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γ-β-CONNECTEDNESS

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Abstract: The purpose of this paper is to introduce and study γ-β-connectedness in terms of γ-β-open sets [6] defined using an operator γ due to Ogata [14]. Several characterizations, basic properties and preservation theorems of γ-β-connectedness are obtained. The concepts like γ-β-component, γ-β-quasi-component and local γ-β connectedness are also investigated.

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1. Introduction

The core foundation of general topology rests on the Kuratowski's closure operator along with its dual interior operator. Judicious usage of such, emerged certain ramifications of generalized open sets. Apart from semiopen sets, preopen sets, in recent years, the study of another generalized open sets, called β-open sets, has become very popular and is acting as a focus of attention by many. As to the applications of β-open sets in very recent times, we refer to specially the papers [3, 4, 5].

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Earlier, in 1994, Popa and Noiri [13] introduced the notion of β-connectedness using β-open sets [12], which is equivalent to semipre-connectedness due to Aho and Nieminen [1]. It was further characterized by Jafari and Noiri [8]. On the other hand, Császár [7] developed γ-connectedness by using a monotone mapping γ on the power set. Recently, using the operator γ due to Ogata [14], Basu et al.[6] initiated γ-β-open sets for the exhibition of certain separation axioms. In this paper we have introduced and investigated γ-β-connectedness by using the concept of such γ-β-open sets .

In Section 3 we have defined γ-β-separated sets, γ-β-connected spaces along with its various characterizations.

Section 4 concerns with the preservation of γ-β-connected spaces in terms of some defined functions and further characterizations of γ-β-connected spaces with respect to real valued and other allied mappings. Characterizations of γ-β-connectedness in terms of intermediate value property of certain functions is also achieved.

Locally γ-β-connected spaces are introduced and characterized in Section 5.

2. Preliminaries

An operation γ [14] on a topology τ on X is a mapping $\gamma : \tau \to P(X)$, such that $V \subset V^\gamma$ for each $V \in \tau$, where $P(X)$ is the power set of X and V^γ denotes the value of γ at V. A subset A of X with an operation γ on τ is called *γ-open* [14] if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. The set of all γ open sets of (X, τ) is denoted by τ_γ and clearly $\tau_\gamma \subset \tau$. Since the intersection of even two γ-open sets may not be γ-open, τ_γ does not form a topology [14]. The *γ-closure* [14] of subset A of X with an operation γ on τ is denoted by τ_γ-$cl(A)$ and is defined to be the intersection of all γ-closed sets containing A and *τ_γ-interior* [10] of A is denoted by τ_γ-$int(A)$ and is defined as the union of all γ-open sets of X contained in A.

In this paper, (X, τ, γ) and (Y, τ', γ') (or simply X and Y) always mean topological spaces with operations γ on τ and γ' on τ' respectively. No separation axioms are assumed on X and Y unless explicitly stated. For a subset A of X, $cl(A)$ and $int(A)$ represent the closure and interior of A respectively.

A subset A of X is called *β-open* [12] or semi-preopen [2] (resp. *preopen* [11]) if $A \subset cl(int(cl(A)))$ (resp. $A \subset int(cl(A))$). A subset A of (X, τ, γ) is called a *γ-β-open* [6] if $A \subset \tau_\gamma$-$cl(\tau_\gamma$-$int(\tau_\gamma$-$cl(A)))$. γ-$\beta O(X)$ denotes the collection of all γ-β-open sets of (X, τ, γ). A subset A of X is called *γ-β-closed* if and only if its complement is γ-β-open. A set which is both γ-β-open and γ-β-closed is called *γ-β-clopen*. The union of all γ-β-open sets contained in A is called *γ-β-interior* [6] of A and is denoted by γ-$\beta int(A)$. The intersection of all γ-β-closed sets containing A is called the *γ-β-closure* [6] of A and is denoted by γ-$\beta cl(A)$. A topological space X is said to be *β-connected [13]* (*semipreconnected* [1]) if X can not be expressed as the union of two nonempty disjoint β-open sets of X. A function $f : (X, \tau) \to (Y, \tau')$ is said to be *γ-β-continuous* [6] at x if for each open set V containing $f(x)$, there exists an γ-β-open set U in X containing x such that $f(U) \subset V$.

