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A DIRECT METHOD FOR OPTIMAL CONTROL PROBLEM

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Abstract: A new stochastic algorithm called Probabilistic Global Search Johor (PGSJ) has recently been established for global optimization of nonconvex real valued problems on finite dimensional Euclidean space. In this paper we present convergence guarantee for this algorithm in probabilistic sense without imposing any more condition. Then, we jointly utilize this algorithm along with control parameterization technique for the solution of constrained optimal control problem. The numerical simulations are also included to illustrate the efficiency and effectiveness of the PGSJ algorithm in the solution of control problems.

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1. Introduction

The optimal control problems frequently arise in many areas including science,

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engineering, managements, etc. However, these problems, in general, are very complicated to solve. In this study, we are aiming at addressing a general form of these problems which is described as,

$$\begin{aligned} \min\ & J = \varphi(x(t_f)) \\ & \dot{x}\ (t) = f(x(t), u(t), t) \\ & g(x(t), u(t), t) \geq 0 \\ & x(t_0) = x_0 \\ & x(t) \in \Re^n, u(t) \in D \in \Re^m \end{aligned} \tag{1}$$

here,φ is a bounded real valued function, u and x are respectively known as control and state functions on time interval $[t_0, t_f]$, and lastly f and g are functions on $\Re^n \times \Re^m \times \Re$ into respectively $\Re^n$ and $\Re^r$ where the inequality constraint can be defined componentwise.

The objective of the problem is then to identify the best control u to minimize the performance index J, while the interaction between the variables of the problem is governed by an ordinary differential equation, and constrained by both equality and inequality constraints possibly involving state and control variables, initial state is given, and control variables are box constrained of the form $D = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_m, b_m]$.

A tremendous amount of research has gone into the investigation on the solution of aforementioned problem leading to many theoretical as well as practical methods for this problem while each method performs well in some certain circumstances. The theoretical studies offer necessary and sufficient conditions of optimality for this problem [2, 11, 14] which have already resulted in many efficient optimization methods see [3, 6, 18, 19] just to mention a few. Nevertheless, the explicit use of theoretical results is often prohibitive due to complexity of arising problems especially when singularity happens [10] or problems behave in noncompliance with assumption, hence the inevitability of the need for numerical methods.

The available numerical methods for the solution of Problem (1) are typically not without discretization and optimization techniques. According to the order of precedence, numerical methods are divided into two approaches, the one called indirect approach which is also known by the paradigm optimize first, and then discretize, and the alternative direct approach which is known by the converse paradigm discretize first, and then optimize [3]. As the name suggest, in the indirect methods first the optimality principles are used to derive an intermediate problem, and then a suitable discretization techniques is applied to solve the resultant problem which helps to acquire a solution for the original problem. Conversely, in the direct approach discretization happens first.

There are two common frameworks available for the discretization procedure in the direct optimization methods, complete parameterization [5] and control parameterization [9]. Either of them is applicable to convert the optimal control problem into a nonlinear programming problem which is solvable by a suitable optimization theoretic. However, there are also advantageous and disadvantageous in applying each one.

The idea of complete parameterization is to discretize the whole variables of a control problem. As a result, the original problem is reduced to a problem of identifying the best parameters of the approximated variables. One advantage of this formulation is to eliminate the online need for the solution of an initial value problem; however this is at the cost of a very large scale optimization problem need to be solved.

The alternative framework results in relatively lower scale problems as in this approach control variable is solely parameterized using linear combinations of basis functions. This advantage helps the control parameterization to be a more popular framework amongst practitioners. A variety of basis functions have been used in this framework, including piecewise constant functions [9, 12, 18], piecewise linear functions [13], Chebyshev polynomials [20], B-splines functions [17], Lagrange polynomials [4], Legendre wavelets [16], and Bzier curves [8].

