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A FIXED POINT THEOREM FOR UNBOUNDED MAPS

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Abstract: A fixed point theorem is proven for unbounded maps in $\mathbb{R}^n$. The proof is aided by an unconventional compactification. Examples are provided.

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1. Introduction Compactification and the Ultra Extended $\mathbb{R}^n$

Fixed points theorems are important tools in pure and applied mathematics. There exists a voluminuous literature on this subject matter. A partial list of text books and publications on fixed point theory related to nonexpansive maps include: R. Agarwal et al, [1], V. Berinde, [4], J.P. Dedieu [5], K. Goebel; W. A. Kirk,[7], L. Grniewicz, [8], A. Granas; J. Dugundji, [9], Nadler, Sam B., Jr.[23], S. Reich, Simeon; D. Shoikhet, [26], I. A. Rus; G. Petruşel, [28]. A partial list of texts and publications on fixed point theory that focus also on expansive maps include, G. J. Jiang; S. M. Kang, [18], M. Rodrguez-Montes; J. A. Charris, [27], Vasuki, R., [34], R. S. Saini et al, [31], R.K. Gujetiya et al, [10]. Texts on applications to economics include, Carl, Siegfried; Heikkil, Seppo, [32], Urai, Ken[33] and J. Franklin, [11]. It is beyond the scope of this article to record the numeorus excellent works on this important subject matter.

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The goal of this article is to prove a fixed point theorem for unbounded maps $f(y)$ in $\mathbb{R}^n$. The theorem states that certain unbounded maps always possess a fixed point where the value "infinity" could be a fixed point as well. Conditions are provided that stipulate when the fixed point is located in $\mathbb{R}^n$. The crux of the proof is based on a certain compactification proposed by H. Gingold [12], followed by an application of the Brouwer fixed point theorem. The usefulness of this unconventional compactification to dynamical systems was presented by H. Gingold and D. Solomon in [16, 15]. The field of systems of nonlinear polynomial finite difference systems also benefited from the above compactification. See H. Gingold [17, 14]. The technique could be useful for computational purposes as it converts the problem of finding fixed points to unbounded maps to a problem of finding fixed points for continuous maps.

Without much ado I proceed to present the nomenclature and ideas that are needed.

Definition 1. Let y, x, p and q be column vectors in $\mathbb{R}^n$. Let $y^\dagger = (y_1, y_2, ..., y_n)$, $x^\dagger = (x_1, x_2, ..., x_n)$, $p^\dagger = (p_1, p_2, ..., p_n)$ and $q^\dagger = (q_1, q_2, ..., q_n)$ denote row vectors in $\mathbb{R}^n$ that are the transposes of y, x p and q, respectively. In particular let $\hat{0}^\dagger = (0, ..., 0)$ be the transpose of the zero vector. Let $f^\dagger(y) := (f_1(y), f_2(y), ..., f_n(y))$ be a vector field in $\mathbb{R}^n$ where $f_j(y), j = 1, 2, ..., n$ being scalar mappings.

In the spirit of Y. Gingold and H. Gingold, [13, 12], I augment $\mathbb{R}^n$ with a set of ideal points ID at infinity as follows.

Definition 2. Denote by $UE\mathbb{R}^n$ the union of $\mathbb{R}^n$ and a certain set ID and call it the Ultra extended $\mathbb{R}^n$ where

$$ID := \{\infty p \,|\, p^\dagger p = 1\},\ UE\mathbb{R}^n := \mathbb{R}^n \cup ID. \tag{1.1}$$

Denote by U the unit ball and by ∂U its boundary.

$$U := \{x \in \mathbb{R}^n \,|\, x^\dagger x \leq 1\},\ \partial U := \{x \in \mathbb{R}^n \,|\, x^\dagger x = 1\}. \tag{1.2}$$

Denote

$$r := \sqrt{y^\dagger y} = \|y\|\,,\ \|x\| := \sqrt{x^\dagger x} = R. \tag{1.3}$$

The rational transformation

$$y = (1 - R^2)^{-1} x \Longrightarrow r = \frac{R}{1 - R^2} \tag{1.4}$$

is shown in H. Gingold [12] to be a bijection from $\mathbb{R}^n$ onto the interior of U. It is also a bijection from the ideal set $ID := \{\infty p \,|\, p^\dagger p = 1\}$ onto ∂U. The inverse of $y = \frac{x}{1-x^\dagger x}$ in $R < 1$ is defined by the branches

$$x = \frac{2y}{1+\sqrt{1+4y^\dagger y}},\ R = \frac{2r}{1+\sqrt{1+4r^2}}. \tag{1.5}$$

