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HYERS-ULAM-RASSIAS STABILITY OF ORTHOGONALLY CUBIC AND QUARTIC FUNCTIONAL EQUATIONS

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Abstract: In this paper, the Hyers-Ulam-Rassias Stability of the following orthogonal Cubic and Quartic functional equations:

$$f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)=48f(y), \tag{1}$$

$$f(3x+y)+f(x+3y) = 64f(x)+64f(y)+24f(x+y)-6f(x-y), \tag{2}$$

in the setting of orthogonality space is established, where f is a mapping from an orthogonality space $(X, \perp)$ into a real Banach space Y.

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1. Introduction and Preliminaries

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equation is, "How do the solutions of the inequality differ from those of the given functional equation?" The stability problem of functional equations originated from a question of S.M. Ulam [19], concerning the stability of group homomorphism:

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Let $(G_1, *)$ be a group and let $(G_2, \Diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exists a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \Diamond h(y)) < \delta,$$

for all $x, y \in G_1$, then there is a homomorphism

$$H : G_1 \to G_2 \quad \text{with} \quad d(h(x), H(x)) < \varepsilon,$$

for all $x \in G_1$? If the answer is affirmative, we would say that equation of homomorphism $H(xy) = H(x)H(y)$ is stable. In 1941, D.H. Hyers [5] gave the first affirmative answer to the question of S.M. Ulam [19] for Banach spaces.

Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

In 1978, Th.M. Rassias [19] gave the generalization result of Hyers theorem by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$ where $\varepsilon > 0$ and $p \in [0, 1)$. In 1994, P.Gavruta [12] following Th.M. Rassias approach for the stability of the linear mapping between Banach spaces obtained a generalization of the Th.M. Rassias result. The stability problem of several functional equations have been investigated by a number of authors and there are many interesting results concerning this problem (see, [3], [4], [6], [11], [15], [16], [17], [20], [21]).

In the recent decades, the stability of functional equations have been investigated by many mathematicians. They have so many applications in Information Theory, Physics, Economic Theory, Pure Mathematics, and Social and Behavior Sciences. These applications are sources of inspiration, attraction, and influences to bring more and more people to this interesting and ever-growing branch of mathematics.

In 1938, A.G. Pinsker [2] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundersan [8] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation $f(x+y)=f(x)+f(y)$, $x\perp y$ in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [14]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. There are many types of orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Semi-inner product, Carlson, area and Diminnie. In 1985, Rätz [7] introduced a new definition of orthogonality by using more restrictive axioms then those of Gudder and Strawther.

Now, Let us recall the orthogonality in the sense of J. Rätz [7]. Suppose X is a real vector space with dim ≥ 2 and $\perp$ is a binary relation on X with the following properties.

(O_1) *totality of* $\perp$ *for zero*: $x\perp 0$, $0\perp x$ for all $x\in X$.

(O_2) *independence*: if $x,y\in X-\{0\}$, $x\perp y$, then x,y are linearly independent.

(O_3) *homogeneity*: if $x,y\in X$, $x\perp y$, then $\alpha x\perp\beta y$ for $\alpha,\beta\in R$.

(O_4) *the Thalesian property*: if P is a 2-dimensional subspace of X, $x\in P$ and $\lambda\in R_+$, then there exists $y_0\in P$ such that $x\perp y_0$ and $x+y_0\perp\lambda x-y_0$.

The pair $(X,\perp)$ is called an orthogonality space. By an orthogonality space we mean an orthogonality space equipped with a norm. The relation $\perp$ is called symmetric if $x\perp y$ and $y\perp x$ for all $x,y\in X$.

