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SOME CONVEX COMBINATION BOUNDS FOR ARITHMETIC AND THE SECOND SEIFFERT MEANS

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Abstract: We find the greatest value α_1 and the least value β_1 such that the double inequality

$$\alpha_1 C(a,b) + (1-\alpha_1)N(a,b) < A(a,b) < \beta_1 C(a,b) + (1-\beta_1)N(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. We also find the estimate for the least value α_2 and the greatest value β_2 such that the double inequality

$$\alpha_2 C(a,b) + (1-\alpha_2)N(a,b) < T(a,b) < \beta_2 C(a,b) + (1-\beta_2)N(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. Here $C(a,b)$, $N(a,b)$, $A(a,b)$ and $T(a,b)$ denote the contraharmonic, square-root, arithmetic, and the second Seiffert means of two positive numbers a and b, respectively.

AMS Subject Classification: 26D15
Key Words: convex combination bound, arithmetic mean, contraharmonic mean, square-root mean, the second Seiffert mean

1. Introduction

For $a, b > 0$ with $a \neq b$, the first and the second Seiffert means $P(a,b)$ and

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$T(a,b)$ were introduced by Seiffert [1,2] as follows:

$$P(a,b)=\frac{a-b}{4\arctan(\sqrt{a/b})-\pi}=\frac{a-b}{2\arcsin\frac{a-b}{a+b}},\ T(a,b)=\frac{a-b}{2\arctan\frac{a-b}{a+b}}. \quad (1.1)$$

Recently, the inequalities for means have been the subject of intensive research [3-12]. In particular, many remarkable inequalities for the first and the second Seiffert mean can be found in the literature [13-19].

Let $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $C(a,b)=(a^2+b^2)/(a+b)$, $N(a,b)=(\sqrt{a}+\sqrt{b})^2/4$ and $A(a,b)=(a+b)/2$ be the identric, contraharmonic, square-root and arithmetic means of two positive real numbers a and b with $a\neq b$. Then

$$\min\{a,b\}<N(a,b)<I(a,b)<A(a,b)<T(a,b)<C(a,b)<\max\{a,b\}. \quad (1.2)$$

In [3], Seiffert proved

$$P(a,b)>\frac{3A(a,b)G(a,b)}{A(a,b)+2G(a,b)} \text{ and } P(a,b)>\frac{2}{\pi}A(a,b),$$

for all $a,b>0$ with $a\neq b$, where $G(a,b)=\sqrt{ab}$ be the geometric mean for $a,b>0$.

In [4], the authors found the greatest value α and the least value β such that the double inequality $\alpha A(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta A(a,b)+(1-\beta)H(a,b)$ holds for all $a,b>0$ with $a\neq b$, where $H(a,b)=2ab/(a+b)$ be the harmonic mean for $a,b>0$.

In [5], Alzer and Qiu established

$$\alpha A(a,b)+(1-\alpha)G(a,b)<I(a,b)<\beta A(a,b)+(1-\beta)G(a,b)$$

for $\alpha\le 2/3$, $\beta\ge 2/e$ and $a,b>0$ with $a\neq b$.

In the present paper, we find the greatest value α_1 and the least value β_1 such that the double inequality

$$\alpha_1C(a,b)+(1-\alpha_1)N(a,b)<A(a,b)<\beta_1C(a,b)+(1-\beta_1)N(a,b)$$

holds for all $a,b>0$ with $a\neq b$. We also find the estimate for the least value α_2 and the greatest value β_2 such that the double inequality

$$\alpha_2C(a,b)+(1-\alpha_2)N(a,b)<T(a,b)<\beta_2C(a,b)+(1-\beta_2)N(a,b)$$

holds for all $a,b>0$ with $a\neq b$.

2. Main Results

The first result in this paper is an optimal convex combination bounds of the contraharmonic and square-root means for arithmetic mean.

Theorem 2.1. *The double inequality*

$$\alpha_1 C(a,b) + (1-\alpha_1)N(a,b) < A(a,b) < \beta_1 C(a,b) + (1-\beta_1)N(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leqslant \frac{1}{3}$ and $\beta_1 = 1$.

Proof. For all $a, b > 0$ with $a \neq b$, the inequality

$$A(a,b) < C(a,b), \tag{2.1}$$

follows from (1.2).