Next I need

Definition 3. We denote $\lim\limits_{y\in S,\|y\|\to\infty} y = \infty p$ if y varies in a subset $S \subseteq \mathbb{R}^n$ such that

$$\lim_{y\in S,\|y\|\to\infty} \frac{y}{\sqrt{y^\dagger y}} = p.$$

We say that the equation $y = f(y)$ has a fixed point ∞p and denote $\infty p = f(\infty p)$ if

$$p = \lim_{y\to\infty p} \frac{f(y)}{\sqrt{f^\dagger(y)f(y)}}.$$

Remark 4. This condition does not imply or require the asymptotic relation

$$\lim_{y\to\infty p} \frac{\sqrt{f^\dagger(y)f(y)}}{\sqrt{y^\dagger(t)y(t)}} = 1.$$

We also need the following definition that describes the conditions under which an unbounded function is continuous in the ultra extended $\mathbb{R}^n$.

Definition 5. We say that the map $f(y)$ is continuous in the ultra extended $\mathbb{R}^n$ if

$$\lim_{y\to y_0} f(y) = f(y_0)$$

holds whenever y_0 and $f(y_0)$ are finite or if $f(y_0) = \infty q,\ q \in \mathbb{R}^n,\ q^\dagger q = 1$ then

$$\lim_{y\to y_0} f(y) = \infty q$$

whenever y_0 is finite or $y_0 = \infty p, p \in \mathbb{R}^n,\ p^\dagger p = 1$.

Remark 6. Two important questions come up when dealing with unbounded maps. The first question is how to extend the definition of continuity of a map *f(y)* to points where *f(y)* becomes unbounded. The definition provided in here differs from a conventional definition that it means that the limit of the Euclidean norm of *f(y)* diverges to infinity. The second question that arises, that is intimately related to the first, is what is a fixed point “at infinity” of an unbounded map. These questions require new definitions. There is more than one answer to these questions. Such definitions could be sensitive to the type of compactification chosen and the metric utilized in Ultra extended $\mathbb{R}^n$. The fact that the fixed point property of a topological space, namely that ("every continuous mapping from a topological space to itself has a fixed point") is a topological invariant, holds true in compact sets of $\mathbb{R}^n$. It does not hold true in the Ultra extended $\mathbb{R}^n$. Any new definition could make or brake the utility of a fixed point theorem for unbounded maps. In order to drive home this point, consider the map $f(y) = y^{-1} - y$ in $\mathbb{R}^1$. This map does not have a fixed point at infinity according to the definitions above. However, this map has a fixed point at infinity if one utilizes a conventional definition. The main theorem of this article, theorem 7, and its corollary 8, predicts a desired conclusion that the map $f(y) = y^{-1} - y$ in $\mathbb{R}^1$ possesses at least one finite fixed point in $\mathbb{R}^1$ because it does not possess a fixed point at infinity. This conclusion is a product of the particular compactification and metric chosen. Choosing a different compactification, like the stereographic projection, could prevent us from reaching the desired conclusion. It is noteworthy that the map $f(y) = y^{-2} + y$ in $\mathbb{R}^1$ has two fixed points "at infinity" acccording to the definitions provided in here and has no fixed point in $\mathbb{R}^1$. There is an alternative way to describe continuity in the $UE\mathbb{R}^n$ via a metric that measures the distance between two points in the $UE\mathbb{R}^n$. See Gingold [12]. A metric for a family of compactifications that include the Poincare comactification and the stereographic projection as special cases of a family of compactifications is provided in Y. I. Gingold, H. Gingold, [13]. The definition of the $UE\mathbb{R}^n$ helps us define what does it mean to have an unbounded function diverging to infinity to be continuous in a generalized sense. The definition of the so called Ultra extended $\mathbb{R}^n$ was suggested by Y. I. Gingold, H. Gingold, [13]. A useful property of the compactification employed in here is that it takes rational maps into rational maps and it distinguishes among all different directions at infinity. The stereographic projection does not possesses these two properties. Geometrical compactifications go all the way back to Greek Mathematics before the common era. See e.g. Hille [21] for a discussion of the stereographic projection. I. Bendixson, [3] made applications of it to differential equations. Applications of a compactification method proposed by Poincare [25] are given e.g. in the text books on differential equations by A. A. Andronov et al [2], G. Sansone and R. Conti, [30], L. Perko, [24] and D. Jordan & P. Smith [19].

