EXTENDING THE HALL-PORSCHING BOUNDS FOR THE PERRON ROOT

Jorma K. Merikoski
School of Information Sciences
FI-33014, University of Tampere
FINLAND

Abstract: Let $A \in \mathbb{R}^{n \times n}$ be nonnegative with Perron root $r$ and row sums $s_1, \ldots, s_n$, and denote $S = \max_i s_i$, $s = \min_i s_i$. We improve the Frobenius bounds $s \leq r \leq S$ by applying them to $DAD^{-1}$, where $D$ is obtained from the identity matrix $I$ by replacing its certain diagonal entries with a suitably chosen positive number. As a special case, in changing only one entry, we obtain the Hall–Porsching bounds.

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1. Introduction

Let $A = (a_{ik}) \in \mathbb{R}^{n \times n}$, $n \geq 2$, be nonnegative with Perron root $r$ and row sums $s_1, \ldots, s_n$, and denote $S = \max_i s_i$, $s = \min_i s_i$. (The column sums can be considered analogously.) There are many improvements of the Frobenius bounds

$$s \leq r \leq S.$$  \hfill (1)

For "classical" improvements, see [4, Section 3.1], [5, Section 2.1].
Hall and Porsching [3, Theorem 1] proved that, for all \( j = 1, \ldots, n \),

\[
\frac{1}{2} \min_{i \neq j} \left\{ s_i - a_{ij} + a_{jj} + \left[ (s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj}) \right]^{\frac{1}{2}} \right\} \leq r
\]

\[
\leq \frac{1}{2} \max_{i \neq j} \left\{ s_i - a_{ij} + a_{jj} + \left[ (s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj}) \right]^{\frac{1}{2}} \right\}.
\]

They assumed that \( A \) is irreducible, but (2) holds also in the reducible case by continuity. They also showed [3, Theorem 2] that these bounds, called ”HP bounds” in the sequel, improve those found by Brauer [1, Theorem 2].

Brauer’s method based on applying the Frobenius bounds (1) to \( DAD^{-1} \), where \( D \) is obtained from the identity matrix \( I \) by replacing its certain diagonal entry with a suitably chosen positive real number \( x \). However (apparently to simplify the results) he disposed the individual \( a_{ij} \)’s by suitable estimations. Hall’s and Porsching’s method was different. In

\[
r = \max_{0 < z \in \mathbb{R}^n} \max \{ \lambda \in \mathbb{R} | \lambda z \leq Az \} = \min_{0 < z \in \mathbb{R}^n} \min \{ \lambda \in \mathbb{R} | \lambda z \geq Az \}
\]

(see [2, p. 64]), they considered only such vectors \( z \) that are obtained from the vector \((1, \ldots, 1)\) by replacing its \( j \)’th entry (\( j \) fixed) with \( x \) (\( 0 < x \leq 1 \)), and optimized over \( x \).

We generalize Brauer’s method. Let \( 1 \leq h < n \), and change \( h \) diagonal entries of \( I \) into \( x(> 0) \), so obtaining \( D \). Since we will do all the computations under full information, it can be expected that in the case \( h = 1 \) or \( h = n - 1 \) we will improve Brauer’s bounds. Indeed, we will then obtain the Hall–Porsching bounds. So, we will provide an alternative proof and an extension of (2).

**2. The General Case**

Let us first assume that \( A \) is positive. Let \( 1 \leq h < n \). Consider a partition of \( N = \{1, \ldots, n\} \) into two sets \( I = \{i_1, \ldots, i_h\} \) and \( J = \{j_1, \ldots, j_{n-h}\} \). (That is, \( I, J \neq \emptyset \), \( I \cap J = \emptyset \), \( I \cup J = N \).) For \( x \in \mathbb{R} \) with \( x > 0 \), define \( D = \text{diag}(d_i) \) by \( d_{i_1} = \cdots = d_{i_h} = x \) and \( d_{j_1} = \cdots = d_{j_{n-h}} = 1 \) otherwise. Also the matrix \( B = DAD^{-1} \) has Perron root \( r \). If \( i \in I \), then the \( i \)’th row sum of \( B \) is

\[
\sigma_i = \sum_{k \in I} a_{ik} + \left( \sum_{k \in J} a_{ik} \right) x.
\]