In this work we use the Bernstein basis function [7]in the control parameterization framework, and the resultant parameter estimation problem is solved using a new stochastic algorithm called Probabilistic Global Search Johor (PGSJ) [1] where this is described in the next section. The convergence of this algorithm is proved in the subsequent section in probabilistic sense. Then, the proposed method is also implemented numerically to illustrate its efficiency, and finally the conclusion is included.

2. The PGSJ Algorithm

As with many other stochastic optimization techniques which are proposed to address the optimization of real valued functions, the PGSJ method utilizes only the function evaluation of trial solutions to direct the search while in contrast with most of evolutionary approaches, no recombination operation is employed. Instead, the new potential solutions in the PGSJ algorithm are selected through carefully sampling in accordance with some Probability Density Functions (PDF) which are iteratively biased toward a global optimizer. Prior

Table 1: The algorithm inputs functions and parameters

n	Dimension of the problem,
f	The Objective function,
D	The box of feasible region,
N	The number of partitions on each interval,
S	The number of samples in each iteration,
A	The acceptable probability density,
b	The number of bisecting procedure,
σ	The Scale Factor,
ξ	Increment in probability updating procedure,
ϵ	The accuracy required,
M	Maximum Number of iterations,
P	Probability of sampling from complementary search space.

to a full description of this algorithm in Algorithm 2.1., the required input function and parameters are introduced in Table (1).

Algorithm 1. The PGSJ algorithm

Step 0 Pre-allocations

Step 0.1 Read the input functions and set the algorithm parameters appear in Table1.

Step 0.2 Let $k_1 = 1$, and go to Step 1.

Step 1 Initializations

Step 1.1 Partition each interval $[a_i, b_i]$ into N subinterval where $i = 1, \ldots, n$ and $j = 1, \ldots, N$

Step 1.2 Initialize a PDF ψ_i on each interval $[a_i, b_i]$, so that the value of ψ_i on I_{ij} is γ_{ij}.

Step 1.3 Initialize a complementary region $C_i = \pi_i(D) - [a_i, b_i]$ where π_i is the projection on ith axis.

Step 1.4 Set $k_2 = 0$ and go to Step 2.

Step 2 Sampling

Step 2.1 Let $i = 1$,

Step 2.2 Generate a uniform random number $0 < r < 1$.

Step 2.3 If $r \leq P$ uniformly sample S point from C_i and go to Step 2.5 otherwise go to Step 2.4.

Step 2.4 Sample S point from interval $[a_i, b_i]$ according to PDF ψ_i.

Step 2.5 If $i < n$ let $i = i + 1$, go to Step 2.2 otherwise go to Step 3.

Step 3 PDF Updating

Step 3.1 Evaluate the new potential solutions, and detect the best (most promising) and the worst (poorest) sampled points according to the objective of the problem.

Step 3.2 for $i = 1$ to n do
If the worst solution is located in subinterval I_{iw}, and the best one in I_{ib}
$\gamma_{iw} = \gamma_{iw} - \xi\gamma_{iw}$
$\gamma_{ib} = \gamma_{ib} + \xi\gamma_{iw}$
end if
end for

Step 3.3 If $\min_{1 \leq i \leq n}(\max_{1 \leq j \leq N}) \geq A$ then go to Step 4, otherwise go to Step 2.1.

Step 4 Bisecting

Step 4.1 for $i = 1$ to n do
Identify the best subinterval I_{ib} on $[a_i, b_i]$ and bisect it into two new subintervals I_{ib_1} and I_{ib_2}.
Identify the worst subinterval I_{iw}, remove it from $[a_i, b_i]$, and add it to the complementary C_i.
Adjust the PDF ψ_i to the change by setting $\gamma_{ib_1} = \gamma_{ib_2} = (\gamma_{ib} + \gamma_{iw})/2$ and $\gamma_{iw} = 0$.
end for

Step 4.2 Let $k_2 = k_2 + 1$

Step 4.3 If $k_2 > b$ then go to Step 5, otherwise go to Step 2.1.

Step 5 Scaling

Step 5.1 Update the best solution $x = [x_1, \ldots, x_n]$ found so far.

