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A FIXED POINT APPROACH TO THE GENERALIZED HYERS-ULAM STABILITY OF A MIXED TYPE FUNCTIONAL EQUATION

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Abstract: In this paper, we investigate the stability of a functional equation

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) = 3f(x)+f(-x)+3f(y)+f(-y)+3f(z)+f(-z)$$

by using the fixed point theory in the sense of Cădariu and Radu.

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1. Introduction

In 1940, Ulam [12] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: Let G_1 be a group and let G_2 be a metric group with the

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metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The Ulam's problem for the case of approximately additive functions was solved by Hyers [6] under the assumption that G_1 and G_2 are Banach spaces. Indeed, Hyers proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to satisfy the Hyers-Ulam stability.

Rassias [11] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\big(\|x\|^p + \|y\|^p\big)$$

and derived Hyers' theorem for the stability of the additive mapping as a special case. Thus in [11], a proof of the generalized Hyers-Ulam stability for the linear mapping between Banach spaces was obtained. It should be remarked that a paper of Aoki [1] was published concerning the Hyers-Ulam stability of the additive mapping earlier than the paper of Rassias [11].

The stability concept that was introduced by Rassias' theorem provided some influence to a number of mathematicians to develop the notion of what is known today with the term generalized Hyers-Ulam stability of the linear mappings. Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see, for example, [5, 7, 9] and the references therein).

Almost all subsequent proofs, in this very active area, have used the Hyers' method of [6]. Namely, starting from the given mapping f, the solution F of a functional equation is explicitly constructed by

$$F(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \quad \text{or} \quad F(x) = \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big).$$

This method of Hyers is called a direct method.

In 2003, Cădariu and Radu [2] observed that the existence of the solution F for a functional equation and the estimation of the difference from the given mapping f can be obtained by using the fixed point theory alternative. This method is called the fixed point method. In 2004, they [4] applied this method to prove the stability theorem of the Cauchy functional equation

$$f(x+y) = f(x) + f(y). \tag{1}$$

Moreover, they [3] proved the stability of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2}$$

by using the fixed point method.

Notice that if each solution of Eq. (1) is called an additive mapping, while every solution of Eq. (2) is called a quadratic mapping.

Now we consider the following functional equation

$$\begin{aligned} &f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) \\ &\quad =3f(x)+f(-x)+3f(y)+f(-y)+3f(z)+f(-z), \end{aligned} \tag{3}$$

which is called the mixed type functional equation. The mapping $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=ax^2+bx$ is a solution of this functional equation, where a,b are real constants. Any solution of Eq. (3) is called a quadratic-additive mapping.

In 1998, Jung [8] proved the stability of Eq. (3) by decomposing f into the odd and even parts. In his proof, it was necessary to let an additive mapping A and a quadratic mapping Q approximate the odd and even parts of f, respectively, and combine A and Q to prove the existence of a quadratic-additive mapping F which is close to the mapping f.

In this paper, we will prove the stability of the quadratic-additive functional equation (3) by making use of the fixed point method. Indeed, we will approximate the given mapping f by a solution F of Eq. (3) without decomposing f into its odd and even parts, while in the previous papers [8] the mapping f was decomposed into the odd and even parts and they were separately approximated by the corresponding parts of a solution F of Eq. (3), respectively.

More precisely, we will introduce in this paper a strictly contractive mapping with the Liptshitz constant $0<L<1$. Using the fixed point method, we can prove that the contractive mapping has a unique fixed point, namely F. In fact, F is an exact solution of Eq. (3). In Section 2, we will apply the fixed point method and prove the stability of Eq. (3) (see Theorems 2.3 and 2.4). In Section 3, using the results from Section 2, we will also prove the stabilities of Eqs. (1) and (2).

2. Main Results

Let X be a nonempty set. A function $d:X^2\to[0,\infty]$ is called a *generalized metric* on X if and only if d satisfies

(M_1) $d(x,y)=0$ if and only if $x=y$;

(M_2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(M_3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We now introduce one of the fundamental results of the fixed point theory. For the proof, we refer to [10].