3. γ-β-Connected Sets

Definition 3.1. Two subsets A, B of (X, τ, γ) are said to be γ-β-separated (relative to X) if and only if γ-$\beta cl(A) \cap B = \gamma$-$\beta cl(B) \cap A = \emptyset$.

Lemma 3.2. *For subsets A, B of (X, τ, γ), the following statements are equivalent:*

(a) A and B are γ-β-separated.

(b) There exist γ-β-closed sets F_1 and F_2 satisfying $A \subset F_1 \subset (X - B)$ and $B \subset F_2 \subset (X - A)$.

(c) There exist γ-β-open sets G_1 and G_2 satisfying $A \subset G_1 \subset (X - B)$ and $B \subset G_2 \subset (X - A)$.

Theorem 3.3. *If for a γ-β-closed subset S of (X, τ, γ), A and B are γ-β-separated sets (relative to X) satisfying $S = A \cup B$, then A and B are γ-β-closed sets.*

Proof. Let $S = A \cup B$, where γ-$\beta cl(A) \cap B = \emptyset = A \cap \gamma$-$\beta cl(B)$. Now, $S \cap \gamma$-$\beta cl(A) = (A \cup B) \cap \gamma$-$\beta cl(A) = A$. As intersection of γ-β-closed sets is γ-β-closed, A is γ-β-closed. Similarly B is γ-β-closed.

Definition 3.4. A subset S of (X, τ, γ) is said to be γ-β-connected (relative to X) if whenever A and B are γ-β-separated sets with $S = A \cup B$ then either $A = \emptyset$ or $B = \emptyset$. The space (X, τ, γ) is said to be γ-β-connected if and only if it is a γ-β-connected subset of itself.

We state the following theorem without giving the proof:

Theorem 3.5. *The following statements are equivalent in (X, τ, γ):*

(a) The space X is γ-β-connected.

(b) X can not be expressed as a union of two non-empty disjoint γ-β-open sets.

(c) Whenever A and B are disjoint γ-β-closed sets with $X = A \cup B$ then either $A = \emptyset$ or $B = \emptyset$.

(d) $\emptyset$ and X are the only sets which are both γ-β-open and γ-β-closed.

Definition 3.6. A subset A of (X, τ, γ) is called γ-preopen if $A \subset \tau_\gamma$-$int(\tau_\gamma$-$cl(A))$. γ-$PO(X)$ denotes the collection of all γ-preopen sets in X. The complement of a γ-preopen set is called a γ-preclosed set.

Definition 3.7. Let A be subset of (X, τ, γ). The union of all γ-preopen sets contained in A is called the γ-preinterior of A and is denoted by γ-$pint(A)$. The intersection of all γ-preclosed sets containing A is called the γ-preclosure of A and is denoted by γ-$pcl(A)$.

Lemma 3.8. *The following properties hold in (X, τ, γ):*

(i) For each $U(\neq \emptyset) \in \gamma$-$\beta O(X)$, γ-$pint(U) \neq \emptyset$.

(ii) For each $U \in \gamma$-$\beta O(X)$, γ-$\beta cl(U) \in \gamma$-$\beta O(X)$.

Proof. (i) If possible, suppose γ-$pint(U) = \emptyset$, for $U(\neq \emptyset) \in \gamma$-$\beta O(X)$. Now γ-$pint(U) = U \cap \tau_\gamma$-$int(\tau_\gamma$-$cl(U)) = \emptyset$. This implies that τ_γ-$cl(U) \cap \tau_\gamma$-$int(\tau_\gamma$-$cl(U)) = \emptyset$ and hence τ_γ-$int(\tau_\gamma$-$cl(U)) = \emptyset$. So, $U \subset \tau_\gamma$-$cl(\tau_\gamma$-$int(\tau_\gamma$-$cl(U))) = \emptyset$ — which is a contradiction. Therefore, γ-$pint(U) \neq \emptyset$.

