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DEFECTIVE CURVILINEAR SUBSCHEMES IN PROJECTIVE SPACES

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Abstract: Let $Z \subset \mathbb{P}^r$, $r \geq 3$, be a curvilinear zero-dimensional scheme such that $\deg(Z) \leq 4m+r-5$, Z spans $\mathbb{P}^r$, $\deg(Z \cap M) < 3m$ for each 3-dimensional linear subspace $M \subseteq \mathbb{P}^r$ and $h^1(\mathcal{I}_Z(m)) > 0$. Then either there is a line D with $\deg(D \cap Z) \geq m+2$ or there is a conic T with $\deg(T \cap Z) \geq 2m+2$.

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1. Introduction

A zero-dimensional scheme $Z \subset \mathbb{P}^r$ is said to be *curvilinear* if at each $P \in Z_{red}$ the Zariski tangent space of Z has dimension ≤ 1. A zero-dimensional scheme is contained in a smooth curve (easy). A zero-dimensional scheme is curvilinear if and only if it has finitely many subschemes (for the “ only if ” part use that it is contained in a smooth curve, for the “ if ” part use that a non-curvilinear subscheme has infinitely many subschemes with degree 2). In this note we point out the following partial extension of [1], Theorem 1, to the case of non-reduced, but curvilinear subschemes.

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Theorem 1. *Fix an integer $m \geq 3$. Let $Z \subset \mathbb{P}^r$, $r \geq 3$, be a curvilinear zero-dimensional scheme spanning $\mathbb{P}^r$. If $r = 3$, then assume $\deg(Z) < 3m$. If $r \geq 4$, then assume $\deg(Z) \leq 4m + r - 5$ and $\deg(Z \cap M) < 3m$ for all 3-dimensional linear subspaces $M \subset \mathbb{P}^r$. We have $h^1(\mathcal{I}_Z(m)) > 0$ if and only if either there is a line D with $\deg(D \cap Z) \geq m + 2$ or there is a conic D' with $\deg(D' \cap Z) \geq 2m + 2$.*

Easy examples show that in the statement the conic D' may be a double line (even with all connected components of Z with degree 2).

2. The Proof

Lemma 1. *Let $J \subset \mathbb{P}^r$ be a closed subscheme. Let Z be a curvilinear subscheme. Fix an integer $t > 0$ such that $\mathcal{I}_J(t)$ is spanned. Then there is $A \in |\mathcal{I}_J(t)|$ such that $A \cap Z = J \cap Z$ (as schemes).*

Proof. Since we have $A \cap Z \supseteq J \cap Z$, it is sufficient to prove the reverse inclusion. Hence we may assume $J \cap Z \neq Z$. Set $e := \deg(J \cap Z)$. It is sufficient to prove that A contains no subscheme of Z with degree $\geq e + 1$. Since Z is curvilinear, it has only finitely many subschemes. Let $\mathcal{B}$ denote the set of all subschemes of Z with degree $> e$. Since $\mathcal{I}_J(t)$ is spanned, for any $B \in \mathcal{B}$ the projective space $|\mathcal{I}_{J \cup B}(t)|$ is a proper linear subspace of the projective space $|\mathcal{I}_J(t)|$. Use that $\mathcal{B}$ is finite. □

Easy examples show that Lemma 1 is wrong for every non-curvilinear zero-dimensional scheme (for some t and J) (e.g. $t = 1$, $J = \{P\}$ and Z containing the first infinitesimal neighborhood of $\{P\}$ in a surface contained in $\mathbb{P}^r$).

Lemma 2. *Fix an integer $t > 0$, zero-dimensional schemes $W \subsetneq Z \subset \mathbb{P}^r$ and a linear subspace $V \subseteq |\mathcal{I}_W(t)|$. Assume $k := \dim(V) > 0$. Then there is $A \in |\mathcal{I}_W(t)|$ such that $\deg(Z \cap A) \geq \deg(W) + 1$.*

Proof. Let J be the scheme-theoretic base locus of V. If $Z \cap J \neq W$, then we may take any $A \in V$. Now assume $Z \cap J = W$. Let W' be any zero-dimensional scheme such that $W \subset W' \subseteq Z$ and $\deg(W') = \deg(W) + 1$ (it exists, because $W \subsetneq Z$). We have $\dim(V \cap |\mathcal{I}_{W'}(t)|) = k - 1 \geq 0$. Take any $A \in V \cap |\mathcal{I}_{W'}(t)|$. □