Theorem 2.1. *Let (X,d) be a complete generalized metric space. Assume that $\Lambda : X\to X$ is a strict contraction with the Lipschitz constant $0<L<1$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}x,\Lambda^{n_0}x)<\infty$ for some $x\in X$, then the following statements are true:*

(i) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;

(ii) x^* is the unique fixed point of Λ in $X^*=\big\{y\in X \mid d(\Lambda^{n_0}x,y)<\infty\big\}$;

(iii) If $y\in X^*$, then
$$d(y,x^*)\le \frac{1}{1-L}d(\Lambda y,y).$$

Throughout this paper, let V be a (real or complex) linear space and Y a Banach space. For a given mapping $f:V\to Y$, we use the following abbreviation

$$\begin{aligned}&Df(x,y,z)\\&\quad:=f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)\\&\qquad-3f(x)-f(-x)-3f(y)-f(-y)-3f(z)-f(-z)\end{aligned}$$

for all $x,y,z\in V$. If f is a solution of the functional equation $Df(x,y,z)=0$ for all $x,y,z\in V$, then f is called a quadratic-additive mapping.

We first prove the following lemma.

Lemma 2.2. *If a mapping $f:V\to Y$ satisfies $Df(x,y,z)=0$ for all $x,y,z\in V\backslash\{0\}$ and $f(0)=0$, then f is a quadratic-additive mapping.*

Proof. By the hypothesis, we get

$$\begin{aligned}&f(2x)-3f(x)-f(-x)\\&\quad=\frac{11}{112}\big(Df(4x,3x,x)-Df(4x,2x,2x)-Df(2x,2x,2x)\\&\qquad\qquad+2Df(2x,x,x)+3Df(x,x,x)+Df(-x,-x,-x)\big)\end{aligned}$$

$$
\begin{aligned}
&- \frac{3}{112}\big(Df(-4x,-3x,-x) - Df(-4x,-2x,-2x) \\
&\qquad - Df(-2x,-2x,-2x) + 2Df(-2x,-x,-x) \\
&\qquad + 3Df(-x,-x,-x) + Df(x,x,x)\big) \\
=&0
\end{aligned}
$$

for all $x \in V\backslash\{0\}$. From this, we have

$$
\begin{aligned}
Df(x,y,0) =&Df\Big(x, \frac{y}{2}, \frac{y}{2}\Big) + Df\Big(y, \frac{x}{2}, \frac{x}{2}\Big) - 2f(x) + 6f\Big(\frac{x}{2}\Big) \\
&+ 2f\Big(-\frac{x}{2}\Big) - 2f(y) + 6f\Big(\frac{y}{2}\Big) + 2f\Big(-\frac{y}{2}\Big) \\
=&0
\end{aligned}
$$

for all $x, y \in V\backslash\{0\}$. By the symmetry, $Df(0,y,z) = Df(x,0,z) = 0$ holds for all $x, y, z \in V\backslash\{0\}$. It is also easy to show that $Df(x,0,0) = 0$, $Df(0,y,0) = 0$, $Df(0,0,z) = 0$, and $Df(0,0,0) = 0$ for all $x, y, z \in V\backslash\{0\}$. $\square$

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (3) by making use of the fixed point method.

Theorem 2.3. *Let $\varphi : (V\backslash\{0\})^3 \to [0,\infty)$ be a mapping for which there exists a constant $0 < L < 1$ such that*

$$\varphi(3x, 3y, 3z) \le 3L\varphi(x,y,z) \tag{4}$$

for all $x, y, z \in V\backslash\{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and

$$\|Df(x,y,z)\| \le \varphi(x,y,z) \tag{5}$$

for all $x, y, z \in V\backslash\{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{2\big(\varphi(x,x,x) + \varphi(-x,-x,-x)\big)}{9(1-L)} \tag{6}$$

for all $x \in V\backslash\{0\}$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{f(3^n x) + f(-3^n x)}{2\cdot 9^n} + \frac{f(3^n x) - f(-3^n x)}{2\cdot 3^n}\right) \tag{7}$$

for all $x \in V$. Moreover, if $0 < L < 1/3$ and φ is continuous, then f is itself a quadratic-additive mapping.