Later on, in 1995, Ger and Sikorska [13] investigated the orthogonal stability of Cauchy functional equation $f(x+y)=f(x)+f(y)$ in the sense of Rätz, for the map $f:X\to Y$ where X is a orthogonality space and Y is a real Banach space for all $x,y\in X$ with $x\perp y$. This result was generalized by M.S. Moslehian [10] in the framework of Banach modules. In 2005, M.S. Moslehian proved the Hyers-Ulam stability of the conditional equation of Pexider type $f(x+y)+f(x-y)=2g(x)+2h(y), x\perp y$. The functional equation

$$f(x+3y)-3f(x+y)+3f(x-y)-f(x-3y)=48f(y) \tag{3}$$

is said to be cubic functional equation since $f(x)=cx^3$ is its solution. Every solution of the cubic functional equation is said to be cubic mapping. The stability problem for this equation was proved by Wiwatwanich and Nakmahalhlasint [1] for mapping $f:X\to Y$, where X and Y are real Banach spaces. The functional equation

$$f(3x+y)+f(x+3y)$$

$$= 64f(x) + 64f(y) + 24f(x+y) - 6f(x-y) \tag{4}$$

is said to be quartic functional equation since $f(x) = cx^4$ is its solution. The stability problem for the quartic equation was proved by Montakarn Petapirak and Paisan Nakmahachalasint [9].

Definition 1.1. Let X be an orthogonality normed space and Y be a real Banach space. A mapping $f : X \to Y$ is said to orthogonally cubic if it satisfies the so-called orthogonally cubic functional equation (1) for all $x, y \in X$ with $x \perp y$.

Definition 1.2. Let X be an orthogonality normed space and Y be a real Banach space. A mapping $f : X \to Y$ is said to orthogonally quartic if it satisfies the so-called orthogonally quartic functional equation (2) for all $x, y \in X$ with $x \perp y$.

In this paper, we generalize the Hyers-Ulam-Rassias stability of the so called orthogonally Cubic functional equation (1) and of the orthogonally Quartic functional equation (2) in the sense of Rätz orthogonality.

2. Main Results

2.1. Stability of the Orthogonally Cubic Functional Equation

In this section, we deal with the Hyers-Ulam-Rassias stability problem for the orthogonal cubic functional equation

$$f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) = 48f(y)$$

for all $x, y \in X$ with $x \perp y$.

Throughout this paper $(X, \perp)$ is orthogonality normed space with norm $\|\cdot\|_X$ and $(Y, \|\cdot\|_Y)$ is a real Banach space.

Theorem 1.1. *Let θ and p $(p < 3)$ be non-negative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying*

$$\|f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{5}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $T : X \to Y$ such that

$$\|f(y) - T(y)\|_Y \leq \frac{\theta(27 + 3^p)\|y\|_X^p}{48(27 - 3^p)} \tag{6}$$

for all $x, y \in X$ with $y \perp 0$.

Proof. Replacing y by $-y$ in equation (1.1), we get

$$\|f(x-3y) - 3f(x-y) + 3f(x+y) - f(x+3y) - 48f(-y)\|_Y \\ \leq \theta(\|x\|_X^p + \|-y\|_X^p). \tag{7}$$

Adding (1.1) and (1.3), we get

$$\begin{aligned} \|-48f(-y) - 48f(y)\|_Y &\leq \theta(\|x\|_X^p + \|-y\|_X^p) + \theta(\|x\|_X^p + \|y\|_X^p) \\ &\leq 2\theta(\|x\|_X^p + \|y\|_X^p) \\ \|f(-y) + f(y)\|_Y &\leq \frac{\theta}{24}(\|x\|_X^p + \|y\|_X^p) \end{aligned} \tag{8}$$

for all $x, y \in X$ with $x \perp y$. From equation (1.1) and (1.4) and fix $x = 0$,

$$\begin{aligned} \|f(3y) - 3f(y) + 3f(-y) - f(-3y) - 48f(y)\|_Y &\leq \theta\|y\|_X^p \\ \|2f(3y) - 54f(y)\|_Y &\leq \theta\|y\|_X^p + \|f(3y) + f(-3y) - 3(f(y) + f(-y))\|_Y \\ &\leq \theta\|y\|_X^p + \|f(3y) + f(-3y)\|_Y + 3\|f(y) + f(-y)\|_Y \\ &\leq \theta\|y\|_X^p + \frac{\theta}{24}\|3y\|_X^p + \frac{3\theta}{24}\|y\|_X^p \\ &\leq \frac{27\theta}{24}\|y\|_X^p + \frac{\theta}{24}\|3y\|_X^p \end{aligned} \tag{9}$$

for all $y \in X$, since $y \perp 0$ by using (O_3), So

$$\left\|\frac{f(3y)}{27} - f(y)\right\|_Y = \frac{\theta}{48}\left(\|y\|_X^p + \frac{\|3y\|_X^p}{27}\right) \text{ for all } y \in X \text{ with } y \perp 0.$$