We firstly prove the inequality

$$A(a,b) > \frac{1}{3}C(a,b) + \frac{2}{3}N(a,b), \tag{2.2}$$

for all $a, b > 0$ with $a \neq b$. Without loss of generality, we assume $a > b$. Let $p = \frac{1}{3}$ and $t^2 = \frac{a}{b} > 1$. Then (1.1) leads to

$$\begin{aligned} & \frac{1}{b}\left\{A(a,b) - [pC(a,b) + (1-p)N(a,b)]\right\} \\ = \quad & A(t^2,1) - [pC(t^2,1) + (1-p)N(t^2,1)] = \frac{f(t)}{4(t^2+1)}, \end{aligned} \tag{2.3}$$

where

$$f(t) = (3-5p)t^4 + 2(1-p)t^3 + 2(3-p)t^2 + 2(1-p)t + 3 - 5p. \tag{2.4}$$

Since $p = \frac{1}{3}$, we clearly see that $f(t) > 0$ for $t > 1$. Thus, (2.2) follows from (2.3).

Secondly, we prove that $\frac{1}{3}C(a,b) + \frac{2}{3}N(a,b)$ is the best possible lower convex combination bound of the contraharmonic and square-root means for arithmetic mean.

If $\alpha_1 > \frac{1}{3}$, then

$$\lim_{t \to +\infty} \frac{\alpha_1 C(t,1) + (1-\alpha_1)N(t,1)}{A(t,1)} = \frac{1+3\alpha_1}{2} > 1,$$

thus there exists $T_1 = T_1(\alpha_1) > 1$, such that

$$\alpha_1 C(t,1) + (1-\alpha_1)N(t,1) > A(t,1)$$

for $t \in (T, +\infty)$.

Finally, we prove that $1 \cdot C(a,b) + 0 \cdot N(a,b)$ is the best possible upper convex combination bound of the contraharmonic and square-root means for arithmetic mean.

If $\beta_1 < 1$, then from (2.4) (with β_1 in place of p) one has

$$\lim_{t \to 1^+} f(t) = 16(1 - \beta_1) > 0.$$

From this result and the continuity of $f(t)$, we derive that there exists $\delta_1 = \delta_1(\beta_1)$ such that $f(t) > 0$ for $t \in (1, 1 + \delta_1)$. Thus

$$A(t^2, 1) > \beta_1 C(t^2, 1) + (1 - \beta_1) N(t^2, 1)$$

for $t \in (1, 1 + \delta)$. □

The second result in this paper is convex combination bounds of the contraharmonic and square-root means for the second Seiffert mean.

Theorem 2.2. *The double inequality*

$$\alpha_2 C(a,b) + (1 - \alpha_2) N(a,b) < T(a,b) < \beta_2 C(a,b) + (1 - \beta_2) N(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if $\alpha_2 \leqslant \frac{\sqrt{202801} - 371}{180} = 0.440746\cdots$ and $\beta_2 \geqslant \frac{8-\pi}{3\pi}$. The number $\beta_2 = \frac{8-\pi}{3\pi}$ is optimal.

Proof. Firstly, we prove

$$T(a,b) < \frac{8 - \pi}{3\pi} C(a,b) + \frac{4(\pi - 2)}{3\pi} N(a,b) \tag{2.5}$$

and

$$T(a,b) > \frac{\sqrt{202801} - 371}{180} C(a,b) + \frac{551 - \sqrt{202801}}{180} N(a,b) \tag{2.6}$$

for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume $a > b$. Let $t^2 = \sqrt{a/b} > 1$ and $p \in \{\frac{\sqrt{202801} - 371}{180}, \frac{8-\pi}{3\pi}\}$. Then (1.1) leads to

$$\begin{aligned} &\frac{1}{b} \{T(a,b) - [pC(a,b) + (1-p)N(a,b)]\} \\ &\qquad = T(t^2, 1) - [pC(t^2, 1) + (1-p)N(t^2, 1)] \\ &= \frac{(1+3p)t^4 + 2(1-p)t^3 + 2(1-p)t^2 + 2(1-p)t + 1 + 3p}{4(t^2+1)\arctan\frac{t^2+1}{t^2-1}} h(t), \end{aligned} \tag{2.7}$$

where

$$h(t) = \frac{2(t^4-1)}{(1+3p)t^4+2(1-p)t^3+2(1-p)t^2+2(1-p)t+1+3p} - \arctan\frac{t^2-1}{t^2+1}. \quad (2.8)$$