If \( j \in J \), the \( j \)’th row sum of \( B \) is

\[
\tau_j = \left( \sum_{k \in I} a_{jk} \right) x^{-1} + \sum_{k \in J} a_{jk}.
\]
We study first the lower bound. We require that the set \( \{ \sigma_i \mid i \in I \} \) contains the smallest row sum of \( B \). This happens if, for all \( j \in J \), there exists \( i \in I \) such that

\[
\sum_{k \in I} a_{ik} + \left( \sum_{k \in J} a_{ik} \right) x \leq \left( \sum_{k \in I} a_{jk} \right) x^{-1} + \sum_{k \in J} a_{jk}. \tag{3}
\]

Then, applying (1) to \( B \),

\[
\min_{i \in I} \sigma_i \leq r. \tag{4}
\]

Fix \( j \in J \). Write (3) as

\[
\left( \sum_{k \in J} a_{ik} \right) x^2 + \left( \sum_{k \in I} a_{ik} - \sum_{k \in J} a_{jk} \right) x - \sum_{k \in I} a_{jk} \leq 0; \tag{5}
\]

then

\[
(0 <) x \leq e_{ij} = \frac{1}{2} \left( \sum_{k \in J} a_{ik} \right)^{-1} \left\{ \sum_{k \in J} a_{jk} - \sum_{k \in I} a_{ik} + \left[ \left( \sum_{k \in J} a_{jk} - \sum_{k \in I} a_{ik} \right)^2 + 4 \left( \sum_{k \in J} a_{ik} \right) \left( \sum_{k \in J} a_{jk} \right) \right]^{\frac{1}{2}} \right\}. \tag{6}
\]

Optimally

\[
x = e_j = \max_{i \in I} e_{ij}
\]

for this particular \( j \) and

\[
x = e = \min_{j \in J} e_j = \min_{j \in J} \max_{i \in I} e_{ij}
\]

for all \( j \in J \). Then (4) reads

\[
r \geq \min_{i \in I} \left[ \sum_{k \in I} a_{ik} + \left( \sum_{k \in J} a_{ik} \right) e \right] = \\
\min_{i \in I} \left( \sum_{k \in I} a_{ik} + \frac{1}{2} \min_{j \in J} \max_{i \in I} \left\{ \sum_{k \in J} a_{jk} - \sum_{k \in I} a_{ik} + \left[ \left( \sum_{k \in J} a_{jk} - \sum_{k \in I} a_{ik} \right)^2 + 4 \left( \sum_{k \in J} a_{ik} \right) \left( \sum_{k \in J} a_{jk} \right) \right]^{\frac{1}{2}} \right\} \right). \tag{6}
\]
As to the upper bound, we proceed similarly. We state that the set \( \{ \tau_j \mid j \in J \} \) contains the largest row sum of \( B \). That is, for all \( i \in I \), there exists \( j \in J \) such that (3) holds. Then

\[
r \leq \max_{j \in J} \tau_j, \tag{7}
\]

Fix \( i \in I \). Writing (5) as

\[
(\sum_{k \in I} a_{jk}) x^{-2} + \left( \sum_{k \in J} a_{jk} - \sum_{k \in I} a_{ik} \right) x^{-1} - \sum_{k \in J} a_{ik} \geq 0,
\]

we have

\[
x^{-1} \geq f_{ij} = \frac{1}{2} \left( \sum_{k \in I} a_{jk} \right)^{-1} \left\{ \sum_{k \in I} a_{ik} - \sum_{k \in J} a_{jk} + \left[ \left( \sum_{k \in I} a_{ik} - \sum_{k \in J} a_{jk} \right)^2 + 4 \left( \sum_{k \in J} a_{ik} \right) \left( \sum_{k \in I} a_{jk} \right) \right]^{\frac{1}{2}} \right\}.
\]

Optimally

\[
x^{-1} = f_i = \min_{j \in J} f_{ij}
\]

for this \( i \) and

\[
x^{-1} = f = \max_{i \in I} f_i = \max_{i \in I} \min_{j \in J} f_{ij}
\]

for all \( i \in I \). Then (7) reads

\[
r \leq \max_{j \in J} \left[ \left( \sum_{k \in I} a_{jk} \right) f + \sum_{k \in J} a_{jk} \right] = \max_{j \in J} \left( \sum_{k \in J} a_{jk} + \frac{1}{2} \max_{i \in I} \min_{j \in J} \left[ \sum_{k \in I} a_{ik} - \sum_{k \in J} a_{jk} + \left[ \left( \sum_{k \in I} a_{ik} - \sum_{k \in J} a_{jk} \right)^2 + 4 \left( \sum_{k \in J} a_{ik} \right) \left( \sum_{k \in I} a_{jk} \right) \right]^{\frac{1}{2}} \right] \right). \tag{8}
\]