Step 5.2 If the best solution has not yet arrived at ϵ neighborhood of optimizer or $k_1 < M$ go to Step 5.2, otherwise stop.

Step 5.3 for $i = 1$ to n do

$d_i = b_i - a_i$

$a_i = x_i - \sigma d_i/2$

$b_i = x_i + \sigma d_i/2$

if $[a_i, b_i] \nsubseteq \pi_i(D)$, set $[a_i, b_i] = [a_i, b_i] \cap \pi_i(D)$.

Step 5.4: Let $k_1 = k_1 + 1$ and go to Step 1.1.

In the earlier descriptions, the PGSJ algorithm is thoroughly described. The convergence analysis of the algorithm is addressed in the next subsection.

3. Convergence Analyzes

We assume PGSJ algorithm can be run indefinitely. Then a sequence of random vectors $\{X_n\}_{n\geq 1}$ can be attained successively. In terms of improvement, we can also define the sequence of random vectors $\{X_n^*\}_{n\geq 1}$ where $X_1^* = X_1$, and $X_n^* = X_n$, if $f(X_n) < (X_{n-1})$ almost surely, otherwise $X_n^* = X_{n-1}$. Additionally, for convenience, we define the following notations,

i X_i^j indicates the random variable which is the jth component of random vector X_i.

ii E_n is the set of $\{X_1, X_2, \ldots, X_n\}$ and $\sigma(E_n)$ is the $\sigma-$algebra generated by all random vectors X_j for $j = 1, 2, \ldots, n-1$.

Theorem 1. *Let Ω be a compact hypercube in $\Re^d$ and f be a continuous real-value function on Ω such that $f^* = essinf f(x) > -\infty$. Consider the sequence $\{X_n^*\}_{n\geq 1}$ and suppose that there is a subsequence $\{X_{n_i}^*\}_{i\geq 1}$ that $X_{n_i}^{*1}, \ldots, X_{n_i}^{*d}$ are conditionally independent given $\sigma-$algebra $\sigma(E_n)$ for $i = 1, 2, \ldots$. Then, $f(X_n^*) \to f^*$ almost surely. In addition, if the global optimizer x^* is unique, then $X_n^*) \to x^*$ almost surely.*

Proof. Let μ be the Lebesgue measure on $\Re^d$, and $\delta > 0$ be any given positive real number. Clearly, we can suppose that there exists a finite subset $\{z_1, \ldots, z_n\} \subseteqq \Omega$ such that,

$$\Omega \subseteqq \cup_{i=1}^{n} B(z_i, \delta/2)$$

As a result, for all $z_i \in \Omega$ there exists an integer k to satisfy $B(z, \delta) \supset B(z_k, \delta/2)$. Consequently, we have,

$$inf_{z\in\Omega}\{\mu(B(z, \delta) \cap \Omega)\} \geq \min_{i=1,\ldots,n}\{\mu(B(z_i, \delta) \cap \Omega)\} > 0$$

In addition, the conditional density related to $X^i_{n_k}$ in PGSJ algorithm is,

$$\Psi^i_{n_k}(x|\sigma(E_{n_k-1})) = \begin{cases} \dfrac{P}{\mu(\pi_i(D) - I_{n_k})} & x \in \pi_i(D) - I_{n_k} \\ \dfrac{\psi^i(x)}{\mu(I_{n_i})} & x \in I^i_{n_k} \end{cases}$$

where, D indicates the search space,π_i is the projection on ith axis, $I^i_{n_k}$ is the interval in the n_kth scaled search space, lastly ψ is described by,

$$\psi^i(x) = \Sigma_{j=1}^N \gamma^i_j \chi_{I^i_j}(x) \ for \ x \in I^i_{n_k} \ where \ \int_{I^i_{n_k}} \psi^i d\mu = 1 - P$$

Then we can deduce,

$$\begin{aligned} &\inf_{i \geq 1} \Psi^i_{n_k}(x|\sigma(E_{n_k-1})) \geq \\ &\min\{P, \gamma^j_i \ for \ i = 1, \ldots, n, \ j = 1, \ldots, N\} > 0 \end{aligned}$$

Therefore, the assumption presumed in [15] is now satisfied for PGSJ algorithm, hence the convergence results in [15] help to complete the proof. □

Apart from the theoretical convergence of the Algorithm 2.1, in the next section this algorithm is practically evaluated on the solution of constrained optimal control problem.