Proof. It follows from (4) that

$$\lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y, 3^n z)}{3^n} = \lim_{n\to\infty} L^n \varphi(x,y,z) = 0$$

for all $x, y, z \in V\backslash\{0\}$. Let S be the set of all mappings $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$\begin{aligned} &d(g,h) \\ &\quad = \inf\big\{K \geq 0 \ : \ \|g(x) - h(x)\| \leq K\big(\varphi(x,x,x) + \varphi(-x,-x,-x)\big) \\ &\qquad\qquad \text{for all } x \in V\backslash\{0\}\big\}. \end{aligned}$$

It is easy to show that (S, d) is a generalized complete metric space.

Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := \frac{g(3x) - g(-3x)}{6} + \frac{g(3x) + g(-3x)}{18}$$

for all $x \in V$. Notice that

$$J^n g(x) = \frac{g(3^n x) - g(-3^n x)}{2 \cdot 3^n} + \frac{g(3^n x) + g(-3^n x)}{2 \cdot 9^n}$$

for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g,h) \leq K$. Then it follows from the definition of d and (4) that

$$\begin{aligned} \|Jg(x) - Jh(x)\| &\leq \frac{2}{9}\|g(3x) - h(3x)\| + \frac{1}{9}\|g(-3x) - h(-3x)\| \\ &\leq \frac{K}{3}\big(\varphi(3x,3x,3x) + \varphi(-3x,-3x,-3x)\big) \\ &\leq LK\big(\varphi(x,x,x) + \varphi(-x,-x,-x)\big) \end{aligned}$$

for all $x \in V\backslash\{0\}$, which implies that

$$d(Jg, Jh) \leq L d(g,h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant $0 < L < 1$.

Moreover, by (5), we see that

$$\begin{aligned} \|f(x) - Jf(x)\| &= \left\| \frac{-2Df(x,x,x) + Df(-x,-x,-x)}{9} \right\| \\ &\leq \frac{2\big(\varphi(x,x,x) + \varphi(-x,-x,-x)\big)}{9} \end{aligned}$$

for all $x \in V\backslash\{0\}$. It follows from the definition of d that $d(f, Jf) \leq 2/9 < \infty$. Therefore, according to Theorem 2.1 (i) and (ii), the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (7) for all $x \in V$. Moreover, in view of Theorem 2.1 (iii), we have

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{2}{9(1-L)}$$

which together with the definition of d implies the validity of (6).

By the definition of F, together with (4), (5), and (7), we have

$$\begin{aligned}
&\|DF(x,y,z)\| \\
&= \lim_{n\to\infty} \left\| \frac{Df(3^n x, 3^n y, 3^n z) - Df(-3^n x, -3^n y, -3^n z)}{2 \cdot 3^n} \right. \\
&\qquad \left. + \frac{Df(3^n x, 3^n y, 3^n z) + Df(-3^n x, -3^n y, -3^n z)}{2 \cdot 9^n} \right\| \\
&\leq \lim_{n\to\infty} \frac{3^n + 1}{2 \cdot 9^n} \big(\varphi(3^n x, 3^n y, 3^n z) + \varphi(-3^n x, -3^n y, -3^n z) \big) \\
&= 0
\end{aligned}$$

for all $x, y, z \in V\backslash\{0\}$. By Lemma 2.2, we get

$$DF(x, y, z) = 0 \tag{8}$$

for all $x, y, z \in V$.

If $0 < L < 1/3$ and φ is continuous, then it follows from (4) that

$$\begin{aligned}
&\lim_{n\to\infty} \varphi\big((a_1 \cdot 3^n + a_2)x, (b_1 \cdot 3^n + b_2)y, (c_1 \cdot 3^n + c_2)z\big) \\
&\leq \lim_{n\to\infty} (3L)^n \varphi\left(\left(a_1 + \frac{a_2}{3^n}\right)x, \left(b_1 + \frac{b_2}{3^n}\right)y, \left(c_1 + \frac{c_2}{3^n}\right)z \right) \\
&= 0
\end{aligned}$$

for all $x, y, z \in V\backslash\{0\}$ and for all fixed integers $a_1, a_2, b_1, b_2, c_1, c_2$ with $a_1, b_1, c_1 \neq 0$. Therefore, by (4), (5), (6), and by (8), we obtain