Simple computations lead to

$$\lim_{t\to 1^+} h(t) = 0, \quad \lim_{t\to +\infty} h(t) = \frac{2}{1+3p} - \frac{\pi}{4}, \quad (2.9)$$

$$h'(t) = \frac{\psi(t)}{(t^4+1)[(1+3p)t^4+2(1-p)t^3+2(1-p)t^2+2(1-p)t+1+3p]^2}$$
$$= \frac{2(t-1)^2 g(t)}{(t^4+1)[(1+3p)t^4+2(1-p)t^3+2(1-p)t^2+2(1-p)t+1+3p]^2}, \quad (2.10)$$

where

$$\psi(t) = 4(t^4+1)\left[(1-p)t^6+2(1-p)t^5+3(1-p)t^4+4(1+3p)t^3 + 3(1-p)t^2+2(1-p)t+1-p\right] - 2t[(1+3p)t^4+2(1-p)t^3+2(1-p)t^2+2(1-p)t+1+3p]^2$$

and

$$g(t) = 2(1-p)t^8+(-9p^2-14p+7)t^7+2(-3p^2-20p+7)t^6 + (5p^2-42p+21)t^5+4(5p^2-11p+6)t^4+(5p^2-42p+21)t^3 + 2(-3p^2-20p+7)t^2+(-9p^2-14p+7)t+2(1-p). \quad (2.11)$$

Some tedious, but not difficult, calculations lead to

$$\lim_{t\to 1^+} g(t) = 16(7-15p), \quad \lim_{t\to +\infty} g(t) = +\infty. \quad (2.12)$$

$$g'(t) = 16(1-p)t^7+7(-9p^2-14p+7)t^6+12(-3p^2-20p+7)t^5 + 5(5p^2-42p+21)t^4+16(5p^2-11p+6)t^3+3(5p^2-42p+21)t^2 + 4(-3p^2-20p+7)t+(-9p^2-14p+7), \quad (2.13)$$

$$\lim_{t\to 1^+} g'(t) = 64(7-15p), \lim_{t\to +\infty} g'(t) = +\infty, \quad (2.14)$$

$$g''(t) = 2\left[56(1-p)t^6 + 21(-9p^2 - 14p + 7)t^5 + 30(-3p^2 - 20p + 7)t^4 + 10(5p^2 - 42p + 21)t^3 + 24(5p^2 - 11p + 6)t^2 + 3(5p^2 - 42p + 21)t + 2(-3p^2 - 20p + 7)\right]. \quad (2.15)$$

$$\lim_{t\to 1^+} g''(t) = 8(-25p^2 - 450p + 211), \lim_{t\to +\infty} g''(t) = +\infty, \quad (2.16)$$

$$g'''(t) = 6\left[112(1-p)t^5 + 35(-9p^2 - 14p + 7)t^4 + 40(-3p^2 - 20p + 7)t^3 + 10(5p^2 - 42p + 21)t^2 + 16(5p^2 - 11p + 6)t + (5p^2 - 42p + 21)\right]. \quad (2.17)$$

$$\lim_{t\to 1^+} g'''(t) = 24(-75p^2 - 510p + 241), \lim_{t\to +\infty} g'''(t) = +\infty, \quad (2.18)$$

$$g^{(4)}(t) = 24\left[140(1-p)t^4 + 35(-9p^2 - 14p + 7)t^3 + 30(-3p^2 - 20p + 7)t^2 + 5(5p^2 - 42p + 21)t + 4(5p^2 - 11p + 6)\right]. \quad (2.19)$$

$$\lim_{t\to 1^+} g^{(4)}(t) = 96(-90p^2 - 371p + 181), \lim_{t\to +\infty} g^{(4)}(t) = +\infty, \quad (2.20)$$

$$g^{(5)}(t) = 120\left[112(1-p)t^3 + 21(-9p^2 - 14p + 7)t^2 + 12(-3p^2 - 20p + 7)t + (5p^2 - 42p + 21)\right]. \quad (2.21)$$