(ii) Let U be a γ-β-open in X. Then γ-$\beta cl(U) = U \cup \tau_\gamma$-$int(\tau_\gamma$-$cl(\tau_\gamma$-$int(U))) \subset \tau_\gamma$-$cl(\tau_\gamma$-$int(\tau_\gamma$-$cl(U))) = \tau_\gamma$-$cl(\tau_\gamma$-$int[\tau_\gamma$-$cl(U) \cup \tau_\gamma$-$cl(\tau_\gamma$-$int(U))]) = \tau_\gamma$-$cl(\tau_\gamma$-$int(\tau_\gamma$-$cl[U \cup \tau_\gamma$-$int(\tau_\gamma$-$cl(\tau_\gamma$-$int(U)))])) = \tau_\gamma$-$cl(\tau_\gamma$-$int(\tau_\gamma$-$cl(\gamma$-$\beta cl(U))))$.

Aho and Nieminen [1] characterized β-connectedness in terms of preopen and semi-preopen sets. Subsequently Jafari and Noiri [8] further characterized β-connectedness in Theorem 3.1 [8]. In the following two theorems we have been able to characterize γ-β-connectedness with respect to γ-β-open and γ-preopen sets.

Theorem 3.9. *The following properties are equivalent in (X, τ, γ):*

(i) X is γ-β-connected.

(ii) $V_1 \cap V_2 \neq \emptyset$ for every nonempty γ-β-open sets V_1, V_2 of X.

(iii) $U_1 \cap U_2 \neq \emptyset$ for every nonempty γ-preopen sets U_1, U_2 of X.

Proof. $(i) \Rightarrow (ii)$: Let X be γ-β-connected and suppose that $V_1 \cap V_2 = \emptyset$ for some pair of nonempty γ-β-open sets V_1, V_2 of X. Therefore, γ-β-$cl(V_1) \cap V_2 = \emptyset$.

Let γ-β-$cl(V_1) = V^\star$. This $V^\star$ is non-empty proper γ-β-open (by above Lemma 3.9 (ii)) as well as γ-β-closed in X. Which contradicts that X is γ-β-connected (because of Theorem 3.5 (d)).

$(ii) \Rightarrow (iii)$: Follows from the fact that every γ-preopen set is γ-β-open.

$(iii) \Rightarrow (i)$: If X is not γ-β-connected. Then for two disjoint nonempty γ-β-open sets V_1, V_2 of X, $X = V_1 \cup V_2$. Now by Lemma 3.8 (i), there exist two nonempty γ-preopen sets $U_i = \gamma\text{-}pint(V_i) \subset V_i$, for $i = 1, 2$. So $U_1 \cap U_2 = \emptyset$ — a contradiction.

Theorem 3.10. *The following properties are equivalent in (X, τ, γ):*

(i) X is γ-β-connected.

(ii) For each nonempty $U \in \gamma\text{-}PO(X)$, $\gamma\text{-}pcl(U) = X$.

(iii) For each nonempty $U \in \gamma\text{-}\beta O(X)$, $\gamma\text{-}pcl(U) = X$.

(iv) For each nonempty $U \in \gamma\text{-}PO(X)$, $\gamma\text{-}\beta cl(U) = X$.

(v) For each nonempty $U \in \gamma\text{-}\beta O(X)$, $\gamma\text{-}\beta cl(U) = X$.

Proof. $(i) \Rightarrow (ii)$: Let X be γ-β-connected. Then by Theorem 3.9, every pair of γ-preopen sets in X are intersecting and therefore $\gamma\text{-}pcl(U) = X$ for each nonempty $U \in \gamma\text{-}PO(X)$.