Proof of Theorem 1. Until step (c) we assume $r = 3$. We may assume $h^1(\mathcal{I}_A(m)) = 0$ for all $A \subsetneq Z$. Since Z_{red} is finite and for each $P \in Z_{red}$ the Zariski tangent space of Z at P has dimension ≤ 2, there is a non-empty open subset Ω of $\mathbb{P}^3 \subset Z_{red}$ such that for each $P \in \Omega$ the linear projection $\ell_P : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$ maps Z isomorphically onto $\ell_P(Z)$. Since $\ell_P(Z) \cong Z$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{\ell_P(Z)}(m)) > 0$ for each $P \in \Omega$. Hence by [3], Corollaire 2, either there is a line D such that $\deg(D \cap \ell_P(Z)) \geq m+2$ or there is a conic D' such that $\deg(D' \cap \ell_P(Z)) \geq 2m+2$. If we need to stress the dependence of the plane curves D, D' we write them as $D(P)$ and $D'(P)$. Notice that if $P \in \Omega$, then no line spanned by a degree 2 subscheme of Z contains P. Since Z is curvilinear, it has only finitely many subschemes. Hence there is a non-empty open subset Θ of Ω such that no plane spanned by a subscheme of Z contains P.

(a) In this step we assume the existence of the line D for at least one $P \in \Theta$. Let $H \subset \mathbb{P}^3$ be the unique plane such that $P \in H$ and $\ell_P(H \setminus \{P\}) = D$. Set $Z_1 := \ell_P(Z) \cap D$. Since $\ell_P| : Z \to \ell_P(Z)$ is an isomorphism, there is a unique $Z' \subseteq Z$ such that$\ell_P(Z') = Z_1$. We have $\deg(Z') = \deg(Z_1) \geq m+2$. Since $Z_1 \subset D$, we have $Z' \subset H$. Since $P \in \Theta$ and Z_1 is contained in a line, then Z' is contained in a line. Hence we are in case (a).

(b) By Step (a) we may assume the existence of $D' = D'(P)$ for all $P \in \Theta$.

(b1) First assume that D' is not smooth, i.e. $D' = R + R'$ with R and R' lines (we allow the case $R = R'$). If $R \neq R'$ we assume $\deg(\ell_P(Z) \cap R) \geq \deg(\ell_P(Z) \cap R')$. Since we excluded case (a) and $\deg(\ell_P(Z) \cap D') \geq 2m+2$, we may assume $\deg(\ell_P(Z) \cap R) \leq m+1$. Hence $\deg(\ell_P(Z) \cap R) = m+1$ and $\deg(\ell_P(Z) \cap D')) = 2m+2$. Since $\ell_P|Z$ is an embedding, there is are unique $W_1 \subseteq W \subseteq Z$ such that $\ell_P(W) = \ell_P(Z) \cap D$ and $\ell_P(W_1)) = \ell_P(Z) \cap R$ and $\deg(W_1) = m+1$. Since $P \in \Theta$, as in step (a) we get the existence of a line $L \subset \mathbb{P}^3$ such that $W_1 \subset L$.

Claim 1. *There is a plane $H \subset \mathbb{P}^3$ containing W_1 and with* $\deg(H \cap Z) \geq m+2$.

Proof of Claim 1. Let $M \subset \mathbb{P}^3$ be the plane containing P with $\ell_P(M \setminus \{P\}) = R$. We have $W_1 \subset M$. If $Z \cap M \supsetneq W_1$, then we may take $H := M$. Assume $Z \cap M = W_1$ as schemes. If either $R' \neq R$ or $Z_{red} \neq L$, then we may take as H a suitable plane containing L. Hence to prove Claim 1 we may assume $D' = 2R$ and $Z_{red} = L$. Since $\deg(Z) \geq 2m+2 > \deg(W_1)$, there is $O \in Z_{red}$ such that the connected component Z_O of Z with O as its reduction is not contained in L. Since $h^0(\mathcal{I}_L(1)) = 2$, Remark 1 gives the existence of a plane $H \supset L$ such that $\deg(Z \cap H) \supsetneq W_1$, proving Claim 1.

Take H as in Claim 1. Notice that $L \subset H$. Since $\deg(\mathrm{Res}_H(Z)) \le 3m - 1 - m - 2 = 2(m-2) - 1$, either $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) = 0$ or there is a line L' such that $\deg(L' \cap \mathrm{Res}_H(Z)) \ge m+1$. First assume $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) = 0$. From the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(m-1) \to \mathcal{I}_Z(m) \to \mathcal{I}_{H,H\cap Z}(m) \to 0$$

we get $h^1(H, \mathcal{I}_{H\cap Z,H}(m)) > 0$. Hence we may apply [3], Remarques at page 116) to get that we are either in case (a) or in case (b) with a line or a conic containing in H. Hence we may assume $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) = 0$. Since $\deg(\mathrm{Res}_H(Z)) \le 3m - 1 - m - 2 \le 2(m-2) - 1$, there is a line $J \subset \mathbb{P}^3$ such that $\deg(J \cap \mathrm{Res}_H(Z)) \ge m+1$ (we allow the case $J = L$).

(b1.1) Assume for the moment $J \ne L$ and $J \cap L \ne \emptyset$. Since $\mathrm{Res}_H(W_1)) = \emptyset$, we have $\deg(Z \cap (J \cup L)) \ge 2m+2$. Hence we are in case (b).

