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THE NÖRLUND ORLICZ SPACE OF DOUBLE ENTIRE RATE SEQUENCES

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Abstract: Let Γ_π^2 denote the space of all double entire rate sequences. Let Λ_π^2 denote the space of all double analytic rate sequences. This paper is devoted to a study of the general properties of Nörlund double Orlicz space $\eta\left(\Gamma_{f\pi}^2\right)$ of entire rate sequences and Nörlund double Orlicz space $\eta\left(\Lambda_{f\pi}^2\right)$ of analytic rate sequences.

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Key Words: analytic sequence, double sequence, Nörlund space, Orlicz space and entire sequence and Rate sequences

1. Introduction

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence (S_{mn}) is convergent where

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$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \qquad (m, n = 1, 2, 3, \dots)$$

(see [1]).

We denote ω^2 as the class of all complex double sequences (x_{mn}). A sequence $x = (x_{mn})$ is said to be double analytic rate if $\sup_{m,n} \left| \frac{x_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} < \infty$. The vector spaces of all Pringsheim double analytic rate sequences are usually denoted by Λ_π^2. A sequence $x = \left(\frac{x_{mn}}{\pi_{mn}}\right)$ is called double entire rate sequences if $\left|\frac{x_{mn}}{\pi_{mn}}\right|^{\frac{1}{m+n}} \to 0$ as m, n $\to \infty$. The vector spaces of all Pringsheim double entire rate sequences are usually denoted by Γ_π^2. The space Λ_π^2 is a metric space with the metric

$$d(x, y) = \sup_{m,n} \left\{ \left| \frac{x_{mn} - y_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} ;\ m, n = 1, 2, 3, \dots \right\}$$

for all $x = \left(\frac{x_{mn}}{\pi_{mn}}\right)$ and $y = \left(\frac{y_{mn}}{\pi_{mn}}\right)$ in Λ_π^2. The space Γ_π^2 is a metric space with the metric

$$d(x, y) = \sup_{m,n} \left\{ \left| \frac{x_{mn} - y_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} ;\ m, n = 1, 2, 3, \dots \right\}$$

for all $x = \left(\frac{x_{mn}}{\pi_{mn}}\right)$ and $y = \left(\frac{y_{mn}}{\pi_{mn}}\right)$ in Γ_π^2

Let $(p_{mn})_{m,n=0}^{\infty}$ be a sequence of non-negative real numbers with $p_{00} > 0$.

Consider the transformation

$$\frac{y_{mn}}{\pi_{mn}} = \frac{1}{\sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}} \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij} \left(\frac{x_{m-i,n-j}}{\pi_{m-i,n-j}} \right)$$

for $m = n = 0, 1, 2, \dots$. The set of all $\left(\frac{x_{mn}}{\pi_{mn}}\right)$ for which $\left(\frac{y_{mn}}{\pi_{mn}}\right) \in \Gamma_\pi^2$ is called the Nörlund Orlicz space of double entire rate sequence. The Nörlund Orlicz space of double entire rate sequence is denoted by $\eta\left(\Gamma_{f\pi}^2\right)$. Similarly the set of all $\left(\frac{x_{mn}}{\pi_{mn}}\right)$ for which $\left(\frac{y_{mn}}{\pi_{mn}}\right) \in \Lambda_\pi^2$ is called the Nörlund Orlicz space of double analytic rate sequence and is denoted by $\eta\left(\Lambda_{f\pi}^2\right)$. We write $P_{mn} = p_{00} + \cdots + p_{mn}$ for $m = n = 0, 1, 2, 3, \dots$. An absolutely convex absorbent closed subset of a locally convex topological vector space X is called barrel. X is called barrelled space if each barrel is a neighbourhood of zero.

A locally convex topological vector space X is said to be semi-reflexive if each bounded closed set in X is $\sigma(X, X')$-compact.