By continuity, we can drop out the assumption on positivity. We have thus proved the following

**Theorem 1.** Inequalities (6) and (8) hold.
If $A$ is positive and the $s_i$'s are not all equal, then this theorem can always be applied to improve strictly the Frobenius bounds (1). Simply choose $I$ and $J$ so that $s_i < s_j$ for all $i \in I$, $j \in J$. Then all the $\sigma_i$'s increase and $\tau_j$'s decrease strictly with $x$, and $e > 1$. If $A$ has zero entries, then strict improvement does not necessarily happen (for a trivial counterexample, consider a diagonal matrix) but in most cases does.

3. The Special Cases $h = 1$ and $h = n - 1$

Let us consider the case $h = 1$. That is, we fix $i \in N$ and put $I = \{i\}$, $J = N \setminus \{i\}$. Then (6) simplifies into

$$r \geq a_{ii} + \frac{1}{2} \min_{\substack{1 \leq j \leq n \atop j \neq i}} \left\{ \sum_{k=1}^{n} a_{jk} - a_{ii} + \left[ \left( \sum_{k=1}^{n} a_{jk} - a_{ii} \right)^2 + 4 \left( \sum_{k=1}^{n} a_{ik} \right) a_{ji} \right]^{1/2} \right\} = a_{ii} + \frac{1}{2} \min_{\substack{1 \leq j \leq n \atop j \neq i}} \left\{ s_j - a_{ji} - a_{ii} + \left[ (s_j - a_{ji} - a_{ii})^2 + 4a_{ji}(s_i - a_{ii}) \right]^{1/2} \right\}$$

$$= \frac{1}{2} \min_{\substack{1 \leq j \leq n \atop j \neq i}} \left\{ s_j - a_{ji} + a_{ii} + \left[ (s_j - a_{ji} - a_{ii})^2 + 4a_{ji}(s_i - a_{ii}) \right]^{1/2} \right\}$$

for all $i = 1, \ldots, n$. Similarly, substitute $h = n - 1$ (i.e., fix $j \in N$ and put $J = \{j\}$, $I = N \setminus \{j\}$). Then (8) simplifies into

$$r \leq a_{jj} + \frac{1}{2} \max_{\substack{1 \leq i \leq n \atop i \neq j}} \left\{ \sum_{k=1}^{n} a_{ik} - a_{jj} + \left[ \left( \sum_{k=1}^{n} a_{ik} - a_{jj} \right)^2 + 4 \left( \sum_{k=1}^{n} a_{jk} \right) a_{ij} \right]^{1/2} \right\} = a_{jj} + \frac{1}{2} \max_{\substack{1 \leq i \leq n \atop i \neq j}} \left\{ a_{ii} - a_{i} - a_{jj} + \left[ (a_{ii} - a_{i} - a_{jj})^2 + 4a_{ij}(a_j - a_{jj}) \right]^{1/2} \right\}$$

$$= \frac{1}{2} \max_{\substack{1 \leq i \leq n \atop i \neq j}} \left\{ a_{ii} - a_{i} + a_{jj} + \left[ (a_{ii} - a_{i} - a_{jj})^2 + 4a_{ij}(a_j - a_{jj}) \right]^{1/2} \right\}$$
for all \( j = 1, \ldots, n \). The bounds (9) and (10) are just the HP bounds (2).