Making use of triangle inequality it follows that

$$\begin{aligned} \left\|\frac{f(3^n y)}{27^n} - f(y)\right\|_Y &= \left\|\sum_{k=0}^{n-1} \frac{f(3^{k+1}y)}{27^{k+1}} - \frac{f(3^k y)}{27^k}\right\|_Y \\ &\leq \sum_{k=0}^{n-1} \frac{1}{27^k}\left\|\frac{f(3^{k+1}y)}{27} - f(3^k y)\right\|_Y \\ &\leq \frac{\theta}{48}\sum_{k=0}^{n-1} \frac{1}{27^k}\left(\|3^k y\|_X^p + \frac{\|3^{k+1}y\|_X^p}{27}\right) \end{aligned} \tag{10}$$

for all $y \in X$, with $y \perp 0$ and $n \geq 1$, for any positive integer m, we divide by 27^m and replacing y by $3^m y$.

$$\begin{aligned}
&\left\| \frac{f(3^{n+m}y)}{27^{n+m}} - \frac{f(3^m y)}{27^m} \right\|_Y \\
&\qquad \leq \frac{\theta}{48} \sum_{k=0}^{n-1} \frac{1}{27^{k+m}} \left(\|3^{k+m}y\|_X^p + \frac{\|3^{k+m+1}y\|_X^p}{27} \right) \qquad (11)
\end{aligned}$$

for all $y \in X$ with $y \perp 0$, this shows that $\{f(3^n y)/27^n\}$ is a Cauchy sequence in Y. Since Y is a complete normed space, there exists a mapping $T : X \to Y$ defined by

$$T(y) = \lim_{n\to\infty} \frac{f(3^n y)}{27^n}$$

for all $y \in X$ with $y \perp 0$. Taking $m = 0$ and $n \to \infty$ in (1.7), we get the inequality (1.2).

To show that T satisfies the equation (1.1) for all $x, y \in X$ we replace x and y by $3^n x$ and $3^n y$ respectively in (1.1) and then dividing by 27^n, we get

$$\begin{aligned}
&\|T(x + 3y) - 3T(x + y) + 3T(x - y) - T(x - 3y) - 48T(y)\|_Y \\
&\quad = \lim_{n\to\infty} \frac{1}{27^n} \|f(3^n(x + 3y)) - 3f(3^n(x + y)) + 3f(3^n(x - y)) \\
&\qquad\qquad - f(3^n(x - 3y)) - 48f(3^n y)\|_Y \\
&\quad \leq \lim_{n\to\infty} \frac{1}{27^n} \theta(\|3^n x\|_X^p + \|3^n y\|_X^p) \\
&\quad = \lim_{n\to\infty} \frac{3^{np}}{27^n} \theta(\|x\|_X^p + \|y\|_X^p) = 0
\end{aligned}$$

for all $x, y \in X$ and $x \perp y$. So

$$\|T(x + 3y) - 3T(x + y) + 3T(x - y) - T(x - 3y) - 48T(y)\|_Y = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $T : X \to Y$ is an orthogonal Cubic mapping satisfying (1.1). For uniqueness let $C : X \to Y$ be another orthogonal Cubic mapping satisfying (1.1) hence it follows that

$$\begin{aligned}
&\|T(y) - C(y)\|_Y \\
&\quad \leq \frac{1}{27^n} \|T(3^n y) - C(3^n y)\|_Y \\
&\quad \leq \frac{1}{27^n} (\|T(3^n y) - f(3^n y)\|_Y + \|f(3^n y) - C(3^n y)\|_Y)
\end{aligned}$$

$$\leq \frac{1}{24} \sum_{k=0}^{\infty} \frac{\theta}{27^{n+k}} \left(\frac{(27+3^p)\|y\|_X^p}{(27-3^p)} \right)$$

which tends to zero for all $y \in X$. So, we obtained $T(x) = C(x)$ for all $x \in X$. This proves the uniqueness of T. Which completes the proof. □