$$\begin{aligned}
&\|f(x) - F(x)\| \\
&\leq \lim_{n\to\infty} \Big(\|Df((2 \cdot 3^n - 1)x, 3^n x, 3^n x) - DF((2 \cdot 3^n - 1)x, 3^n x, 3^n x)\| \\
&\qquad + \|(F - f)((4 \cdot 3^n - 1)x)\| + \|(f - F)((2 \cdot 3^n - 1)x)\| \\
&\qquad + \|(f - F)(-(2 \cdot 3^n - 1)x)\| + 6\|(f - F)(3^n x)\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\|(f - F)(-3^n x)\|\Big) \\
\le& \lim_{n\to\infty} \Big(\varphi((2\cdot 3^n - 1)x, 3^n x, 3^n x) + \frac{2}{9(1-L)}\big(\psi((4\cdot 3^n - 1)x) \\
&\quad + 2\psi((2\cdot 3^n - 1)x) + 8\psi(3^n x)\big)\Big) \\
=& 0
\end{aligned}
$$

for all $x \in V\backslash\{0\}$, where we define $\psi(x) = \varphi(x, x, x) + \varphi(-x, -x, -x)$. Since $f(0) = 0 = F(0)$, we have shown that $f(x) = F(x)$ for all $x \in V$. This completes the proof of our theorem. □

In the following theorem, we prove another version of Theorem 2.3.

Theorem 2.4. *Let $\varphi : (V\backslash\{0\})^3 \to [0, \infty)$ be a mapping. Assume that a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequality* (5) *for all $x, y, z \in V\backslash\{0\}$. If there exists a constant $0 < L < 1$ with*

$$
L\varphi(3x, 3y, 3z) \ge 9\varphi(x, y, z) \tag{9}
$$

for all $x, y, z \in V\backslash\{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that

$$
\|f(x) - F(x)\| \le \frac{L}{9(1-L)}\big(\varphi(x, x, x) + \varphi(-x, -x, -x)\big) \tag{10}
$$

for all $x \in V\backslash\{0\}$. In particular, F is represented by

$$
F(x) = \lim_{n\to\infty}\Big(\frac{3^n}{2}\Big(f\Big(\frac{x}{3^n}\Big) - f\Big(-\frac{x}{3^n}\Big)\Big) + \frac{9^n}{2}\Big(f\Big(\frac{x}{3^n}\Big) + f\Big(-\frac{x}{3^n}\Big)\Big)\Big) \tag{11}
$$

for all $x \in V$.

Proof. Let (S, d) be a generalized complete metric space defined in the proof of Theorem 2.3. We now define a mapping $J : S \to S$ by

$$
Jg(x) := \frac{3}{2}\Big(g\Big(\frac{x}{3}\Big) - g\Big(-\frac{x}{3}\Big)\Big) + \frac{9}{2}\Big(g\Big(\frac{x}{3}\Big) + g\Big(-\frac{x}{3}\Big)\Big)
$$

for all $g \in S$ and $x \in V$. Notice that

$$
J^n g(x) = \frac{3^n}{2}\Big(g\Big(\frac{x}{3^n}\Big) - g\Big(-\frac{x}{3^n}\Big)\Big) + \frac{9^n}{2}\Big(g\Big(\frac{x}{3^n}\Big) + g\Big(-\frac{x}{3^n}\Big)\Big)
$$

for all $n \in \mathbb{N}$ and $x \in V$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d and (9), we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| =& 6\Big\|g\Big(\frac{x}{3}\Big) - h\Big(\frac{x}{3}\Big)\Big\| + 3\Big\|g\Big(-\frac{x}{3}\Big) - h\Big(-\frac{x}{3}\Big)\Big\| \\ \leq& 9K\bigg(\varphi\Big(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\Big) + \varphi\Big(-\frac{x}{3}, -\frac{x}{3}, -\frac{x}{3}\Big)\bigg) \\ \leq& LK\big(\varphi(x, x, x) + \varphi(-x, -x, -x)\big)\end{aligned}$$

for all $x \in V \backslash \{0\}$. Thus, we get

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant $0 < L < 1$.