The rest of this article proceeds with Section 2 where a fixed point theorem is proven. It is followed by Section 3 that provides examples and applications.

2. A Fixed Point Theorem

The main result of this article is

Theorem 7. *Let $f(y)$ be a continuous map in the $UE\mathbb{R}^n$. Then, $f(y)$ has a fixed point in the $UE\mathbb{R}^n$.*

Moreover, assume that there exists $\rho > 0$ such that

$$\sqrt{y^\dagger y} \geq \rho \Longrightarrow sup \sqrt{f^\dagger(y)f(y)} < \infty.$$

Then, $f(y)$ has a fixed point in $\mathbb{R}^n$.

Proof. It is easily verified that the transformation $y = (1-R^2)^{-1}x$ takes the equation $y = f(y)$ into the relations

$$y = (1-R^2)^{-1}x = f((1-R^2)^{-1}x) = f(y), \; x = (1-R^2)f((1-R^2)^{-1}x). \quad (2.1)$$

The equation $y = f(y)$ and the forms above are equivalent for $(1-R^2) \neq 0$. The relations in (2.1) imply that

$$R^2 = x^\dagger x = (1-R^2)^2 y^\dagger y = (1-R^2)^2 f^\dagger(y)f(y).$$

it can be written as the quadratic equation for $(1-R^2)$ that is

$$(1-R^2) = 1-(1-R^2)^2 f^\dagger f, \quad (1-R^2)^2 f^\dagger f + (1-R^2) - 1 = 0.$$

Its solution is given by

$$(1-R^2) = \frac{-1+\sqrt{1+4f^\dagger(y)f(y)}}{2f^\dagger(y)f(y)} = \frac{2}{1+\sqrt{1+4f^\dagger(y)f(y)}} \leq 1,$$

with equality iff $f^\dagger(y)f(y) = 0$. Thus, (2.1) can be written as

$$x = \frac{2f(y)}{1+\sqrt{1+4f^\dagger(y)f(y)}}.$$

I claim that the composite map

$$H = H(f) := \frac{2f(y)}{1+\sqrt{1+4f^\dagger(y)f(y)}}, \; y = \frac{x}{1-x^\dagger x},$$

as a function of x is a continuous map from U into U. Notice first that the boundedness of H in U follows from the relation

$$H^\dagger H = \frac{4f^\dagger(y)f(y)}{[1+\sqrt{1+4f^\dagger(y)f(y)}]^2} = [\frac{2\sqrt{f^\dagger(y)f(y)}}{1+\sqrt{1+4f^\dagger(y)f(y)}}]^2 \leq 1,$$

with equality iff $f^\dagger(y)f(y)=\infty$. Next we focus on the continuity of H in U. Consider first the case that y_0 and $f(y_0)$ are finite. Then,

$$y_0=(1-R_0^2)^{-1}x_0,\ R_0^2:=x_0^\dagger x_0<1,\ x_0=\frac{2y_0}{1+\sqrt{1+4y_0^\dagger y_0}}.$$

$H(f)$ is a continuous function of f at $f=f(y_0)$, $f(y)$ is a continuous function of y at $y=y_0$ and $y=\frac{x}{1-x^\dagger x}$ is a continuous function of x at $x=x_0=\frac{2y_0}{1+\sqrt{1+4y_0^\dagger y_0}}$ where $x_0^\dagger x_0<1$. Therefore, $H(f)$ is a continuous function of x at $x=x_0=\frac{2y_0}{1+\sqrt{1+4y_0^\dagger y_0}}$. I focus now on the continuity of $H(f)$ at points x_0 where $x_0^\dagger x_0<1$ and $\infty q=f(y_0)$. Continuity follows from the relation

$$\lim_{y\to y_0}\frac{2f(y)}{1+\sqrt{1+4f^\dagger(y)f(y)}}=\lim_{y\to y_0}\frac{f(y)}{\sqrt{f^\dagger(y)f(y)}}=q,$$

because

$$\lim_{y\to y_0}\sqrt{f^\dagger(y)f(y)}=\infty\Longrightarrow\lim_{y\to y_0}\frac{2\sqrt{f^\dagger(y)f(y)}}{1+\sqrt{1+4f^\dagger(y)f(y)}}=1.$$