4. Numerical Simulation

In order to employ the described PGSJ algorithm, on the solution of Problem (1) this problem has first to be converted to an approximate form of the following optimization problem,

$$\min_{\nu \in D} \ f(\nu) \tag{2}$$

where, f is a bounded real valued function and D is a box in a finite Euclidean space. With this end in view, we use Bernstein basis function [7] in the control parameterization frameworks [9] in that the control function can efficiently be parameterized with polynomials of the form,

$$u(t) = \frac{n!}{(t_f - t_0)^n} \Sigma_{i=1}^n \frac{u_i}{i!(n-i)!}(t - t_0)^i (t_f - t)^{n-i}$$

with unknown coefficients that have to correctly be identified. One advantage of this parameterization is the elimination of discretizing time intervals, hence the trouble of dealing with switching times. This method is then illustrated in the following examples.

Example 1. The first problem is a state constrained optimal control problem which is described by,

$$\begin{aligned}\min\ & x_3(t_f))\\ & \dot{x}_1\ (t) = x_2(t)\\ & \dot{x}_2\ (t) = -x_2(t) + u(t)\\ & \dot{x}_3\ (t) = x_1^2(t) + x_2^2(t) + 0.005u^2(t)\\ & \dot{x}_4\ (t) = 1\\ & x_1(t) - 8(x_4 - 0.5)^2 + 0.5 \le 0\\ & 0 \le t \le 1,\ -5 \le u(t) \le 15,\ and\ x(0) = [0, -1, 0, 0]^T.\end{aligned}$$

In order to handle the constraint in the problem we introduce a new state variable,

$$\dot{x}_5\ (t) = \max\ \{0,\ x_1(t) - 8(x_4 - 0.5)^2 + 0.5\}$$

for, $0 \le t \le 1$, and $x_5(0) = 0$, while the objective of the problem is now changed to minimizing $x_3(t_f) + \theta x_5(t_f)$ where θ is a penalty parameter and can be any suitable positive real number.

The Bernstein-based control parameterization (BCP) described above is now used to transform this problem into the form of Problem (2). The resultant problem is then solved using PGSJ method. In this study we set $n = 5$ in BCP, and the parameters in Table 1.1, set as follow, $N = 7, S = 50, A = 0.5, b = 5, and\sigma = 0.9$ We implement this method using C++ programing language while the PGSJ algorithm only allowed for up to 4000 function evaluations, after several running the algorithm, this method averagely acquired the solution of 0.7712. This figure improved to 0.7409 when the allowable number of function evaluations increased up to 8000 which is also obtained using Chebyshev polynomial method reported in [20]. The graph of optimal control as well as related graphs of states variables and constraint is available in Figures, (1), (2), (3).

