weakened, to "Inac is nonempty".

3. If $H_{1\boldsymbol{\alpha}}(\mathrm{Inac})$ is stationary then $H(H_{1,\boldsymbol{\alpha}}(\mathrm{Inac}))$ is stationary. This can be seen as collecting the universe, then applying axiom 2.
4. If η is a limit ordinal, α_ξ for $\xi < \eta$ is an ascending chain with limit α, and $H_{1\boldsymbol{\alpha}_\xi}(\mathrm{Inac})$ is stationary for all $\xi < \eta$, then $H_{1\boldsymbol{\alpha}}(\mathrm{Inac})$ is stationary. This follows by the principle of collecting the universe using arguments as in Section 2, theorem 18, and axiom 2.
5. If $\alpha\xi$ for $\xi <$ Ord is an ascending chain with limit α, and $H_{1\boldsymbol{\alpha}_\xi}(\mathrm{Inac})$ is stationary for all ξ, then $H_{1\boldsymbol{\alpha}}(\mathrm{Inac})$ is stationary.
6. $H_{n\boldsymbol{\alpha}} \in \mathcal{S}_n$. Axioms 4 and 5 follow for $\mathcal{S}_n$ for $n \geq 2$, and closure under composition is clear. Closure under iteration through schemes follows. Closure under $*$ may be postulated as an application of collecting the universe.

Axiom M_ϵ follows from the above axioms, by induction on α, as the reader may verify. To justify axiom $M_{\phi(1,\epsilon_0)}$, note that the above axioms may be justified for BV_{1c} terms (indeed any reasonable scheme term system). Arguments like those already given justify postulating that $P_1^0 \in \mathcal{S}_1$, if $H_{n\alpha} \in \mathcal{S}_n$ for all n then $P_{n+1,\boldsymbol{\alpha}}(\vec{H}_\alpha)(\in \mathcal{S}_n$ for all n.

9. Mitchell Order

let $\rho(\kappa)$ denote the Mahlo rank of κ; and let $o(\kappa)$ denote the Mitchell order. Let κ_c denote the least measurable cardinal κ such that $o(\kappa) > \rho(\kappa)$. Recall the following theorem from [4].

Theorem 19. *Suppose κ_c does not exist. Then for a cardinal κ, $o(\kappa) < \kappa^{++}$.*

Proof. The proof is essentially unchanged from the proof of theorem 3 of [4]. For step (2) a core model is unnecessary; the model $L[\mathcal{F}]$ of [7] suffices. □

Theorem 20. $o(\kappa_c) = \rho(\kappa_c) + 1$.

Proof. For an ultrafilter U on the cardinal κ, and a function $f : \kappa \mapsto V$, let $[f]_U$ denote the element of the transitive collapse M of the ultrapower V^κ/U, corresponding to the function f. Suppose U is normal. By standard arguments $[o \upharpoonright \kappa]_U$ equals $o(\kappa)^M$ and $[\rho \upharpoonright \kappa]_U$ equals $\rho(\kappa)^M$. Also, since $\mathrm{Pow}(\kappa)^M = \mathrm{Pow}(\kappa)$, the stationary subsets of κ are the same in M as in V, $\rho(S)^M = \rho(S)$ for any stationary subset $S \subseteq \kappa$, and $\rho(\kappa)^M = \rho(\kappa)$. Finally, for any measurable (indeed weakly compact) cardinal κ, $\rho(\kappa)$ is a limit ordinal.

κ_c cannot be the smallest measurable cardinal κ, since $o(\kappa) = 1$ but $\rho(\kappa)$ is a limit ordinal. For larger κ, inductively $o(\kappa)^M \leq \rho(\kappa)^M$. But $\rho(\kappa)^M = \rho(\kappa)$, and it is well-known that $o(\kappa)^M = o(U)$, and so $o(U) \leq \rho(\kappa)$. □

It follows that there is a normal ultrafilter U in κ_c such that $o(U) = \rho(\kappa)$. Let $S = \{\lambda : o(\lambda) = \rho(\lambda)\}$. Since $o(U) = \rho(\kappa)$, $o(\kappa)^M = \rho(\kappa)^M$; and so $S \in U$. Clearly it is of interest what can be said about $\rho(S)$.

The results of Section 5 yield the following.

Theorem 21. *If $o(\kappa) < \Gamma_{0\kappa}$ then $o(\kappa) \leq \rho(\kappa)$.*

Proof. This follows by theorems 4-8 and theorem 14 of [5]. □

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