THE NÖRLUND ORLICZ SPACE OF DOUBLE ENTIRE RATE SEQUENCES

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Abstract: Let $\Gamma^2_\pi$ denote the space of all double entire rate sequences. Let $\Lambda^2_\pi$ denote the space of all double analytic rate sequences. This paper is devoted to a study of the general properties of Nörlund double Orlicz space $\eta(\Gamma^2_\pi)$ of entire rate sequences and Nörlund double Orlicz space $\eta(\Lambda^2_\pi)$ of analytic rate sequences.

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1. Introduction

Let $(x_{mn})$ be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence $(S_{mn})$ is convergent where

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$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \quad (m, n = 1, 2, 3, \ldots)$$

(see [1]).

We denote $\omega^2$ as the class of all complex double sequences $(x_{mn})$. A sequence $x = (x_{mn})$ is said to be double analytic rate if $\sup_{m,n} \left| \frac{x_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} < \infty$. The vector spaces of all Pringsheim double analytic rate sequences are usually denoted by $\Lambda^2_\pi$. A sequence $x = \left(\frac{x_{mn}}{\pi_{mn}}\right)$ is called double entire rate sequences if

$$\left| \frac{x_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$ 

The vector spaces of all Pringsheim double entire rate sequences are usually denoted by $\Gamma^2_\pi$. The space $\Lambda^2_\pi$ is a metric space with the metric

$$d(x, y) = \sup_{m,n} \left\{ \left| \frac{x_{mn} - y_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} ; \ m, n = 1, 2, 3, \ldots \right\}$$

for all $x = \left(\frac{x_{mn}}{\pi_{mn}}\right)$ and $y = \left(\frac{y_{mn}}{\pi_{mn}}\right)$ in $\Lambda^2_\pi$. The space $\Gamma^2_\pi$ is a metric space with the metric

$$d(x, y) = \sup_{m,n} \left\{ \left| \frac{x_{mn} - y_{mn}}{\pi_{mn}} \right|^{\frac{1}{m+n}} ; \ m, n = 1, 2, 3, \ldots \right\}$$

for all $x = \left(\frac{x_{mn}}{\pi_{mn}}\right)$ and $y = \left(\frac{y_{mn}}{\pi_{mn}}\right)$ in $\Gamma^2_\pi$.

Let $(p_{mn})_{m,n=0}^\infty$ be a sequence of non-negative real numbers with $p_{00} > 0$. Consider the transformation

$$y_{mn} = \frac{1}{\sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}} \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij} \left(\frac{x_{m-i,n-j}}{\pi_{m-i,n-j}}\right)$$

for $m = n = 0, 1, 2, \ldots$. The set of all $\left(\frac{x_{mn}}{\pi_{mn}}\right)$ for which $\left(\frac{y_{mn}}{\pi_{mn}}\right) \in \Gamma^2_\pi$ is called the Nörlund Orlicz space of double entire rate sequence. The Nörlund Orlicz space of double entire rate sequence is denoted by $\eta \left(\Gamma^2_\pi\right)$. Similarly the set of all $\left(\frac{x_{mn}}{\pi_{mn}}\right)$ for which $\left(\frac{y_{mn}}{\pi_{mn}}\right) \in \Lambda^2_\pi$ is called the Nörlund Orlicz space of double analytic rate sequence and is denoted by $\eta \left(\Lambda^2_\pi\right)$. We write $P_{mn} = p_{00} + \cdots + p_{mn}$ for $m = n = 0, 1, 2, 3, \ldots$. An absolutely convex absorbent
closed subset of a locally convex topological vector space $X$ is called barrel. $X$ is called barrelled space if each barrel is a neighbourhood of zero.

A locally convex topological vector space $X$ is said to be semi-reflexive if each bounded closed set in $X$ is $\sigma(X,X')$-compact.