$(ii) \Rightarrow (iii)$: Let $U(\neq \emptyset) \in \gamma\text{-}\beta O(X)$. Taking $V = \gamma\text{-}pint(U)$ (which is nonempty by Lemma 3.8), we get $X = \gamma\text{-}pcl(V) \subset \gamma\text{-}pcl(U)$ and hence we have $\gamma\text{-}pcl(U) = X$.

$(iii) \Rightarrow (iv)$: Let $U(\neq \emptyset)$ be a γ-preopen set in X and if possible $\gamma\text{-}\beta cl(U) \neq X$. Then if we take $V = X - \gamma\text{-}\beta cl(U)$, then $V \in \gamma\text{-}\beta O(X)$ and $V \cap U = \emptyset$. Since U is nonempty γ-preopen, we have $U \cap \gamma\text{-}pcl(V) = \emptyset$. Thus $\gamma\text{-}pcl(V) \neq X$ — a contradiction.

$(iv) \Rightarrow (v)$: Let $U(\neq \emptyset) \in \gamma\text{-}\beta O(X)$. Since $V = \gamma\text{-}pint(U)$ is nonempty (by Lemma 3.8) we have $X = \gamma\text{-}\beta cl(V) \subset \gamma\text{-}\beta cl(U)$ and hence $\gamma\text{-}\beta cl(U) = X$.

$(v) \Rightarrow (i)$: Suppose there exist $U(\neq \emptyset)$, $V(\neq \emptyset) \in \gamma\text{-}\beta O(X)$ with $U \cap V = \emptyset$. Clearly $\gamma\text{-}\beta cl(U) \neq X$ — a contradiction. So X is γ-β-connected.

Theorem 3.11. *If a γ-β-connected set S of (X, τ, γ) is contained in $A \cup B$, where A and B are γ-β-separated sets then either $S \subset A$ or $S \subset B$.*

Proof. We have $S = (S \cap A) \cup (S \cap B)$ where $S \cap A$ and $S \cap B$ are γ-β-separated. So either $S \cap A = \emptyset$ or $S \cap B = \emptyset$ and hence either $S \subset B$ or $S \subset A$.

Theorem 3.12. *A subset M of (X, τ, γ) is γ-β-connected if there exists a γ-β-connected set C satisfying $C \subset M \subset \gamma$-$\beta cl(C)$.*

Proof. Let $M = A \cup B$, where A and B are γ-β-separated sets. Then either $C \subset A$ and $C \subset B$ and hence either $M \subset \gamma$-$\beta cl(C) \subset \gamma$-$\beta cl(A) \subset (X - B)$ or $M \subset (X - A)$. Therefore either $B = \emptyset$ or $A = \emptyset$.

Corollary 3.13. *If C is γ-β-connected in (X, τ, γ), then γ-$\beta cl(C)$ is also γ-β-connected.*

Theorem 3.14. *If $\{M_\alpha : \alpha \in I\}$ is a family of γ-β-connected sets of (X, τ, γ) satisfying the property that any two of which are not γ-β-separated, then $M = \cup_{\alpha \in I} M_\alpha$ is γ-β-connected.*

Proof. Let $M = A \cup B$ where A and B are γ-β-separated. Then for each $\alpha \in I$ either $M_\alpha \subset A$ or $M_\alpha \subset B$. Since any two members of the family $\{M_\alpha : \alpha \in I\}$ are not γ-β-separated then either $M_\alpha \subset A$ for each $\alpha \in I$ or $M_\alpha \subset B$ for each $\alpha \in I$. So either $B = \emptyset$ or $A = \emptyset$.