A similar argument holds if $p^\dagger p=1$ and $\lim_{y\to\infty p}\frac{f(y)}{\sqrt{f^\dagger(y)f(y)}}=q$. Then,

$$\begin{aligned}\lim_{x\to p}\frac{2f(y)}{1+\sqrt{1+4f^\dagger(y)f(y)}}&=\lim_{y\to\infty p}\frac{2f(y)}{1+\sqrt{1+4f^\dagger(y)f(y)}}\\&=\lim_{y\to\infty p}\frac{f(y)}{\sqrt{f^\dagger(y)f(y)}}=q.\end{aligned}$$

By Brouwer's fixed point theorem $H(f(y(x)))$ has a fixed point in U, namely for some x_0, $x_0=H(f(y(x_0)))$. The equations $y=f(y)$ and $x_0=H(f(y(x_0)))$ are not completely equivalent. Therefore, a few cases need to be considered and properly interpreted.

Case 1. $x_0^\dagger x_0<1$, $y_0=(1-R_0^2)^{-1}x_0$ and $f(y_0)$ are finite. Then, y_0 is a desired fixed point of $f(y)$ in $\mathbb{R}^n$.

Case 2. $x_0^\dagger x_0<1$, $y_0=(1-R_0^2)^{-1}x_0$ is finite and $f(y_0)=\infty q$. Then, x_0 cannot be a fixed point of $H(f(y(x_0)))$ because then we will have

$$y_0=(1-R_0^2)^{-1}x_0=f(y_0)=\infty q,$$

a contradiction.

Case 3. $x_0^\dagger x_0 = 1$, $y_0 = (1 - R_0^2)^{-1} x_0 = \infty p$ and $f(y_0)$ is finite. Then, although x_0 is a fixed point of $H(f(y(x)))$, y_0 is not a fixed point of $f(y)$ because then we will have the contradiction

$$y_0 = (1 - R_0^2)^{-1} x_0 = f(y_0).$$

Case 4. $x_0^\dagger x_0 = 1$, $y_0 = (1 - R_0^2)^{-1} x_0 = \infty p$ and $f(y_0) = \infty q$. Then, although x_0 is a fixed point of $H(f(y(x)))$, y_0 is not a fixed point of $f(y)$ in a conventional sense. It is not claimed then that $\lim_{y \to \infty p} \frac{\sqrt{f^\dagger(y) f(y)}}{\sqrt{y^\dagger(t) y(t)}} = 1$. The interpretation then is that

$$\infty p = y_0 = (1 - R_0^2)^{-1} x_0 = f(y_0) = \infty q$$

and the vector fields y and $f(y)$ share the same direction $p = q = x_0$. Thus, either Case 1 or Case 4 must occur. However, if the condition $\sqrt{y^\dagger y} \geq \rho \Longrightarrow$ *sup* $\sqrt{f^\dagger(y) f(y)} < \infty$ holds, then Case 4 is ruled out and the conclusion of the theorem follows. □

Corollary 8. *A continuous map f(y) in the $UE\mathbb{R}^n$ that has no fixed point at infinity has a fixed point in $\mathbb{R}^n$.*

Also note,

Remark 9. The interpretation of Case 4 in theorem 7, is supported by the fact that the vector x is a multiple of the vector $f(y)$ by the scalar $\frac{2}{1+\sqrt{1+4f^\dagger(y)f(y)}} \geq 0$. This needs further elaboration as $\frac{2}{1+\sqrt{1+4f^\dagger(\infty p)f(\infty p)}} = 0$ and $f(\infty p) = \infty q$. However, in this case the indeterminate form $0[\infty q]$ is well defined as a limit. Therefore the following alternative definition to Definition 3 could be useful. We say that ∞p is a fixed point of $f(y)$ and denote $\infty p = f(\infty p)$ if the equation

$$x = \frac{2f(y)}{1 + \sqrt{1 + 4f^\dagger(y) f(y)}}, \; y = \frac{x}{1 - x^\dagger x}$$

possesses a fixed point x with $x^\dagger x = 1$. One can show that by defining an appropriate metric in the $UE\mathbb{R}^n$ like in Y. Gingold and H. Gingold [13] and like in H. Gingold [12], the $UE\mathbb{R}^n$ is a complete metric space and then continuity of $f(y)$ in the $UE\mathbb{R}^n$ is made possible.