4. Examples

Example 1a. Let

\[
A = \begin{pmatrix}
2 & 1 & 3 & 4 \\
0 & 4 & 0 & 3 \\
2 & 0 & 4 & 0 \\
1 & 1 & 2 & 1
\end{pmatrix},
\]
cited from [3, p. 163]. Then \( r = 6.784 \). The HP bounds give

\[
6 \leq r \leq 7.123,
\]
see [3]. Do we find better bounds applying \( h = 2 \)? It is reasonable to put in \( I \) (respectively, in \( J \)) the two indices with smallest (largest) row sums. Since \( s_1 = 10, s_2 = 7, s_3 = 6, s_4 = 5 \), we set \( I = \{3, 4\}, J = \{1, 2\} \). For \( i = 3 \) and \( j = 1 \), we have

\[
\sum_{k \in J} a_{jk} = a_{31} + a_{32} = 3, \quad \sum_{k \in I} a_{ik} = a_{33} + a_{34} = 4,
\]
\[
\sum_{k \in J} a_{ik} = a_{31} + a_{32} = 2, \quad \sum_{k \in I} a_{jk} = a_{13} + a_{14} = 7,
\]
and so

\[
e_{31} = \frac{3 - 4 + \sqrt{(3 - 4)^2 + 4 \cdot 2 \cdot 7}}{2 \cdot 2} = \frac{\sqrt{57} - 1}{4} = 1.637.
\]
Similarly,

\[
e_{41} = \frac{\sqrt{14}}{2} = 1.871, \quad e_{32} = \frac{\sqrt{6}}{2} = 1.225, \quad e_{42} = \frac{3}{2}.
\]
Further \( e_1 = e_{41}, e_2 = e_{42} \), and so \( e = e_{42} \). The minimum of

\[
a_{33} + a_{34} + (a_{31} + a_{32})e = 4 + 2 \cdot \frac{3}{2} = 7
\]
and

\[
a_{43} + a_{44} + (a_{41} + a_{42})e = 3 + 2 \cdot \frac{3}{2} = 6
\]
gives, by (6), the lower bound 6.
To find the upper bound, we have

\[ f_{31} = \frac{4}{\sqrt{57} - 1} = 0.611, \quad f_{32} = \frac{2}{\sqrt{6}} = 0.816, \]

\[ f_{41} = \frac{2}{\sqrt{14}} = 0.534, \quad f_{42} = \frac{2}{3}, \]

and further \( f_3 = f_{31}, \ f_4 = f_{41}, \ f = f_{31} \). The maximum of

\[ (a_{13} + a_{14})f + a_{11} + a_{12} = 7 \cdot \frac{4}{\sqrt{57} - 1} + 3 = \frac{1}{2}(\sqrt{57} + 7) = 7.275 \]

and

\[ (a_{23} + a_{24})f + a_{21} + a_{22} = 3 \cdot \frac{4}{\sqrt{57} - 1} + 4 = \frac{1}{14}(59 + 3\sqrt{57}) = 5.832 \]

gives, by (8), the upper bound 7.275. In all,

\[ 6 \leq r \leq 7.275. \tag{12} \]

The lower bound is equal to that in (11) but the upper a little worse.

**Example 1b.** Changing the order of the rows does not effect on \( r \) but effects on the bounds discussed here. We look what happens if we reverse the order of the rows of \( A \). So let

\[
A = \begin{pmatrix}
1 & 1 & 2 & 1 \\
2 & 0 & 4 & 0 \\
0 & 4 & 0 & 3 \\
2 & 1 & 3 & 4
\end{pmatrix}.
\]

Now \( s_1 = 5, \ s_2 = 6, \ s_3 = 7, \ s_4 = 10 \). We apply first the HP bounds. Denote

\[ \lambda_{ij} = \frac{1}{2} \left\{ s_i - a_{ij} + a_{jj} + [(s_i - a_{ij} - a_{jj})^2 + 4a_{ij}(s_j - a_{jj})]^{\frac{1}{2}} \right\}; \]

then

\[ \lambda_{21} = 2 + \sqrt{10} = 5.162, \ \lambda_{31} = 7, \ \lambda_{41} = 9, \]

\[ \lambda_{12} = 2 + \sqrt{10} = 5.162, \ \lambda_{32} = \frac{1}{2}(3 + \sqrt{105}) = 6.623, \]

\[ \lambda_{42} = \frac{1}{2}(9 + \sqrt{105}) = 9.623, \]

\[ \lambda_{13} = \frac{1}{2}(3 + \sqrt{65}) = 5.531, \ \lambda_{23} = 1 + \sqrt{15} = 4.873, \]

\[ \lambda_{43} = \frac{1}{2}(7 + \sqrt{143}) = 9.479, \]
\[ \lambda_{14} = 4 + \sqrt{6} = 6.449, \; \lambda_{24} = 6, \; \lambda_{34} = 4 + 3\sqrt{2} = 8.243. \]

Hence, by (2),

\[ 6 \leq r \leq 8.243. \]

Compared with (11), the lower bound remains and the upper worsens.