Theorem 1.2. *Let θ and p $(p > 3)$ be non-negative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying*

$$\|f(x+3y) - 3f(x+y) + 3f(x-y) - f(x-3y) - 48f(y)\|_Y \\ \leq \theta(\|x\|_X^p + \|y\|_X^p) \tag{12}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $T : X \to Y$ such that

$$\|f(y) - T(y)\|_Y \leq \frac{\theta(27+3^p)\|y\|_X^p}{48(3^p - 27)} \tag{13}$$

for all $x, y \in X$.

Proof. Replacing y by $y/3$ and dividing by 2 in (1.5) we get

$$\left\| f(y) - 27f\left(\frac{y}{3}\right) \right\|_Y \leq \frac{27\theta}{48} \left\| \frac{y}{3} \right\|_X^p + \frac{\theta}{48}\|y\|_X^p \tag{14}$$

for all $y \in X$, since $y \perp 0$, So

$$\left\| 27^{n+m} f\left(\frac{y}{3^{n+m}}\right) - 27^m f\left(\frac{y}{3^m}\right) \right\|_Y \\ \leq \frac{\theta}{48} \sum_{k=0}^{n-1} 27^{k+m} \left(\left\| \frac{y}{3^{k+m}} \right\|_X^p + 27 \left\| \frac{y}{3^{k+m+1}} \right\|_X^p \right) \tag{15}$$

for all $y \in X$ this shows that $\{27^n f(y/3^n)\}$ is a Cauchy sequence in Y as $m \to \infty$ and Y is a complete normed space, there exists a mapping $T : X \to Y$ defined by

$$T(y) = \lim_{n \to \infty} 27^n f(y/3^n)$$

for all $y \in X$. Taking $n = 0$ and $m \to \infty$ we get (1.9). The rest of the proof is similar to Theorem 1.1. □

1.2. Stability of the Orthogonally Quartic Functional Equations

In this section, we deal with the Hyers-Ulam-Rassias stability problem for the orthogonal functional equation

$$\begin{aligned} &f(3x+y)+f(x+3y)\\ &\quad = 64f(x)+64f(y)+24f(x+y)-6f(x-y) \end{aligned}$$

for all $x, y \in X$ with $x \perp y$.

Theorem 1.3. *Let θ and $p(p<4)$ be non-negative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0)=0$ satisfying*

$$\begin{aligned} &\|f(3x+y)+f(x+3y)-64f(x)-64f(y)-24f(x+y)+6f(x-y)\|_Y\\ &\quad \le \varepsilon(\|x\|_X^p+\|y\|_X^p) \end{aligned} \tag{16}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Quartic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\|_Y \le \frac{\varepsilon}{|81-3^p|}\|x\|_X^p \tag{17}$$

for all $x \in X$ with $x \perp 0$.

Proof. Substituting $y=0$ in (2.1), we get

$$\begin{aligned} &\|f(3x)+f(x)-64f(x)-24f(x)+6f(x)\|_Y \le \varepsilon\|x\|_X^p,\\ &\|f(3x)-81f(x)\|_Y \le \varepsilon\|x\|_X^p \end{aligned} \tag{18}$$

for all $x \in X$ since $x \perp 0$. So

$$\left\|\frac{f(3x)}{81}-f(x)\right\|_Y \le \frac{\varepsilon}{81}\|x\|_X^p$$

for all $x \in X$ with $x \perp 0$
Again substituting $x = 3x$ and dividing by 81, we get

$$\left\|\frac{f(3^2x)}{81^2}-\frac{f(3x)}{81}\right\|_Y \le \frac{\varepsilon}{81^2}\|3x\|_X^p \ .$$

By using induction, we get

$$\left\|\frac{f(3^nx)}{81^n}-\frac{f(3^mx)}{81^m}\right\|_Y \le \frac{\varepsilon}{81}\sum_{k=m}^{n-1}\frac{3^{kp}}{81^k}\|x\|_X^p \tag{19}$$

for all non negative integers n, m with $n > m$. This implies that sequence $f(3^n x)/81^n$ is a Cauchy sequence in Y. Since Y is complete. Therefore there exists a mapping $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty} f(3^n x)/81^n$$

for all $x \in X$ with $x \perp 0$. Taking $m = 0$ and $n \to \infty$ in (2.4) we get the inequality