Consider a double sequence $x = (x_{ij})$. The (m,n)th section $x^{[m,n]}$ of the sequence is defined by

$$x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij} \text{ for all } m,n \in N = \{\text{all possible integers}\}$$

where

$$\delta_{mn} = \begin{cases} 1; & \text{if } m = n \\ 0; & \text{otherwise} \end{cases}$$

An FK-space (or a metric space) X is said to have AK-property if (δ_{mn}) is a Schauder basis for X or equivalently, $x^{[m,n]} \to x$. We need the following inequality in the sequel of the paper:

Lemma 1.1. *For $a, b \geq 0$ and $0 < p < 1$, we have $(a+b)^P \leq a^P + b^P$.*

2. Preliminaries and Definitions

Some initial works on double sequence spaces are found in Bromwich [4]. Later on the same was investigated by Hardy [7], Moricz [11], Moricz and Rhoades [12], Basarir and Solankan [2], Tripathy [18], Tturkmenoglu [20] and many others. Orlicz [15] used the idea of Orlicz function to construct the space (L^M) Lindenstrauss and Tzafriri [9] investigated Orlicz sequence space in more detail and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p $(1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [16], Mursaleen et al. [13], Bektas and Altin [3], Tripathy et al. [19], Rao and Subramanian [5] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [8].

Recall [15] and [8], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) = 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called modulus function, defined by Nakano [14] and further discussed by Ruckle [17] and Maddox [10] and many others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u) (u \geq 0)$. The Δ_2-condition is equivalent to $M(lu) \leq KlM(u)$, for all values of u and for all $l > 1$.

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function M to construct Orlicz sequence space

$l_M = \left\{ x \in w : \sum_{K=1}^{\infty} M\left(\frac{|x_K|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$, where $w = \{$all complex sequences$\}$. The space l_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{K=1}^{\infty} M\left(\frac{|x_K|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space, which is called an Orlicz sequence space. For $M(t) = t^P$ $(1 \leq p < \infty)$, the space l_M coincides with the classical sequence space l_P.

Hardy [7] gave regularity conditions for a Nörlund matrix. Based on this Nörlund matrix transformation, in this paper the Nörlund Orlicz space of double entire rate sequence $\eta\left(\Gamma^2_{f\pi}\right)$ is introduced similar results hold for the Orlicz space of analytic rate sequences. We also examine whether the space $\Gamma^2_{f\pi}$ is barrelled and semi-reflexive.

3. Definitions

Throughout the article ω^2 denotes the space of all sequences. Γ^2_M and Λ^2_M denoted the Pringsheim sense of double Orlicz space of entire sequences and Pringsheim sense of double Orlicz space of bounded sequences respectively.

Let ω^2 denote the set of all complex double sequences

$x = (x_{mn})_{m,n=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function.

$$\Gamma^2_M = \left\{ x \in w^2 : \left[M\left(\frac{|x_{mn}|^{\frac{1}{m+n}}}{\rho} \right) \right] \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0 \right\}$$

and

$$\Lambda^2_M = \left\{ x \in w^2 : \sup_{m,n \geq 1} \left[M\left(\frac{|x_{mn}|^{\frac{1}{m+n}}}{\rho} \right) \right] < \infty \text{ for some } \rho > 0 \right\}$$

The space Λ^2_M is a metric space with the metric

$$d(x,y) = \inf\left\{\rho > 0 : \sup_{m,n\geq 1}\left[M\left(\frac{|x_{mn} - y_{mn}|}{\rho}\right)\right] \leq 1\right\} \forall x, y \in \Lambda^2_M$$

The space Γ^2_M is a metric space with the metric

$$d(x,y) = \inf\left\{\rho > 0 : \sup_{m,n\geq 1}\left[M\left(\frac{|x_{mn} - y_{mn}|}{\rho}\right)\right] \leq 1\right\}, \forall x, y \in \Gamma^2_M$$

$\Gamma^2_{f\pi}$ and $\Lambda^2_{f\pi}$ denote the Pringsheim sense of double Orlicz space of entire rate sequences and Pringsheim sense of double Orlicz space of bounded analytic rate sequences respectively.