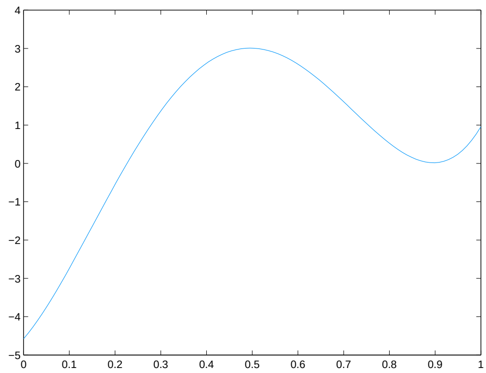

Figure 1: The optimal control for Example 1

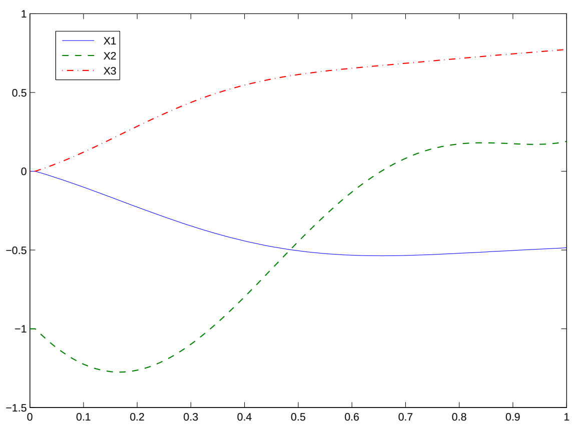

Figure 2: The optimal state x_1 for Example 1

Example 2. The next problem is from [12], known as Rayleigh's problem and described by,

$$\begin{aligned} \min\ & x_3(t_f)) \\ & \dot{x}_1\ (t) = x_2(t) \\ & \dot{x}_2\ (t) = -x_1(t) + x_2(t)(1.4 + 0.14x_2^2(t)) + 4u(t) \end{aligned}$$

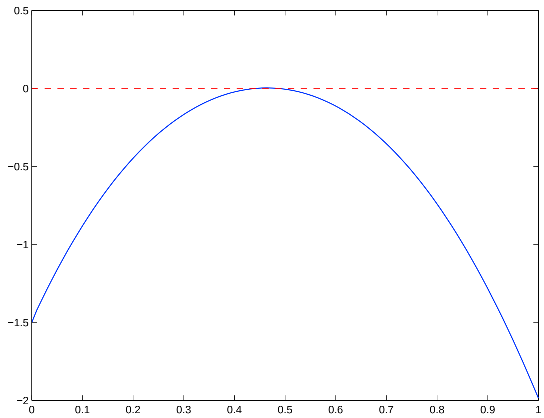

Figure 3: The graph of the constraint for Example 1

$$\dot{x}_3\ (t) = u(t) + x_1^2(t)$$
$$u(t) + \frac{1}{6}x_1(t) \le 0$$
$$0 \le t \le 4.5,\ \ -1 \le u(t) \le 1,\ \ and\ x(0) = [-5, -5, 0]^T.$$

Using the same approach as in previous Example, we introduce the following additional state variable,

$$\dot{x}_4\ (t) = \max\ \{0,\ \ u(t) + \frac{1}{6}x_1(t)\}$$

along with augmenting the problem objective with $\theta x_4(t_f)$ for any large enough positive real number θ as a penalty parameter. With the same parameter setting this problem is solved using the PSGJ method and the solution of 47.6947 is obtained while the tolerable number of the function evaluations is set for up to 4000 The results of simulations are shown in Figures, (4), (5), (6), (7), (8).

5. Conclusion

In this study, we analyzed the convergence of a newly established stochastic method known as Probabilistic Global Search Johor (PGSJ) in probabilistic sense. The method, then practically evaluated for the solution of constrained