Corollary 3.15. *If $M = \cup_{\alpha \in I} M_\alpha$, where each M_α is γ-β-connected in (X, τ, γ) and also $M_\alpha \cap M_{\alpha'} \neq \emptyset$ for $\alpha, \alpha' \in I$, then M is γ-β-connected.*

Corollary 3.16. *If $M = \cup_{\alpha \in I} M_\alpha$, where each M_α is γ-β-connected in (X, τ, γ) and $\cap_{\alpha \in I} M_\alpha \neq \emptyset$ for $\alpha \in I$, then M is γ-β-connected.*

4. Preservation of γ-β-Connected Sets

Definition 4.1. A function $f : (X, \tau, \gamma) \to (Y, \tau')$ is said to be

(a) γ-β-continuous [6] if, inverse image of each open set in Y is γ-β-open in X.

(b) Contra γ-β-continuous if, inverse image of each closed set in Y is γ-β-open in X.

Theorem 4.2. *Let $f : (X, \tau, \gamma) \to (Y, \tau')$ be γ-β-continuous or contra γ-β-continuous. If $S \subset X$ is γ-β-connected then $f(S)$ is connected in Y.*

Proof. We prove this theorem for γ-β-continuous function only, because the proof for contra γ-β-continuous is quite similar.

Suppose $f(S)$ is not connected. Let U and V are separated sets in Y such that $f(S) = U \cup V$. Clearly $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Now, we have $S = A \cup B$, where $\emptyset \neq A = S \cap f^{-1}(U)$ and $\emptyset \neq B = S \cap f^{-1}(V)$. Again, f being γ-β-continuous, we have γ-$\beta cl(f^{-1}(U)) \subset f^{-1}(cl(U))$ and γ-$\beta cl(f^{-1}(V)) \subset$

$f^{-1}(cl(V))$. Also, $A \subset f^{-1}(U) \subset \gamma$-$\beta cl(f^{-1}(U))$, which implies γ-$\beta cl(A) \subset \gamma$-$\beta cl(f^{-1}(U)) \subset f^{-1}(cl(U))$. Therefore, γ-$\beta cl(A) \cap B = \emptyset$. Similarly, $A \cap \gamma$-$\beta cl(B) = \emptyset$. So, A and B are γ-β-separated sets in X such that $S = A \cup B$. Hence S is not γ-β-connected — a contradiction. Therefore $f(S)$ is connected.

Theorem 4.3. *A space (X, τ, γ) is γ-β-connected if and only if no γ-β-continuous or contra γ-β-continuous function on X into the discrete two point space $\{a, b\}$ is surjective.*

Theorem 4.4. *A space (X, τ, γ) is γ-β-connected if and only if every real valued γ-β-continuous or contra γ-β-continuous function $f : X \to \mathbb{R}$ takes on all the values between any two that it assumes, where $\mathbb{R}$ denotes the sets of real numbers.*

Proof. We prove this theorem for γ-β-continuous function only, because the proof for contra γ-β-continuous is quite similar.

Suppose (X, τ, γ) is γ-β-connected and $f : X \to \mathbb{R}$ is γ-β-continuous. Then by Theorem 4.2, $f(X) \subset \mathbb{R}$ is connected and hence it is an interval. If $f(a) = x$ and $f(b) = y$ for $a, b \in X$, then $[x, y] \subset f(X)$. Thus for each z such that $x \leq z \leq y$, there exists a point c in X such that $f(c) = z$.

Conversely, let (X, τ, γ) be not γ-β-connected. Then by Theorem 3.5, $X = U \cup V$, for some nonempty disjoint γ-β-open sets U and V of X. Now we define a function $f : X \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} a & \text{if } x \in U \\ b & \text{if } x \in V \quad (a \neq b) \end{cases}$$

Clearly f is γ-β-continuous and this function does not take any value between a and b, contrary to the hypothesis. Hence (X, τ, γ) is γ-β-connected.

Theorem 4.5. *A space (X, τ, γ) is γ-β-connected if and only if every contra γ-β-continuous function from (X, τ, γ) into any T_1 space (Y, τ_1) is constant.*

Proof. Let (X, τ, γ) be γ-β-connected. Since (Y, τ_1) is T_1 space, $\mathcal{U} = \{f^{-1}(y) : y \in Y\}$ is disjoint γ-β-open partition of X. If $|\mathcal{U}| \geq 2$ (where $|\mathcal{U}|$ denotes the cardinality of $\mathcal{U}$), then X is the union of two nonempty disjoint γ-β-open sets. Since (X, τ, γ) is γ-β-connected, $|\mathcal{U}| = 1$. So, f is constant.