Let ω^2 denote the set of all complex double rate sequences

$$x = \left(\frac{x_{mn}}{\pi_{mn}}\right)_{m,n=1}^{\infty} \text{ and } f : [0,\infty) \to [0,\infty)$$

be an Orlicz function or a modulus function.

$$\Gamma^2_{f\pi} = \left\{x \in w^2 : \left(f\left(\left|\frac{x_{mn}}{\pi_{mn}}\right|^{\frac{1}{m+n}}\right)\right) \to 0 \text{ as } m,n \to \infty\right\}$$

and

$$\Lambda^2_{f\pi} = \left\{x \in w^2 : \sup_{m,n\geq 1}\left(f\left(\left|\frac{x_{mn}}{\pi_{mn}}\right|^{\frac{1}{m+n}}\right)\right) < \infty\right\}$$

The spaces $\Lambda^2_{f\pi}$, $\Gamma^2_{f\pi}$ are metric spaces with the metric

$$d(x,y) = \inf\left\{\sup_{m,n\geq 1}\left(f\left(\left|\frac{x_{mn} - y_{mn}}{\pi_{mn}}\right|^{\frac{1}{m+n}}\right)\right) \leq 1\right\}$$

for all x, y belongs to $\Lambda^2_{f\pi}$ and $\Gamma^2_{f\pi}$ respectively.

4. Main Results

Proposition 4.1.

$$\eta\left(\Gamma^2_{f\pi}\right) = \Gamma^2_{f\pi}.$$

Proof. Let $x = (x_{mn}) \in \eta\left(\Gamma^2_{f\pi}\right)$. Then $y \in \Gamma^2_{f\pi}$ so that for every $\varepsilon > 0$, we have a positive integer n_0 such that

$$\left(f\left(\left|\frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{\pi_{mn}P_{mn}}\right|\right)\right) < \varepsilon^{m+n}, \forall m, n \geq n_0.$$

Take $p_{00} = 1$, $p_{11} = \cdots = p_{mn} = 0$.

We then have $\left(f\left(\left|\frac{x_{mn}}{\pi_{mn}}\right|\right)\right) < \varepsilon^{m+n}$, $\forall m, n \geq n_0$.

Therefore, $x = (x_{mn}) \in \Gamma^2_{f\pi}$.

Hence

$$\eta\left(\Gamma^2_{f\pi}\right) \subset \Gamma^2_{f\pi} \tag{1}$$

On the other hand, let $x = (x_{mn}) \in \Gamma^2_{f\pi}$.

But for any given $\varepsilon > 0$, there exists a positive integer n_o such that $\left(f\left(\left|\frac{x_{mn}}{\pi_{mn}}\right|\right)\right) < \varepsilon^{m+n}$, $\forall m, n \geq n_0$.

We have

$$\begin{aligned}
\left(f\left(\left|\frac{y_{mn}}{\pi_{mn}}\right|\right)\right) &\leq \left(f\left(\left|\frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{P_{mn}\pi_{mn}}\right|\right)\right) \\
&\leq \frac{1}{P_{mn}}\left[p_{00}\left(f\left|\frac{x_{mn}}{\pi_{mn}}\right|\right) + \cdots + p_{mn}\left(f\left|\frac{x_{00}}{\pi_{00}}\right|\right)\right] \\
&\leq \frac{1}{P_{mn}}\left[p_{00}\varepsilon^{m+n} + \cdots + p_{mn}\varepsilon^{0+0}\right] \\
&\leq \frac{\varepsilon^{m+n}}{P_{mn}}\left(p_{00} + \cdots + p_{mn}\right) \\
&\leq \frac{\varepsilon^{m+n}}{P_{mn}}P_{mn} = \varepsilon^{m+n}, \forall m, n \geq n_0
\end{aligned}$$

Therefore, $(y_{mn}) \in \Gamma^2_{f\pi}$.