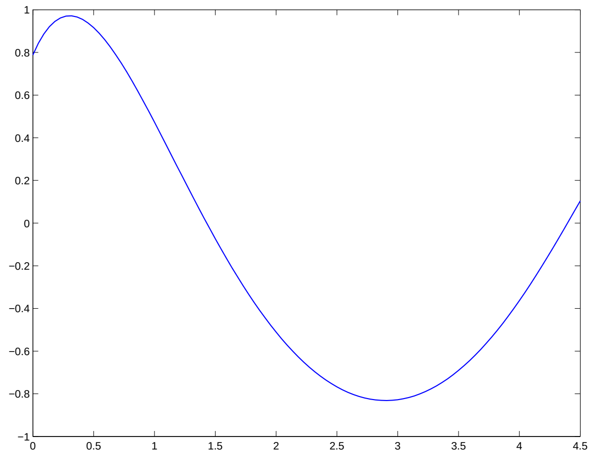

Figure 4: The optimal control for Example 2

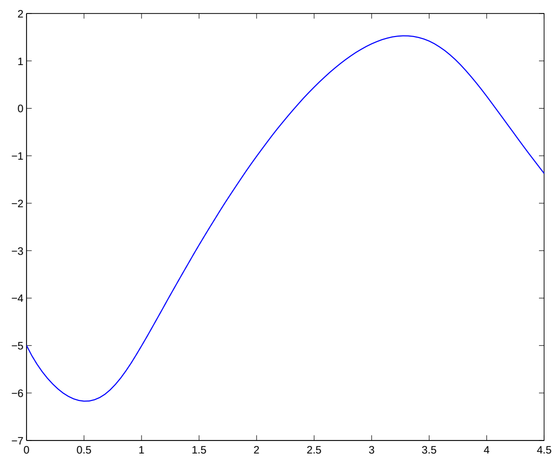

Figure 5: The optimal state x_1 for Example 2

optimal control problem while a Bernstein-based control parameterization used to convert the original problem into nonlinear programing problem. The sim-

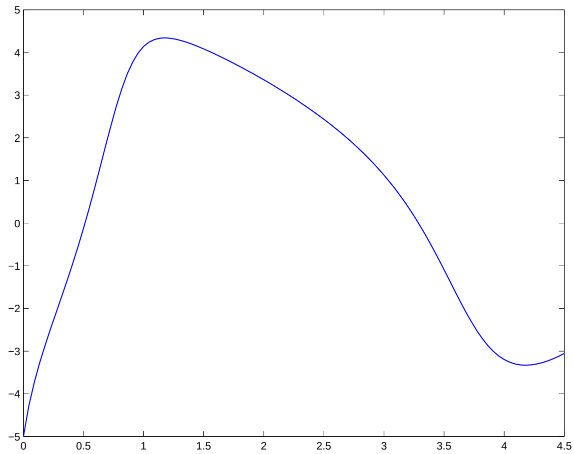

Figure 6: The optimal state x_2 for Example 2

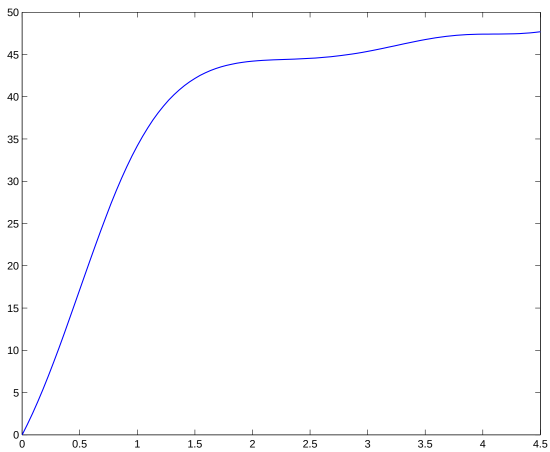

Figure 7: The optimal state x_3 for Example 2

ulations of benchmark problems collected from the literature illustrate the effectiveness of this method.

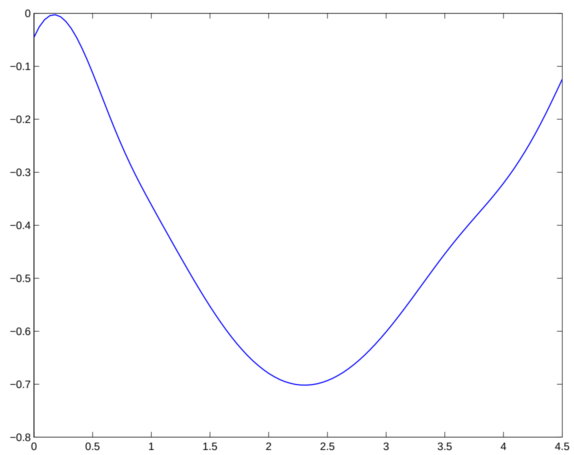

Figure 8: The graph of the constraint for Example 2

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