Consider a double sequence $x = (x_{ij})$. The $(m,n)$th section $x^{[m,n]}$ of the sequence is defined by

$$ x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij} \text{ for all } m, n \in \mathbb{N} = \{\text{all possible integers}\} $$

where

$$ \delta_{mn} = \begin{cases} 1; & \text{if } m = n \\ 0; & \text{otherwise} \end{cases} $$

An FK-space (or a metric space) $X$ is said to have AK-property if $(\delta_{mn})$ is a Schauder basis for $X$ or equivalently, $x^{[m,n]} \to x$. We need the following inequality in the sequel of the paper:

**Lemma 1.1.** For $a, b \geq 0$ and $0 < p < 1$, we have $(a + b)^p \leq a^p + b^p$.

2. Preliminaries and Definitions

Some initial works on double sequence spaces are found in Bromwich [4]. Later on the same was investigated by Hardy [7], Moricz [11], Moricz and Rhoades [12], Basarir and Solankan [2], Tripathy [18], Turkmenoglu [20] and many others. Orlicz [15] used the idea of Orlicz function to construct the space $(L^M)$ Lindenstrauss and Tzafriri [9] investigated Orlicz sequence space in more detail and they proved that every Orlicz sequence space $l_M$ contains a subspace isomorphic to $l_p$ ($1 \leq p < \infty$). Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [16], Mursaleen et al. [13], Bektaş and Altin [3], Tripathy et al. [19], Rao and Subramanian [5] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [8].

Recall [15] and [8], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) = 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function, defined by Nakano [14] and further discussed by Ruckle [17] and Maddox [10] and many others.
An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$ if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)(u \geq 0)$. The $\Delta_2$-condition is equivalent to $M(lu) \leq KlM(u)$, for all values of $u$ and for all $l > 1$.

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function $M$ to construct Orlicz sequence space

$$l_M = \{x \in w : \sum_{K=1}^{\infty} M\left(\frac{|x_K|}{\rho}\right) < \infty, \text{ for some } \rho > 0\},$$

where $w = \{\text{all complex sequences}\}$. The space $l_M$ with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{K=1}^{\infty} M\left(\frac{|x_K|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space, which is called an Orlicz sequence space. For $M(t) = t^p$ $(1 \leq p < \infty)$, the space $l_M$ coincides with the classical sequence space $l_p$.

Hardy [7] gave regularity conditions for a Nörlund matrix. Based on this Nörlund matrix transformation, in this paper the Nörlund Orlicz space of double entire rate sequence $\eta \left(\Gamma_{f\pi}^2\right)$ is introduced similar results hold for the Orlicz space of analytic rate sequences. We also examine whether the space $\Gamma_{f\pi}^2$ is barrelled and semi-reflexive.

3. Definitions

Throughout the article $\omega^2$ denotes the space of all sequences. $\Gamma_M^2$ and $\Lambda_M^2$ denoted the Pringsheim sense of double Orlicz space of entire sequences and Pringsheim sense of double Orlicz space of bounded sequences respectively.

Let $\omega^2$ denote the set of all complex double sequences $x = (x_{mn})_{m,n=1}^\infty$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function.

$$\Gamma_M^2 = \left\{x \in \omega^2 : M\left(\frac{|x_{mn}|}{\rho}\right) \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0\right\}$$

and

$$\Lambda_M^2 = \left\{x \in \omega^2 : \sup_{m,n \geq 1} M\left(\frac{|x_{mn}|}{\rho}\right) < \infty \text{ for some } \rho > 0\right\}$$
The space $Λ^2_M$ is a metric space with the metric
\[ d(x, y) = \inf \left\{ \rho > 0 : \sup_{m,n \geq 1} \left[ M \left( \frac{|x_{mn} - y_{mn}|}{\rho} \right) \right] \leq 1 \right\} \forall x, y \in Λ^2_M \]

The space $Γ^2_M$ is a metric space with the metric
\[ d(x, y) = \inf \left\{ \rho > 0 : \sup_{m,n \geq 1} \left[ M \left( \frac{|x_{mn} - y_{mn}|}{\rho} \right) \right] \leq 1 \right\}, \forall x, y \in Γ^2_M \]

$Γ^2_{fπ}$ and $Λ^2_{fπ}$ denote the Pringsheim sense of double Orlicz space of entire rate sequences and Pringsheim sense of double Orlicz space of bounded analytic rate sequences respectively.