$$\lim_{t\to 1^+} g^{(5)}(t) = 480(-55p^2 - 172p + 91), \lim_{t\to +\infty} g^{(5)}(t) = +\infty, \quad (2.22)$$

$$g^{(6)}(t) = 720\left[56(1-p)t^2 + 7(-9p^2 - 14p + 7)t + 2(-3p^2 - 20p + 7)\right] \quad (2.23)$$

$$\lim_{t\to 1^+} g^{(6)}(t) = 720(-69p^2 - 194p + 119), \lim_{t\to +\infty} g^{(6)}(t) = +\infty, \quad (2.24)$$

$$g^{(7)}(t) = 720\left[112(1-p)t + 7(-9p^2 - 14p + 7)\right] \quad (2.25)$$

$$\lim_{t\to 1^+} g^{(7)}(t) = 720(-63p^2 - 210p + 161), \lim_{t\to +\infty} g^{(7)}(t) = +\infty, \quad (2.26)$$

$$g^{(8)}(t) = 80640(1-p) > 0. \quad (2.27)$$

From (2.27) one can deduce that $g^{(7)}(t)$ is strictly increasing for $t > 1$. Now we distinguish between two cases.

Case 1 $p = \frac{\sqrt{202801}-371}{180}$. From (2.12), (2.14), (2.16), (2.18), (2.20), (2.22), (2.24) and (2.26), we derive

$$\lim_{t\to 1^+} g^{(i)}(t) > 0,$$

for $i = 0, 1, 2, 3, 5, 6, 7$ and

$$\lim_{t \to 1^+} g^{(4)}(t) = 0,$$

where we have denoted $g^{(0)}(t) = g(t)$.

Since $g^{(7)}(t)$ is strictly increasing for $t > 1$ and $\lim_{t \to 1^+} g^{(7)}(t) > 0$, then $g^{(7)}(t) > 0$ for $t > 1$. The same reasoning applied to the ith ($i = 6, 5, 4, 3, 2, 1, 0$) order derivative of $g(t)$ imply $g(t) > 0$ for $t > 1$. Thus, by (2.10), $h'(t) > 0$, which implies $h(t)$ is strictly increasing for $t > 1$. Notice the second equality in (2.9) becomes

$$\lim_{t \to +\infty} f(t) = \frac{120}{\sqrt{202801} - 311} - \frac{\pi}{4} > 0,$$

thus (2.6) follows from (2.7).

Case 2 $p = \frac{8-\pi}{3\pi}$. From (2.12), (2.14), (2.16), (2.18), (2.20), (2.22), (2.24) and (2.26), we have

$$\lim_{t \to 1^+} g(t) = \frac{64(3\pi - 10)}{\pi} < 0, \tag{2.28}$$

$$\lim_{t \to 1^+} g'(t) = \frac{256(3\pi - 10)}{\pi} < 0, \tag{2.29}$$

$$\lim_{t \to 1^+} g''(t) = \frac{64}{9\pi^2}\left(403\pi^2 - 1300\pi - 200\right) < 0, \tag{2.30}$$

$$\lim_{t \to 1^+} g'''(t) = \frac{64}{\pi^2}\left(151\pi^2 - 460\pi - 200\right) < 0, \tag{2.31}$$

$$\lim_{t \to 1^+} g^{(4)}(t) = \frac{128}{\pi^2}\left(221\pi^2 - 622\pi - 480\right) < 0, \tag{2.32}$$

$$\lim_{t \to 1^+} g^{(5)}(t) = \frac{2560}{3\pi^2}\left(80\pi^2 - 203\pi - 220\right) < 0, \tag{2.33}$$

$$\lim_{t \to 1^+} g^{(6)}(t) = \frac{3840}{\pi^2}\left(33\pi^2 - 74\pi - 92\right) > 0, \tag{2.34}$$

$$\lim_{t \to 1^+} g^{(7)}(t) = \frac{161280}{\pi^2}(\pi^2 - 2\pi - 2) > 0, \tag{2.35}$$