Conversely, let (X, τ, γ) be not γ-β-connected and every contra γ-β-continuous function from (X, τ) into any T_1 space Y is constant. Since (X, τ) is not γ-β-connected, there exists a nonempty proper γ-β-open as well as γ-β-closed set V in X. Consider the space $Y = \{0, 1\}$ with the discrete topology τ_1. The function $f : (X, \tau) \to (Y, \tau_1)$ defined by $f(V) = \{0\}$ and $f(X - V) = \{1\}$ is obviously contra γ-β-continuous and which is non-constant — a contradiction. Therefore (X, τ, γ) is γ-β-connected.

Definition 4.6. Let $S \subset X$ containing the point x. The γ-β-component of S containing x is defined to be the set $S_x = \cup\{C \subset S : x \in C, C$ is γ-β-connected relative to $X\}$.

Theorem 4.7. *For a space (X, τ, γ), the following properties hold:*

(a) Each γ-β-component of S is a maximal γ-β-connected subset of S with respect to X.

(b) Each γ-β-component of a γ-β-closed set S in X is γ-β-closed in X.

(c) Any subset S of X is the union of all its γ-β-components.

(d) Two different γ-β-components of S are disjoint.

Proof. (a) Let C be a γ-β-connected subset of S with respect X containing x. Now by Corollary 3.16, S_x is γ-β-connected. Since $C \subset S_x$, then S_x is a maximal γ-β-connected subset of S containing x.

(b) Let S be γ-β-closed in X and $x \in S$. Then by Corollary 3.13, γ-$\beta cl(S_x) \subset S$ is a γ-β-connected set with respect to X containing x. Now by (a), γ-$\beta cl(S_x) \subset S_x$ and hence S_x is γ-β-closed.

(c) For $x \in S$, $\{x\}$ is a γ-β-connected subset of S containing x. Hence $\{x\} \subset S_x$ i.e. $x \in S_x$. Since $S = \cup_{x \in S} S_x$, then the proof follows.

(d) If $x, y \in S$ and $S_x \cap S_y \neq \emptyset$ then $S_x \cup S_y$ is a γ-β-connected subset of S containing both x and y. By the maximality of S_x and S_y, $S_x = S_x \cup S_y = S_y$.

Definition 4.8. Let S be a subset of a space (X, τ, γ). Two elements x and y are called γ-β-equivalent, written as $x \sim y$, whenever U and V are γ-β-separation relative to X and $S = U \cup V$ then either $x, y \in U$ or $x, y \in V$.

The relation $\sim$ is an equivalence relation on S. The equivalence class $S[x]$ of S containing $x \in S$ is called a γ-β-quasi component of S containing the point x.

Theorem 4.9. *Let S be a nonempty subset of a space (X, τ, γ). Then $S[x]$ is an γ-β-component of S for each $x \in S$, for which $S[x]$ is γ-β-connected.*

Proof. Let $x \in S$ for which $S[x]$ is γ-β-connected and $M \subset S$ be a γ-β-connected set containing $S[x]$. If, for $z \in M$, U and V are γ-β-separated sets relative to X such that $S = U \cup V$, then either $M \subset U$ or $M \subset V$. Thus either $x, z \in U$ or $x, z \in V$. Therefore $z \in S[x]$ and hence $M = S[x]$. So, $S[x]$ is a γ-β-component of S.

Definition 4.10. For a nonempty set S of a space (X, τ, γ) with $x \in S$, we define $[S]_x = \cap\{K \subset X$: for some $M \subset X$, K and M are γ-β-separated sets relative to X with $S = K \cup M$ and $x \in K\}$.