3. Applications and Examples

First I consider families of functions in the $UE\mathbb{R}^1$, namely $n = 1$. The first, example 10, can be worked out independent of theorem 7 and demonstrates the usefulness of the presented compactification and the relevance of corollary 8.

Example 10. The map $f(y) = y^{-1} - y$ in $\mathbb{R}^1$ has no fixed points at infinity. Theorem 7 and especially its corollary 8 predicts the existence of at least one fixed point in $\mathbb{R}^1$. In fact, two fixed points exist in $\mathbb{R}^1$. They are of course $y = \pm\sqrt{\frac{1}{2}}$.

Next consider

Example 11. The map

$$f(y) = \begin{cases} 4, & \text{if } y \leq -m\pi,\ m, l \in \mathbb{N}, \\ 4sec^2 y, & \text{if } -m\pi < y < l\pi, \\ 4, & \text{if } l\pi \leq y \end{cases}$$

satisfies all conditions of Theorem 7. However, the fixed points are not readily available.

A geometrical argument in the example above that the graph of the functions y and $f(y)$ must intersect at a finite point may be stated as follows. The only manner that y and $f(y)$ would not intersect is if the function y would be parallel to one of the vertical asymptotes of $f(y)$, which is of course impossible.

Another example is provided by certain families of rational functions.

Example 12. Let $f(y)$ be a rational function of the form $f(y) = \frac{p(y)}{q(y)}$ where the numerator $p(y)$ and the denominator $q(y)$ are polynomials in y such that: a) the degree of $p(y)$ does not exceed the degree of $q(y)$, b) $p(y)$ and $q(y)$ do not share a common real root and c) all roots of $q(y)$ are of even multiplicity. Then $f(y)$ has a fixed point in $\mathbb{R}$. An interested reader may want to verify that all conditions of theorem 7 are satisfied.

Next I consider families of functions in the $UE\mathbb{R}^2$.

Example 13. Let $F(z) = u(y) + iv(y)$ be a meromorphic function of $z = y_1 + iy_2$ on the real axis. Assume that the poles of $F(z)$ that are located on the real axis $y_2 = 0$ are of even order. Assume that $F(z)$ is regular at $z = \infty$. Then, the mapping $f(y), \quad f^\dagger(y) = (u(y), v(y))$ has a fixed point in $\mathbb{R}^2$. In order to confirm this notice that if $z_0 \in \mathbb{R}$ is a pole of even order then $F(z) = (z - z_0)^{-2m} G(z),\ m \in \mathbb{N}$ where $G(z)$ is an analytic function at $z_0 \in \mathbb{R}$

with $G(z_0) \neq 0$. Put $G(z) = \hat{u}(y) + i\hat{v}(y)$. Then,

$$f^\dagger(y) = (u(y), v(y)) = (y_1 - z_0)^{-2m}(\hat{u}(y), \hat{v}(y)),$$

with

$$\lim_{(y_1,y_2)\to(z_0,0)} (\hat{u}^2(y) + \hat{v}^2(y)) = (\hat{u}^2((z_0,0)) + \hat{v}^2((z_0,0))) \neq 0.$$

Consequently,there exist the limit

$$\lim_{(y_1,y_2)\to(z_0,0)} \frac{(y_1 - z_0)^{-2m}(\hat{u}(y), \hat{v}(y))}{\sqrt{(y_1 - z_0)^{-4m}(\hat{u}^2(y) + \hat{v}^2(y))}} = \frac{(\hat{u}((z_0,0)), \hat{v}((z_0,0)))}{\sqrt{(\hat{u}^2((z_0,0)) + \hat{v}^2((z_0,0)))}}.$$

The regularity of $F(z)$ at $z = \infty$ implies that the fixed point of $f(y)$ that is guaranteed by theorem 7 is indeed in $\mathbb{R}^2$. Namely, the meromorphic function $F(z)$ has a fixed point on the real line.

Lastly, I discuss an example in the $UE\mathbb{R}^n$.