Moreover, we see that

$$\begin{aligned}\|f(x) - Jf(x)\| =& \Big\|Df\Big(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\Big)\Big\| \leq \varphi\Big(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\Big) \\ \leq& \frac{L}{9}\big(\varphi(x, x, x) + \varphi(-x, -x, -x)\big)\end{aligned}$$

for all $x \in V \backslash \{0\}$, which implies that $d(f, Jf) \leq L/9 < \infty$. Therefore, according to Theorem 2.1 (i) and (ii), the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S \,:\, d(f, g) < \infty\}$, where F is given by (11). In view of Theorem 2.1 (iii), we have

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{L}{9(1-L)},$$

which together with the definition of d implies the validity of the inequality (10).

By the definition of F, (5), (9), and by (10), it holds that

$$\begin{aligned}&\|DF(x, y, z)\| \\ &= \lim_{n\to\infty} \bigg\|\frac{3^n}{2}\bigg(Df\Big(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\Big) - Df\Big(-\frac{x}{3^n}, -\frac{y}{3^n}, -\frac{z}{3^n}\Big)\bigg) \\ &\qquad + \frac{9^n}{2}\bigg(Df\Big(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\Big) + Df\Big(-\frac{x}{3^n}, -\frac{y}{3^n}, -\frac{z}{3^n}\Big)\bigg)\bigg\| \\ &\leq \lim_{n\to\infty} \frac{3^n + 9^n}{2}\bigg(\varphi\Big(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\Big) + \varphi\Big(-\frac{x}{3^n}, -\frac{y}{3^n}, -\frac{z}{3^n}\Big)\bigg) \\ &= 0\end{aligned}$$

for all $x, y, z \in V\backslash\{0\}$. By Lemma 2.2, F is quadratic-additive. □

Remark 2.5. If φ satisfies the additional condition $\varphi(x, y, z) = \varphi(-x, -y, -z)$ for all $x, y, z \in V\backslash\{0\}$ in Theorems 2.3 and 2.4, then the inequalities (6) and (10) can be replaced by

$$\|f(x) - F(x)\| \leq \frac{\varphi(x, x, x)}{3(1 - L)}$$

resp.

$$\|f(x) - F(x)\| \leq \frac{L}{9(1 - L)}\varphi(x, x, x)$$

for all $x \in V\backslash\{0\}$.

3. Applications

For a given mapping $f : V \to Y$, we use the following abbreviations

$$\begin{aligned} Af(x, y) &:= f(x + y) - f(x) - f(y), \\ Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y) \end{aligned}$$

for all $x, y \in V$.

In the following corollaries, by using Theorems 2.3 and 2.4, we will investigate the stability problems of the additive functional equation $Af(x, y) = 0$ and the quadratic functional equation $Qf(x, y) = 0$.

Corollary 3.1. *Let $f_i : V \to Y$ be a mapping for which there exists a mapping $\phi_i : V^2 \to [0, \infty)$ such that $f_i(0) = 0$ and*

$$\|Af_i(x, y)\| \leq \phi_i(x, y) \tag{12}$$

for all $x, y \in V$ and $i = 1, 2$. If there exists a constant $0 < L < 1$ such that

$$\phi_1(3x, 3y) \leq 3L\phi_1(x, y), \tag{13}$$

$$9\phi_2(x, y) \leq L\phi_2(3x, 3y) \tag{14}$$

for all $x, y \in V$, then there exist (uniquely determined) additive mappings $F_1, F_2 : V \to Y$ such that

$$\|f_1(x) - F_1(x)\| \leq \frac{2\Phi_1(x)}{9(1 - L)}, \tag{15}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L\Phi_2(x)}{9(1-L)} \tag{16}$$

for all $x \in V$*, where* $\Phi_i : V \to Y$ *is given by*

$$\begin{aligned}\Phi_i(x) =&\phi_i(2x, x) + \phi_i(2x, -x) + 2\phi_i(0, x) + 2\phi_i(x, x) + 2\phi_i(x, -x)\\ &+ 2\phi_i(-x, x) + \phi_i(-2x, -x) + \phi_i(-2x, x) + 2\phi_i(0, -x)\\ &+ 2\phi_i(-x, -x)\end{aligned}$$

for all $x \in V$*. In particular, the mappings* F_1, F_2 *can be expressed by*

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(3^n x)}{3^n}, \tag{17}$$

$$F_2(x) = \lim_{n\to\infty} 3^n f_2\Big(\frac{x}{3^n}\Big) \tag{18}$$

for all $x \in V$*. Moreover, if* $0 < L < 1/3$ *and* ϕ_1 *is continuous, then* f_1 *is itself an additive mapping.*