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{|81 - 3^p|}\|x\|_X^p .$$

Now, To show that $Q : X \to Y$ satisfies the equation (1.1) for all $x, y \in X$ with $x \perp y$, we replace x and y by $3^n x$ and $3^n y$ respectively and then dividing by 81^n as follows

$$\begin{aligned}
&\|Q(3x+y)+Q(x+3y)-64Q(x)-64Q(y)-24Q(x+y)+6Q(x-y)\|_Y\\
&= \lim_{n\to\infty} \frac{1}{81^n}\|f(3^n(3x+y)) + f(3^n(x+3y)) - 64f(3^n x)\\
&\qquad - 64f(3^n y) - 24f(3^n(x+y)) + 6f(3^n(x-y))\|_Y\\
&\leq \lim_{n\to\infty} \frac{1}{81^n}(\|3^n x\|_X^p + \|3^n y\|_X^p)\\
&\leq \lim_{n\to\infty} \frac{3^{np}}{81^n}(\|x\|_X^p + \|y\|_X^p) = 0
\end{aligned}$$

for all $x \in X$ with $x \perp y$. So

$$\|Q(3x+y)+Q(x+3y)-64Q(x)-64Q(y)-24Q(x+y)+6Q(x-y)\|_Y = 0$$

for all $x \in X$ with $x \perp y$. Hence $Q : X \to Y$ is an orthogonally Quartic mapping.

To prove uniqueness of orthogonally Quartic mapping $Q : X \to Y$, we assume that there exists another Quartic mapping $P : X \to Y$ satisfying (2.2), then we have

$$\begin{aligned}
&\|Q(x) - P(x)\|_Y\\
&\leq \frac{1}{81^n}\|Q(3^n x) - P(3^n x)\|_Y\\
&\leq \frac{1}{81^n}(\|f(3^n x) - Q(3^n x)\|_Y + \|f(3^n x) - P(3^n x)\|_Y)\\
&\leq \frac{1}{81^n}\left(\frac{\varepsilon}{|81 - 3^p|}\|3^n x\|_X^p + \frac{\varepsilon}{|81 - 3^p|}\|3^n x\|_X^p\right)
\end{aligned}$$

$$\leq \frac{1}{81^n}\left(\frac{2\varepsilon}{|81-3^p|}\|3^n x\|_X^p\right)$$
$$\leq \frac{2}{81^n}\left(\frac{3^{np}\varepsilon}{|81-3^p|}\|x\|_X^p\right)$$

which tends to zero for all $x \in X$. So we get $Q(x) = P(x)$ for all $x \in X$. This proves the uniqueness of Q. Which completes the proof. □

Theorem 1.4. *Let θ and p $(p > 4)$ be non-negative real numbers. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying*

$$\|f(3x+y)+f(x+3y)-64f(x)-64f(y)-24f(x+y)+6f(x-y)\|_Y \leq \varepsilon(\|x\|_X^p + \|y\|_X^p) \quad (20)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Quartic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\varepsilon}{|3^p - 81|}\|x\|_X^p \quad (21)$$

for all $x \in X$.

Proof. Replacing x by $x/3$ in (2.4), we get

$$\left\|81^n f\left(\frac{x}{3^n}\right) - 81^m f\left(\frac{x}{3^m}\right)\right\|_Y \leq \frac{\varepsilon}{81}\sum_{k=m}^{n-1}\frac{81^k}{3^{kp}}\|x\|_X^p$$

for all non-negative integers n, m with $n > m$. This implies that the sequence $81^n f(x/3^n)$ is a Cauchy sequence in Y. Since Y is complete there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{n\to\infty} 81^n f(x/3^n)$$

for all $x \in X$ with $x \perp 0$. Taking $m = 0$ and $n \to \infty$ we get (2.6). The rest of the proof is same as the proof of Theorem 2.1. □

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