By (2.35) and the fact that $g^{(7)}(t)$ is strictly increasing, we know $g^{(7)}(t) > 0$ for $t > 1$. Thus $g^{(6)}(t)$ is strictly increasing for $t > 1$. By (2.34), $g^{(6)}(t) > 0$ for $t > 1$, thus $g^{(5)}(t)$ is strictly increasing for $t > 1$. By (2.33), there exists $\lambda_1 > 1$ such that $g^{(5)}(t) < 0$ for $t \in (1, \lambda_1)$ and $g^{(5)}(t) > 0$ for $t \in (\lambda_1, +\infty)$. Thus $g^{(4)}(t) < 0$ is strictly decreasing for $t \in (1, \lambda_1)$ and strictly increasing for $t \in (\lambda_1, +\infty)$. The same reasoning applied to $g^{(i)}(t)$ ($i = 4, 3, 2, 1, 0$), we can derive that there exists $\lambda_2 > 1$ such that $g(t) < 0$ for $t \in (1, \lambda_2)$ and $g(t) > 0$

for $t \in (\lambda_2, +\infty)$. Thus, by (2.10), $h'(t) < 0$ for $t \in (1, \lambda_2)$ and $h'(t) > 0$ for $t \in (\lambda_2, +\infty)$. Hence $h(t)$ is strictly decreasing for $t \in (1, \lambda_2)$ and strictly increasing for $t \in (\lambda_2, +\infty)$. Notice the second equality in (2.9) becomes

$$\lim_{t\to+\infty} h(t) = 0.$$

Therefore $h(t) < 0$, and (2.5) follows from (2.7).

Secondly, we prove that $\frac{8-\pi}{3\pi}C(a,b)+\frac{4(\pi-2)}{3\pi}N(a,b)$ is the best possible upper convex combination bound of the contraharmonic and square-root means for the second Seiffert mean.

If $\beta_2 < \frac{8-\pi}{3\pi}$, then from (1.1) one has

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\beta_2 C(t,1) + (1-\beta_2)N(t,1)}{T(t,1)} \\
&= \lim_{t\to+\infty} \frac{\left[4\beta_2(t^2+1) + (1-\beta_2)(\sqrt{t}+1)^2(t+1)\right]\arctan\frac{t-1}{t+1}}{2(t^2-1)} = \frac{\pi(3\beta_2+1)}{8} \\
&< 1. \qquad (2.36)
\end{aligned}$$

Inequality (2.36) implies that for any $\beta_2 < \frac{8-\pi}{3\pi}$ there exists $T_2 = T_2(\beta_2) > 1$ such that

$$\beta_2 C(t,1) + (1-\beta_2)G(t,1) < T(t,1)$$

for $t \in (T, +\infty)$. □

Remark Although we can not find the greatest value $(\alpha_2)_{\max}$ such that

$$(\alpha_2)_{\max} C(a,b) + (1-(\alpha_2)_{\max})N(a,b) < T(a,b)$$

holds for all $a, b > 0$ with $a \neq b$, we can give the estimate $(\alpha_2)_{\max} < \frac{7}{15} = 0.466\cdots$. In fact, if $(\alpha_2)_{\max} = \frac{7}{15}$, then from (2.12), (2.14) and (2.16), we have

$$\lim_{t\to1^+} g(t) = 0, \qquad (2.37)$$

$$\lim_{t\to1^+} g'(t) = 0, \qquad (2.38)$$

$$\lim_{t\to1^+} g''(t) = -\frac{320}{9} < 0. \qquad (2.39)$$

From (2.38), (2.39) and the continuity of $g''(t)$ we see that there exists $\delta_2 = \delta_2((\alpha_2)_{\max}) > 0$ such that $g'(t) < 0$ for $t \in (1, 1+\delta_2)$. Thus $g(t)$ is strictly decreasing for $t \in (1, 1+\delta_2)$. (2.37) and (2.10) imply that $h'(t) < 0$ for

$t \in (1, 1+\delta_2)$. Therefore, by the first equality of (2.9), $f(t) < 0$ for $t \in (1, 1+\delta_2)$. This implies, by (2.7), that

$$T(t^2, 1) < \frac{7}{15} C(t^2, 1) + \frac{8}{15} N(t^2, 1)$$

for $t \in (1, 1+\delta_2)$.

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