Theorem 4.11. *For a subset S of a space (X, τ, γ), the γ-β-quasi component $S[x] = \cap_{K \in [S]_x} K$.*

Proof. Let $z \in S[x]$ and let $K \in [S]_x$. Then by definition of $[S]_x$, there is an $M \subset X$, satisfying $S = K \cup M$, where K and M are γ-β-separated sets relative to X with $x \in K$. Since $x \sim z$, we have $x, z \in K$. Therefore $z \in \cap_{K \in [S]_x} K$. Now, if $z \in \cap_{K \in [S]_x} K$ then obviously $z \in S[x]$.

Theorem 4.12. *Each γ-β-quasi component A of the space (X, τ, γ) is the intersection of all γ-β-clopen sets containing a point belonging to A.*

Corollary 4.13. *Each γ-β-quasi component of (X, τ, γ) is γ-β-closed.*

Definition 4.14. A function $f : (X, \tau, \gamma) \to (Y, \tau', \gamma')$ is said to be

(a) $(\gamma$-$\gamma')$-β-irresolute [3] (resp. $(\gamma$-$\gamma')$-pre-irresolute) if the inverse image of each γ'-β-open (resp. γ'-preopen) set in Y is γ-β-open (γ-preopen) set in X.

(b) γ-γ'-β-open if $f(V)$ is γ'-β-open in Y whenever V is γ-β-open in X.

(c) θ-$(\gamma$-$\gamma')$-β-irresolute if for each $x \in X$ and each γ'-β-open set V in Y containing $f(x)$, there exists γ-β-open set U in X containing x such that $f(\gamma\text{-}\beta cl(U)) \subset \gamma'\text{-}\beta cl(V)$.

Theorem 4.15. *If $f : (X, \tau, \gamma) \to (Y, \tau', \gamma')$ be surjective a function which is either*

(i) $(\gamma$-$\gamma')$-β-irresolute or

(ii) θ-$(\gamma$-$\gamma')$-β-irresolute or

(iii) $(\gamma$-$\gamma')$-pre-irresolute, then Y is γ'-β-connected if X is γ-β-connected.

Proof. (i) First suppose that, f is $(\gamma$-$\gamma')$-β-irresolute function. Let if possible that Y is not γ'-β-connected. Then there exist nonempty disjoint γ'-β-open sets V_1, V_2 in Y such that $Y = V_1 \cup V_2$. Then V_1, V_2 becomes γ'-β-open and γ'-β-closed sets in Y. Therefore $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint γ-β-open set of X, since f is $(\gamma$-$\gamma')$-β-irresolute surjective function. Also, $X = f^{-1}(V_1) \cup f^{-1}(V_2)$, showing that X is not γ-β-connected — a contradiction. Hence Y is γ'-β-connected.

(ii) Next suppose that f is θ-$(\gamma$-$\gamma')$-β-irresolute. Let V be a nonempty γ'-β-open set of Y. Now, since f is also surjective, there exists a point $x \in X$ such that $f(x) \in V$. Again since f is θ-$(\gamma$-$\gamma')$-β-irresolute, there exists γ-β-open set

U in X containing x, such that $f(\gamma\text{-}\beta(cl(U)) \subset \gamma^{'}\text{-}\beta cl(V)$. Using Theorem 3.10, we have $\gamma\text{-}\beta(cl(U) = X$ and hence $Y = f(X) = f(\gamma\text{-}\beta(cl(U)) \subset \gamma^{'}\text{-}\beta(cl(V)$. Therefore $\gamma^{'}\text{-}\beta cl(V) = Y$ for any nonempty $\gamma^{'}$-β-open set V of Y. Hence by Theorem 3.10, Y is $\gamma^{'}$-β-connected.