Let $ω^2$ denote the set of all complex double rate sequences
\[ x = \left( \frac{x_{mn}}{π_{mn}} \right)_{m,n=1}^∞ \] and $f : [0, \infty) \to [0, \infty)$ be an Orlicz function or a modulus function.

\[ Γ^2_{fπ} = \left\{ x \in ω^2 : \left( f \left( \frac{|x_{mn}|}{π_{mn}} \right) \right) \to 0 \text{ as } m, n \to ∞ \right\} \]

and
\[ Λ^2_{fπ} = \left\{ x \in ω^2 : \sup_{m,n \geq 1} \left( f \left( \frac{|x_{mn}|}{π_{mn}} \right) \right) < ∞ \right\} \]

The spaces $Λ^2_{fπ}$, $Γ^2_{fπ}$ are metric spaces with the metric
\[ d(x, y) = \inf \left\{ \sup_{m,n \geq 1} \left( f \left( \frac{|x_{mn} - y_{mn}|}{π_{mn}} \right) \right) \leq 1 \right\} \]

for all $x, y$ belongs to $Λ^2_{fπ}$ and $Γ^2_{fπ}$ respectively.

4. Main Results

Proposition 4.1.
\[ η(Γ^2_{fπ}) = Γ^2_{fπ}. \]
Proof. Let \( x = (x_{mn}) \in \eta \left( \Gamma^2_{f\pi} \right) \). Then \( y \in \Gamma^2_{f\pi} \) so that for every \( \varepsilon > 0 \), we have a positive integer \( n_0 \) such that

\[
\left( f \left( \frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{\pi_{mn}P_{mn}} \right) \right) < \varepsilon^{m+n}, \forall m, n \geq n_0.
\]

Take \( p_{00} = 1, p_{11} = \cdots = p_{mn} = 0 \).

We then have

\[
\left( f \left( \frac{|x_{mn}|}{\pi_{mn}} \right) \right) < \varepsilon^{m+n}, \forall m, n \geq n_0.
\]

Therefore, \( x = (x_{mn}) \in \Gamma^2_{f\pi} \).

Hence

\[
\eta \left( \Gamma^2_{f\pi} \right) \subset \Gamma^2_{f\pi} \quad (1)
\]

On the other hand, let \( x = (x_{mn}) \in \Gamma^2_{f\pi} \).

But for any given \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that

\[
\left( f \left( \frac{|x_{mn}|}{\pi_{mn}} \right) \right) < \varepsilon^{m+n}, \forall m, n \geq n_0.
\]

We have

\[
\left( f \left( \frac{|y_{mn}|}{\pi_{mn}} \right) \right) \leq \left( f \left( \frac{p_{00}x_{mn} + \cdots + p_{mn}x_{00}}{P_{mn}\pi_{mn}} \right) \right)
\]

\[
\leq \frac{1}{P_{mn}} \left[ p_{00} \left( f \left( \frac{|x_{mn}|}{\pi_{mn}} \right) \right) + \cdots + p_{mn} \left( f \left( \frac{|x_{00}|}{P_{00}} \right) \right) \right]
\]

\[
\leq \frac{1}{P_{mn}} \left[ p_{00} \varepsilon^{m+n} + \cdots + p_{mn} \varepsilon^{0+0} \right]
\]

\[
\leq \frac{\varepsilon^{m+n}}{P_{mn}} (p_{00} + \cdots + p_{mn})
\]

\[
\leq \frac{\varepsilon^{m+n}}{P_{mn}} P_{mn} = \varepsilon^{m+n}, \forall m, n \geq n_0
\]

Therefore, \( (y_{mn}) \in \Gamma^2_{f\pi} \).