Proof. Notice that

$$\begin{aligned}Df_i(x, y, z) =&Af_i(x + y, z) + Af_i(x + y, -z) + Af_i(x - y, z)\\ &+ Af_i(-x + y, z) + 2Af_i(x, y) + Af_i(x, -y)\\ &+ Af_i(-x, y)\end{aligned}$$

for all $x, y, z \in V$ and $i = 1, 2$. If we put

$$\begin{aligned}\varphi_i(x, y, z) :=&\phi_i(x + y, z) + \phi_i(x + y, -z) + \phi_i(x - y, z)\\ &+ \phi_i(-x + y, z) + 2\phi_i(x, y) + \phi_i(x, -y) + \phi_i(-x, y)\end{aligned}$$

for all $x, y, z \in V$ and $i = 1, 2$, then φ_1 and φ_2 satisfy (4) and (9), respectively.

Therefore, according to Theorem 2.3, there exists a unique mapping $F_1 : V \to Y$ satisfying (15), where F_1 can be expressed as (7).

Observe by (12) and (13) that

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f_1(3^n x) + f_1(-3^n x)}{2\cdot 3^n}\right\| &= \lim_{n\to\infty}\left\|\frac{f_1(3^n x) + f_1(-3^n x) - f_1(0)}{2\cdot 3^n}\right\|\\ &= \lim_{n\to\infty}\frac{1}{2\cdot 3^n}\|Af_1(3^n x, -3^n x)\|\\ &\le \lim_{n\to\infty}\frac{1}{2\cdot 3^n}\phi_1(3^n x, -3^n x)\\ &\le \lim_{n\to\infty}\frac{L^n}{2}\phi_1(x, -x)\end{aligned}$$

$$= 0$$

and

$$\lim_{n\to\infty} \left\| \frac{f_1(3^n x) + f_1(-3^n x)}{2 \cdot 9^n} \right\| \le \lim_{n\to\infty} \frac{3^n L^n}{2 \cdot 9^n} \phi_1(x, -x) = 0$$

for all $x \in V$. From these inequalities and (7), we get (17).

Moreover, we have

$$\left\| \frac{Af_1(3^n x, 3^n y)}{3^n} \right\| \le \frac{\phi_1(3^n x, 3^n y)}{3^n} \le L^n \phi_1(x, y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$AF_1(x, y) = 0$$

for all $x, y \in V$.

As in the proof of Theorem 2.3, if $0 < L < 1/3$ and ϕ_1 is continuous, then f_1 is itself an additive mapping.

On the other hand, according to Theorem 2.4, there exists a (uniquely determined) mapping $F_2 : V \to Y$ satisfying (16), which can be represented by (11). Observe that

$$\begin{aligned} \lim_{n\to\infty} \frac{9^n}{2} \left\| f_2\Big(\frac{x}{3^n}\Big) + f_2\Big(-\frac{x}{3^n}\Big) \right\| &= \lim_{n\to\infty} \frac{9^n}{2} \left\| Af_2\Big(\frac{x}{3^n}, -\frac{x}{3^n}\Big) \right\| \\ &\le \lim_{n\to\infty} \frac{9^n}{2} \phi_2\Big(\frac{x}{3^n}, -\frac{x}{3^n}\Big) \\ &\le \lim_{n\to\infty} \frac{L^n}{2} \phi_2(x, -x) \\ &= 0 \end{aligned}$$