(b1.2) Assume $J \cap L = \emptyset$. Since $\mathcal{I}_{J\cup L}(2)$ is spanned, there is $Q \in |\mathcal{I}_{J\cup L}(2)|$ such that $Q \cap Z = (J \cup L) \cap Z$ (as schemes). Since $\deg(J \cap Z) = \deg(L \cap Z) = m+1$, we have $h^1(Q, \mathcal{I}_{Z\cap Q,Q}(m)) = 0$. Since $\deg(\mathrm{Res}_Q(Z)) \le 3m-1-2m-2 \le m-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(m-2)) = 0$. The exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_Q(Z)}(m-2) \to \mathcal{I}_Z(m) \to \mathcal{I}_{Z\cap Q,Q}(m) \to 0$$

gives $h^1(\mathcal{I}_Z(m)) = 0$.

(b1.3) Assume $J = L$. We get $\deg(2M \cap Z) \ge 2m+2$. Hence $\deg(\mathrm{Res}_{2M}(Z)) \le 3m - 1 - 2m - 2$. Therefore $h^1(\mathcal{I}_{\mathrm{Res}_{2M}(Z)}(m-2)) = 0$. Since $h^1(\mathcal{I}_A(m)) = 0$ for all $A \subsetneq Z$, we get $Z \subset 2M$ and in particular $D' = 2R$. We also get $\deg(Z) = 2m+2$ (since we are not in case (a)). Let $H \supset L$ be a plane such that $\deg(Z \cap H) \ge m+2$ (Lemma 2). Obviously $H \supset L$. If $Z \subset H$, then Z is contained in the double line of H with support on L. Now assume $H \cap Z \ne Z$ and hence $h^1(\mathcal{I}_{Z\cap H}(m)) = 0$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) > 0$ (see [1], Remark 1). Since $\deg(\mathrm{Res}_H(Z)) \le 2m+2-m-2$, we have $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) = 0$, a contradiction.

(b2) Assume that D' is a smooth conic. Let $D'(P) \subset \mathbb{P}^3$ the quadric cone with vertex in P and such that $\ell_P(D'(P) \setminus \{P\}) = D'$. Since $\ell_P|Z$ is an embedding, we have $\deg(Z \cap D'(P)) \ge 2m+2$. Hence $\deg(\mathrm{Res}_{D'(P)}(Z)) \le 3m - 1 - 2m - 2 \le m-1$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{D'(P)}(Z)}(m-2)) = 0$. From the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{D'(P)}(Z)}(m-2) \to \mathcal{I}_Z(m) \to \mathcal{I}_{Z\cap D'(P),D'(P)}(m) \to 0$$

we get $h^1(D'(P), \mathcal{I}_{Z\cap D'(P),D'(P)}(m)) > 0$. Hence $h^1(\mathcal{I}_{Z\cap D'(P),D'(P)}(m)) > 0$. Since $h^1(\mathcal{I}_A(m)) = 0$ for all $A \subsetneq Z$, we get $Z \subset D'(P)$. Take another general $P_1 \in \Theta \setminus \Theta \cap D'(P)$. We get $Z \subset D'(P) \cap D'(P_1)$. Since $P_1 \notin D'(P)$, the scheme $D'(P) \cap D'(P_1)$ contains no line and hence either a multiplicity two structure on a smooth conic or irreducible. First assume that $D'(P)\cap D'(P_1)$ is a multiplicity two structure on a smooth conic $C \subset D'(P)$. Notice that $P \notin C$, because C is smooth. Since $\ell_P|Z$ is an embedding, we get $Z \subset C$. Hence we are in case (b). Now assume that $T := D'(P) \cap D'(P_1)$ is irreducible. In this case T is an arithmetically Cohen-Macaulay curve with arithmetic genus 1. Hence $h^1(\mathcal{I}_Z(m)) = h^1(T, \mathcal{I}_{Z,T}(m))$. We have $h^1(T, \mathcal{I}_{Z,T}(m)) = 0$, because $\deg(\mathcal{O}_T(m)) = 4m > \deg(Z)$ and $p_a(T) = 1$.

(c) From now on we assume $r > 3$ and use induction on r. This is the part of [1] which works verbatim (use deg instead of $\sharp$; we use again Remark 1 and hence that Z is curvilinear only in $\mathbb{P}^4$, with $\mathcal{I}_{L_1\cup L_2\cup L_3}(2)$ (see [1], Lemma 21)). If $r \geq 5$ we do not use that Z is curvilinear in the step $r-1 \Longrightarrow r$. We use several times Lemma 2 in which Z is not assumed to be curvilinear. □

Question 1. Is Theorem 1 true for non-curvilinear zero-dimensional schemes?

It is easy to prove Question 1 assuming that it is true if $r \leq 4$.

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