(iii) Finally suppose that f is (γ-$\gamma^{'}$)-pre-irresolute function. Let W_1 and W_2 be any two nonempty $\gamma^{'}$-preopen set of Y. Since f is (γ-$\gamma^{'}$)-pre-irresolute surjective function $f^{-1}(W_1)$ and $f^{-1}(W_2)$ are γ-preopen set in X. Again, since X is γ-β-connected, using Theorem 3.9, we have $f^{-1}(W_1) \cap f^{-1}(W_2) \neq \emptyset$ and hence $W_1 \cap W_2 \neq \emptyset$. Therefore by Theorem 3.9, Y is γ-β-connected.

Lemma 4.16. *If $f : (X, \tau, \gamma) \to (Y, \tau^{'}, \gamma^{'})$ is a γ-$\gamma^{'}$-β-open injective function and $A, B \subset X$ are γ-β-separated then $f(A)$, $f(B)$ are $\gamma^{'}$-β-separated.*

Proof. Since A, B are γ-β separated, by Lemma 3.2 there exist γ-β-open sets F_1 and F_2 such that $A \subset F_1 \subset (X - B)$ and $B \subset F_2 \subset (X - A)$. Then $f(A) \subset f(F_1) \subset f(X - B)$ and $f(B) \subset F_2 \subset f(X - A)$, where $f(F_1)$ and $f(F_2)$ are $\gamma^{'}$-β-open sets in Y. Since f is injective, we have $f(X - A) \subset Y - f(A)$ and $f(X - B) \subset Y - f(B)$, so that $f(A)$ and $f(B)$ are $\gamma^{'}$-β-separated by Lemma 3.2.

Theorem 4.17. *If $f : (X, \tau, \gamma) \to (Y, \tau^{'}, \gamma^{'})$ is a γ-$\gamma^{'}$-β-open injective function and $f(C)$ is $\gamma^{'}$-β-connected for $C \subset X$, then C is γ-β-connected.*

Proof. Let $C = A \cup B$, where A and B are γ-β-separated in X. Then by Lemma 4.16, $f(A)$ and $f(B)$ are $\gamma^{'}$-β-separated in Y. Also $f(C) = f(A) \cup f(B)$. Therefore either $f(A) = \emptyset$ or $f(B) = \emptyset$ i.e. either $A = \emptyset$ or $B = \emptyset$. Hence C is γ-β-connected.

Corollary 4.18. *If a bijective γ-$\gamma^{'}$-β-open function $f : (X, \tau, \gamma) \to (Y, \tau^{'}, \gamma^{'})$ is either*

(i) (γ-$\gamma^{'}$)-β-irresolute or

(ii) θ-(γ-$\gamma^{'}$)-β irresolute or

(iii) (γ-$\gamma^{'}$)-pre-irresolute, then X is γ-β-connected if and only if Y is $\gamma^{'}$-β-connected.

Theorem 4.19. *Let $f : (X, \tau, \gamma) \to (Y, \tau^{'}, \gamma^{'})$ be a mapping. Then we have the following:*

(a) If f is γ-β-continuous, then the image of each γ-β-component of X lie in a component of Y.

(b) If f is (γ-$\gamma^{'}$)-β-irresolute, then image of each γ-β-component of X must lie in an $\gamma^{'}$-β-component of Y.

5. Locally γ-β-Connected Spaces

Definition 5.1. A space (X, τ, γ) is called locally γ-β-connected if for each $x \in X$ and each γ-β-open set U containing x there exists an open γ-β-connected set V such that $x \in V \subset U$.

Theorem 5.2. *Let (X, τ, γ) be a space. Then X is locally γ-β-connected if and only if each γ-β-component of γ-β-open set is open.*

Proof. Let (X, τ, γ) be locally γ-β- connected and U be a γ-β-open set and C be a γ-β-component of U. If $x \in C$, then there exists an open γ-β-connected set V such that $x \in V \subset U$. By the maximality of C, we have $x \in V \subset C$. So C is an open set.

Conversely, let $x \in X$ and U be a γ-β-open set containing x. Also let C be a γ-β-component of U containing x. Since C is an open (by hypothesis) γ-β-connected set with $x \in C \subset U$, the result follows.

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