Consequently $x \in \eta\left(\Gamma^2_{f\pi}\right)$. Hence

$$\Gamma^2_{f\pi} \subset \eta\left(\Gamma^2_{f\pi}\right). \tag{2}$$

From (1) and (2) we obtain $\eta\left(\Gamma^2_{f\pi}\right) = \Gamma^2_{f\pi}$. This completes the proof. □

Proposition 4.2.

$$\eta\left(\Lambda^2_{f\pi}\right) = \Lambda^2_{f\pi}$$

Proof. Let $(x_{mn}) \in \Lambda^2_{f\pi}$. Then there exists a constant T such that

$$\left(f\left(\left|\frac{x_{mn}}{\pi_{mn}}\right|\right)\right) \leq T^{m+n} \text{ for } m, n = 0, 1, 2, \ldots$$

$$\begin{aligned}
\left(f\left(\left|\frac{y_{mn}}{\pi_{mn}}\right|\right)\right) &\leq \frac{p_{00}T^{m+n} + \cdots + p_{mn}T^{0+0}}{P_{mn}} \\
&\leq \frac{T^{m+n}}{P_{mn}}\left[p_{00} + \cdots + \frac{p_{mn}}{T^{m+n}}\right] \\
&\leq \frac{T^{m+n}}{P_{mn}}\left[p_{00} + \cdots + p_{mn}\right] \\
&\leq \frac{T^{m+n}}{P_{mn}}P_{mn} = T^{m+n}, \text{ for } m, n = 0, 1, 2, \ldots.
\end{aligned}$$

Hence $(y_{mn}) \in \Lambda^2_{f\pi}$. But $x = (x_{mn}) \in \eta\left(\Lambda^2_{f\pi}\right)$. Consequently,

$$\Lambda^2_{f\pi} \subset \eta\left(\Lambda^2_{f\pi}\right). \tag{3}$$

On the other hand, let

$$(x_{mn}) \in \eta\left(\Lambda^2_{f\pi}\right). \text{ Then } (y_{mn}) \in \Lambda^2_{f\pi}.$$

Hence there exists a positive constant T such that

$$\left(f\left(\left|\frac{y_{mn}}{\pi_{mn}}\right|\right)\right) < T^{m+n}, \text{ for } m, n = 0, 1, 2, \ldots.$$

This in turn implies that

$$\left(f\left(\left|\frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{P_{mn}\pi_{mn}}\right|\right)\right) < T^{m+n}.$$

Hence

$$\frac{1}{P_{mn}}\left(f\left|\frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{\pi_{mn}}\right|\right) < T^{m+n},$$

and thus

$$\left(f\left(\left|\frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{\pi_{mn}}\right|\right)\right) < P_{mn}T^{m+n}.$$

Take $p_{00} = 1$; $p_{11} = \cdots = p_{mn} = 0$. Then it follows that $P_{mn} = 1$ and so $\left(f\left|\frac{x_{mn}}{\pi_{mn}}\right|\right) < T^{m+n}$ for all m, n.

Consequently $x = (x_{mn}) \in \Lambda^2_{f\pi}$. Hence

$$\eta\left(\Lambda^2_{f\pi}\right) \subset \Lambda^2_{f\pi}. \tag{4}$$

From (3) and (4), we get $\eta\left(\Lambda^2_{f\pi}\right) = \Lambda^2_{f\pi}$

This completes the proof. □

Proposition 4.3. *$\Gamma^2_{f\pi}$ is not a barrelled space.*

Proof. Let $A = \left\{x \in \Gamma^2_{f\pi} : \left(f\left(\left|\frac{x_{mn}}{\pi_{mn}}\right|^{\frac{1}{m+n}}\right)\right) \leq 1, \forall m, n\right\}$.

Then A is an absolutely convex, closed absorbent in $\Gamma^2_{f\pi}$.

But A is not a neighbourhood of zero.

Hence $\Gamma^2_{f\pi}$ is not barrelled. □

Proposition 4.4. *$\Gamma^2_{f\pi}$ is not semi-reflexive.*

Proof. Let $\{\delta^{(mn)}\} \in U$ be the unit closed ball in $\Gamma^2_{f\pi}$. Clearly no subsequence of $\{\delta^{(mn)}\}$ can converge weakly to any $y \in \Gamma^2_{f\pi}$.

Hence $\Gamma^2_{f\pi}$ is not semi-reflexive. □

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