Consequently \( x \in \eta \left( \Gamma^2_{f\pi} \right) \). Hence

\[
\Gamma^2_{f\pi} \subset \eta \left( \Gamma^2_{f\pi} \right) .
\]

(2)

From (1) and (2) we obtain \( \eta \left( \Gamma^2_{f\pi} \right) = \Gamma^2_{f\pi} \). This completes the proof. \( \square \)

Proposition 4.2.

\[
\eta \left( \Lambda^2_{f\pi} \right) = \Lambda^2_{f\pi}
\]
Proof. Let \((x_{mn}) \in \Lambda_{f,\pi}^2\). Then there exists a constant \(T\) such that
\[
\left( f \left( \left| \frac{x_{mn}}{\pi_{mn}} \right| \right) \right) \leq T^{m+n} \text{ for } m, n = 0, 1, 2, \ldots
\]
\[
\left( f \left( \left| \frac{y_{mn}}{\pi_{mn}} \right| \right) \right) \leq \frac{p_{00} T^{m+n} + \cdots + p_{mn} T^{0+0}}{P_{mn}}
\]
\[
\leq T^{m+n} \left[ \frac{p_{00} + \cdots + \frac{p_{mn}}{T^{m+n}}}{P_{mn}} \right]
\]
\[
\leq \frac{T^{m+n}}{P_{mn}} \left[ p_{00} + \cdots + p_{mn} \right]
\]
\[
\leq \frac{T^{m+n}}{P_{mn}} P_{mn} = T^{m+n}, \text{ for } m, n = 0, 1, 2, \ldots.
\]
Hence \((y_{mn}) \in \Lambda_{f,\pi}^2\). But \(x = (x_{mn}) \in \eta \left( \Lambda_{f,\pi}^2 \right)\). Consequently,
\[
\Lambda_{f,\pi}^2 \subset \eta \left( \Lambda_{f,\pi}^2 \right).
\]
(3)

On the other hand, let
\[
(x_{mn}) \in \eta \left( \Lambda_{f,\pi}^2 \right). \text{ Then } (y_{mn}) \in \Lambda_{f,\pi}^2.
\]
Hence there exists a positive constant \(T\) such that
\[
\left( f \left( \left| \frac{y_{mn}}{\pi_{mn}} \right| \right) \right) < T^{m+n}, \text{ for } m, n = 0, 1, 2, \ldots.
\]
This in turn implies that
\[
\left( f \left( \left| \frac{p_{00} x_{mn} + \cdots + p_{mn} x_{00}}{P_{mn} \pi_{mn}} \right| \right) \right) < T^{m+n}.
\]
Hence
\[
\frac{1}{P_{mn}} \left( f \left( \left| \frac{p_{00} x_{mn} + \cdots + p_{mn} x_{00}}{\pi_{mn}} \right| \right) \right) < T^{m+n},
\]
and thus
\[
\left( f \left( \left| \frac{p_{00} x_{mn} + \cdots + p_{mn} x_{00}}{\pi_{mn}} \right| \right) \right) < P_{mn} T^{m+n}.
\]
Take \(p_{00} = 1; p_{11} = \cdots = p_{mn} = 0\). Then it follows that \(P_{mn} = 1\) and so
\[
\left( f \left| \frac{x_{mn}}{\pi_{mn}} \right| \right) < T^{m+n} \text{ for all } m, n.
\]
Consequently $x = (x_{mn}) \in \Lambda^2_{f \pi}$. Hence

$$\eta (\Lambda^2_{f \pi}) \subset \Lambda^2_{f \pi}.$$  \hspace{1cm} (4)

From (3) and (4), we get $\eta (\Lambda^2_{f \pi}) = \Lambda^2_{f \pi}$

This completes the proof. \qed

**Proposition 4.3.** $\Gamma^2_{f \pi}$ is not a barrelled space.

**Proof.** Let $A = \left\{ x \in \Gamma^2_{f \pi} : \left( f \left( \frac{x_{mn}}{\pi_{mn}} \right) \right) \leq 1, \forall m, n \right\}$.

Then $A$ is an absolutely convex, closed absorbent in $\Gamma^2_{f \pi}$.

But $A$ is not a neighbourhood of zero.

Hence $\Gamma^2_{f \pi}$ is not barrelled. \qed

**Proposition 4.4.** $\Gamma^2_{f \pi}$ is not semi-reflexive.

**Proof.** Let $\{\delta^{(mn)}\} \in U$ be the unit closed ball in $\Gamma^2_{f \pi}$. Clearly no subsequence of $\{\delta^{(mn)}\}$ can converge weakly to any $y \in \Gamma^2_{f \pi}$.

Hence $\Gamma^2_{f \pi}$ is not semi-reflexive. \qed

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**References**