Second, let us set \( h = 2, \; I = \{1, 2\}, \; J = \{3, 4\} \). Then

\[
\begin{align*}
e_{13} &= \frac{4}{3}, \quad e_{23} = \frac{1}{8}(\sqrt{65} + 1) = 1.133, \\
e_{14} &= \frac{1}{6}(\sqrt{61} + 5) = 2.135, \quad e_{24} = \frac{1}{8}(\sqrt{73} + 5) = 1.693, \\
f_{13} &= \frac{3}{4}, \quad f_{14} = \frac{1}{8}(\sqrt{61} - 5) = 0.468, \\
f_{23} &= \frac{1}{8}(\sqrt{65} - 1) = 0.883, \quad f_{24} = \frac{1}{8}(\sqrt{73} - 5) = 0.591, \\
e_3 &= e_{13}, \quad e_4 = e_{14}, \quad e = e_{13}, \quad f_1 = f_{14}, \quad e_2 = f_{24}, \quad f = f_{24}, \\
&\quad a_{11} + a_{12} + (a_{13} + a_{14})e = 6, \\
&\quad a_{21} + a_{22} + (a_{23} + a_{24})e = 7\frac{1}{3}, \\
&\quad (a_{31} + a_{32})f + a_{33} + a_{34} = \frac{1}{7}(2\sqrt{73} - 1) = 5.363, \\
&\quad (a_{41} + a_{42})f + a_{43} + a_{44} = \frac{1}{7}(\sqrt{73} + 9) = 8.772.
\]

Hence, by (6) and (8),

\[ 6 \leq r \leq 8.772. \]

Again, compared with (12), the lower bound remains and the upper worsens.

**Example 2a.** To give an example where the extension to \( h = 2 \) improves the HP bounds, consider

\[
A = \begin{pmatrix}
1 & 2 & 0 & 1 \\
2 & 0 & 2 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1
\end{pmatrix},
\]

again cited from [3, p. 163]. Then \( r = 3.508 \). The HP bounds only repeat the Frobenius bounds

\[ 3 \leq r \leq 4. \quad (13) \]

But setting \( h = 2, \; I = \{3, 4\}, \; J = \{1, 2\} \) yields

\[ e_{31} = e_{41} = e_1 = \frac{1}{2}(\sqrt{5} + 1) = 1.618, \]
\[ e_{32} = e_{42} = e = \sqrt{2} = 1.414, \]
\[ f_{31} = f_{41} = f_{3} = f_{4} = f = \frac{1}{2}(\sqrt{5} - 1) = 0.618, \]
\[ f_{32} = f_{42} = \frac{1}{2}\sqrt{2} = 0.707, \]
\[ a_{33} + a_{34} + (a_{31} + a_{32})e = a_{43} + a_{44} + (a_{41} + a_{42})e = 2 + \sqrt{2} = 3.414, \]
\[ (a_{13} + a_{14})f + a_{11} + a_{12} = \frac{1}{2}(5 + \sqrt{5}) = 3.618, \]
\[ (a_{23} + a_{24})f + a_{21} + a_{22} = \sqrt{5} + 1 = 3.236. \]

So we get better bounds
\[ 3.414 \leq r \leq 3.618 \] (14)

by (6) and (8).

**Example 2b.** Again reversing the order of the rows, let

\[
A = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
2 & 0 & 2 & 0 \\
1 & 2 & 0 & 1
\end{pmatrix}.
\]

In computing the HP bounds, the crucial \( \lambda_{ij} \)'s are \( \lambda_{13} = \lambda_{23} = 2 + \sqrt{2} = 3.414, \lambda_{43} = 4 \), improving (13) into

\[ 3.414 \leq r \leq 4. \]

Finally, let us set \( h = 2, I = \{1, 2\}, J = \{3, 4\} \). Then \( e = \frac{1}{2}\sqrt{6}, f = \frac{1}{4}(\sqrt{17} - 1) \), and (6) and (8) give respectively the lower bound \( 1 + \sqrt{6} = 3.449 \) and the upper bound \( \frac{1}{2}(3 + \sqrt{17}) = 3.562 \). So

\[ 3.449 \leq r \leq 3.562, \]

which beats (14).

**References**