and

$$\lim_{n\to\infty} \frac{3^n}{2} \left\| f_2\Big(\frac{x}{3^n}\Big) + f_2\Big(-\frac{x}{3^n}\Big) \right\| \le \lim_{n\to\infty} \frac{L^n}{2 \cdot 3^n} \phi_2(x, -x) = 0$$

for all $x \in V$. From these inequalities and (11), we get (18). Moreover, we have

$$\left\| 3^n Af_2\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) \right\| \le 3^n \phi_2\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) \le \frac{L^n}{3^n} \phi_2(x, y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$AF_2(x, y) = 0$$

for all $x, y \in V$. $\square$

Corollary 3.2. *Let $\phi_i : V^2 \to [0, \infty)$, $i = 1, 2$, be given mappings. Assume that $f_i : V \to Y$ satisfies*

$$\|Qf_i(x, y)\| \le \phi_i(x, y) \tag{19}$$

for all $x, y \in V$ and $i = 1, 2$. If there exists a constant $0 < L < 1$ such that ϕ_1 and ϕ_2 satisfy (13) and (14) for all $x, y \in V$, respectively, then there exist unique quadratic mappings $F_1, F_2 : V \to Y$ such that

$$\|f_1(x) - F_1(x)\| \le \frac{2\Phi_1(x)}{9(1 - L)}, \tag{20}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L\Phi_2(x)}{9(1 - L)} \tag{21}$$

for all $x \in V$, where $\Phi_i : V \to Y$ is given by

$$\begin{aligned}\Phi_i(x) =&\phi_i(x, 0) + \phi_i(2x, x) + 2\phi_i(x, x) + 3\phi_i(0, x) + \phi_i(-x, 0)\\ &+ \phi_i(-2x, -x) + 2\phi_i(-x, -x) + 3\phi_i(0, -x).\end{aligned}$$

In particular, F_1 and F_2 are represented by

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(3^n x)}{9^n}, \tag{22}$$

$$F_2(x) = \lim_{n\to\infty} 9^n f_2\Big(\frac{x}{3^n}\Big) \tag{23}$$

for all $x \in V$. Moreover, if $0 < L < 1/3$ and ϕ_1 is continuous, then f_1 is itself a quadratic mapping.

Proof. Notice that

$$\begin{aligned}Df_i(x, y, z) =&Qf_i(z, x - y) + Qf_i(x + y, z) + 2Qf_i(x, y)\\ &- Qf_i(0, x) - Qf_i(0, z) - Qf_i(0, y)\end{aligned}$$

for all $x, y, z \in V$ and $i = 1, 2$. Put

$$\begin{aligned}\varphi_i(x, y, z) :=&\phi_i(z, x - y) + \phi_i(x + y, z) + 2\phi_i(x, y)\\ &+ \phi_i(0, x) + \phi_i(0, z) + \phi_i(0, y)\end{aligned}$$

for all $x, y, z \in V$ and $i = 1, 2$. Then, φ_1 and φ_2 satisfy (4) and (9), respectively. Moreover, we have

$$\|Df_i(x, y, z)\| \le \varphi_i(x, y, z)$$

for all $x, y, z \in V$ and $i = 1, 2$.

According to Theorem 2.3, there exists a unique mapping $F_1 : V \to Y$ satisfying (20), where F_1 is given by (7). It follows from (13) and (19) that

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f_1(3^n x) - f_1(-3^n x)}{2\cdot 3^n}\right\| &= \lim_{n\to\infty}\frac{1}{2\cdot 3^n}\|Qf_1(0, 3^n x)\| \\ &\le \lim_{n\to\infty}\frac{1}{2\cdot 3^n}\phi_1(0, 3^n x) \\ &\le \lim_{n\to\infty}\frac{L^n}{2}\phi_1(0, x) \\ &= 0\end{aligned}$$

and

$$\lim_{n\to\infty}\left\|\frac{f_1(3^n x) - f_1(-3^n x)}{2\cdot 9^n}\right\| \le \lim_{n\to\infty}\frac{L^n}{2\cdot 3^n}\phi_1(0, x) = 0$$

for all $x \in V$. From these inequalities and (7), we get (22) for all $x \in V$.