Example 14. Consider the rational map $f(y)$ of the form

$$f^\dagger(y) = (f_1(y), \cdots, f_n(y)),\ f_j(y) = \frac{1}{[1 - \sum_{j=1}^n (y_j)^2]^{2j}}, \qquad j = 1, \cdots, n.$$

Then,

$$f^\dagger(y) = (f_1(y), \cdots, f_n(y))$$
$$= \frac{1}{[1 - \sum_{j=1}^n (y_j)^2]^{2n}}([1-\sum_{j=1}^n (y_j)^2]^{2(n-1)}, [1-\sum_{j=1}^n (y_j)^2]^{2(n-2)}, ..., [1-\sum_{j=1}^n (y_j)^2]^2, 1\}.$$

It can be easily verified that

$$\frac{f(y)}{\sqrt{f^\dagger(y)f(y)}} =$$
$$\frac{([1 - \sum_{j=1}^n (y_j)^2]^{2(n-1)}, [1 - \sum_{j=1}^n (y_j)^2]^{2(n-2)}, ..., [1 - \sum_{j=1}^n (y_j)^2]^2, 1)}{\sqrt{([1 - \sum_{j=1}^n (y_j)^2]^{4(n-1)} + [1 - \sum_{j=1}^n (y_j)^2]^{4(n-2)} +, ..., +[1 - \sum_{j=1}^n (y_j)^2]^4 + 1)}}.$$

Thus $\frac{f(y)}{\sqrt{f^\dagger(y)f(y)}}$ can be defined as a continuous function of y at every point $y_0 \in \mathbb{R}^n$. This includes of course the points y where $1 - \sum_{j=1}^n (y_j)^2 = 0$. Moreover,

$$y \to \infty p \Longrightarrow [1 - \sum_{j=1}^n (y_j)^2] \to -\infty.$$

This implies that $\lim_{y\to\infty p} f^\dagger(y) = \hat{0}$ which in turn guarantees that conditions of theorem 7 are satisfied and its conclusion follows.

Last, I provide an example where a finite fixed point does not exist and where a fixed point or a fixed direction at infinity exists.

Example 15. Consider the scalar relation

$$y = f(y) := 1 + y + y^{2m}, \quad m = 1, 2, 3, 4, \dots .$$

Evidently, it has no real fixed point in $\mathbb{R}$. However, according to theorem 7, it must have a fixed point at infinity. I will proceed to calculate it. Notice that

$$f(y) = 1 + \frac{x}{1-x^2} + [\frac{x}{1-x^2}]^{2m} = \frac{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}}{(1-x^2)^{2m}},$$

and that

$$\begin{aligned} 1 + 4f^\dagger(y)f(y) = 1 + 4[f(y)]^2 &= 1 + 4\frac{\{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}\}^2}{(1-x^2)^{4m}} \\ &= \frac{(1-x^2)^{4m} + 4\{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}\}^2}{(1-x^2)^{4n}}. \end{aligned}$$

The terms above imply the relation

$$\begin{aligned} x = \frac{2f(y)}{1+\sqrt{1+4f^\dagger(y)f(y)}} &= \frac{2\frac{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}}{(1-x^2)^{2m}}}{1+\sqrt{\frac{(1-x^2)^{4m} + 4\{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}\}^2}{(1-x^2)^{4m}}}} \\ = 2&\frac{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}}{(1-x^2)^{2m} + (1-x^2)^{2m}\sqrt{\frac{(1-x^2)^{4m} + 4\{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}\}^2}{(1-x^2)^{4m}}}} \\ = 2&\frac{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}}{(1-x^2)^{2m} + \sqrt{(1-x^2)^{4m} + 4\{(1-x^2)^{2m} + x(1-x^2)^{2m-1} + x^{2m}\}^2}}. \end{aligned}$$

A critical point at infinity is attained iff $(1-x^2) = 0$ so

$$x = \frac{x^{2m}}{\sqrt{\{x^{2m}\}^2}} = 1.$$

This is tantamount to saying that $y = \infty 1$ and that $f(\infty 1) = \infty 1$, namely y and $f(y)$ have the same direction at the fixed point $\infty 1$. Consider a map

$$f^\dagger(y) = (f_1(y), \cdots, f_n(y)), \; f_j(y) = 1 + y_j + y_j^{2m}, \quad m = 1, 2, 3, 4, \dots .$$

Repeating an analogous computation with

$$x^{\dagger} = (x_1, ..., x_n), \quad R^2 = x_1^2 +, \cdots, +x_n^2,$$

I have for each fixed $k \in \{1, 2, \cdots, n\}$, a fixed point ∞x with

$$x_j = 1 \; if \; j = k \quad and \quad x_j = 0 \; if \; j \neq k, \; j = 1, 2, \cdots, n.$$

Thus, the map $f(y)$ has no fixed points in $\mathbb{R}^n$. It has n fixed points "at infinity" that are elements of the standard basis of $\mathbb{R}^n$ as given above.

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