Moreover, we have

$$\left\|\frac{Qf_1(3^n x, 3^n y)}{9^n}\right\| \le \frac{\phi_1(3^n x, 3^n y)}{9^n} \le \frac{L^n}{3^n}\phi_1(x, y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$QF_1(x, y) = 0$$

for all $x, y \in V$. Similarly as in the proof of Theorem 2.3, we can prove that if $0 < L < 1/3$ and ϕ_1 is continuous, then f_1 is itself a quadratic mapping.

On the other hand, due to Theorem 2.4, there exists a unique mapping $F_2 : V \to Y$ satisfying (21), where F_2 is given by (11). By (9) and (19), we get

$$\begin{aligned}9^n\left\|f_2\Big(\frac{x}{3^n}\Big) - f_2\Big(-\frac{x}{3^n}\Big)\right\| &= 9^n\left\|Qf_2\Big(0, \frac{x}{3^n}\Big)\right\| \\ &\le 9^n\phi_2\Big(0, \frac{x}{3^n}\Big) \le L^n\phi_2(0, x)\end{aligned}$$

for all $x \in V$. Hence, we obtain

$$\lim_{n\to\infty} 9^n\left(f_2\Big(\frac{x}{3^n}\Big) - f_2\Big(-\frac{x}{3^n}\Big)\right) = 0$$

and

$$\lim_{n\to\infty} 3^n\left(f_2\Big(\frac{x}{3^n}\Big) - f_2\Big(-\frac{x}{3^n}\Big)\right) = 0$$

for all $x \in V$. From these inequalities and (11), we obtain (23). Moreover, we have

$$\left\| 9^n Q f_2\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) \right\| \le 9^n \phi_2\Big(\frac{x}{3^n}, \frac{y}{3^n}\Big) \le L^n \phi_2(x, y)$$

for all $x, y \in V$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$QF_2(x, y) = 0$$

for all $x, y \in V$. $\square$

Now, we prove the Hyers-Ulam-Rassias stability of Eq. (3) in the framework of normed spaces by using Theorems 2.3 and 2.4.

Corollary 3.3. *Let X be a normed space. Assume that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y, z)\| \le \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$$

for all $x, y, z \in X\backslash\{0\}$, where $\theta \ge 0$ and $p \in (-\infty, 1) \cup (2, \infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{3\theta}{3^p - 9}\|x\|^p & \text{if } \ p > 2, \\ \frac{3\theta}{3 - 3^p}\|x\|^p & \text{if } \ 0 \le p < 1 \end{cases}$$

for all $x \in X\backslash\{0\}$. Moreover, if $p < 0$, then f is itself a quadratic-additive mapping.

Proof. If we put

$$\varphi(x, y, z) = \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$$

for all $x, y, z \in X\backslash\{0\}$ and

$$L = \begin{cases} 3^{p-1} & \text{if } \ p < 1, \\ 3^{2-p} & \text{if } \ p > 2, \end{cases}$$

then our corollary follows from Theorems 2.3 and 2.4 and Remark 2.5. $\square$

Corollary 3.4. *Let X be a normd space. Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y, z)\| \le \theta\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in X\backslash\{0\}$, *where* $\theta \geq 0$ *and* $p+q+r \in (-\infty, 1) \cup (2, \infty)$. *Then there exists a unique quadratic-additive mapping* $F : X \to Y$ *such that*

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta\|x\|^{p+q+r}}{3^{p+q+r}-9} & \text{if } p+q+r > 2, \\ \frac{\theta\|x\|^{p+q+r}}{3-3^{p+q+r}} & \text{if } 0 \leq p+q+r < 1 \end{cases}$$

for all $x \in X\backslash\{0\}$. *Moreover, if* $p+q+r < 0$, *then* f *is itself a quadratic-additive mapping.*

Proof. If we put

$$\varphi(x, y, z) = \theta\|x\|^p\|y\|^q\|z\|^r$$

for all $x, y, z \in X\backslash\{0\}$ and

$$L = \begin{cases} 3^{p+q+r-1} & \text{if } p+q+r < 1, \\ 3^{2-p-q-r} & \text{if } p+q+r > 2, \end{cases}$$

then our corollary follows from Theorems 2.3 and 2.4 and Remark 2.5. □

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