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SOME GLOBAL RESULTS USING BIFURCATION METHODS FOR SECOND ORDER m-POINT BOUNDARY VALUE PROBLEMS

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Abstract: Using bifurcation techniques, we investigate the global behavior of the components for a class of second order m-point boundary value problems. Our results improve those in the literature.

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1. Introduction

Consider the following m-point boundary value problems (BVP, for short)

$$\begin{cases} u''(t) + f(u(t)) = 0, \quad t \in (0,1); \\ u(0) = 0, u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases} \tag{1.1}$$

where $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, \cdots, m-2$ with $\sum\limits_{i=1}^{m-2} \alpha_i < 1$, and $f : \mathbf{R} \to \mathbf{R}$ is a given sign-changing continuous function. The methods we

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use are global results on bifurcation not only from the line of trivial solutions but also from infinity. We assume throughout this paper that the initial value problem

$$\begin{cases} u''(t) + f(u(t)) = 0, & t \in (0,1); \\ u(t_0) = u'(t_0) = 0 \end{cases}$$

has the unique trivial solution $u \equiv 0$ on $[0,1]$ for any $t_0 \in [0,1]$; in fact some suitable conditions such as a Lipschitz assumption or $f \in C^1$ guarantee this.

Multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. In recent years many authors have studied the existence and multiplicity of nontrivial solutions for multi-point boundary value problems, see [1, 2, 4, 8-11] and the references therein. For example, in [4], Ma and O'Regan use a global bifurcation method to investigate the multiplicity of nodal solutions for BVP(1.1). They also obtained some results on the spectrum of the linear operator corresponding to (1.1) under some technical hypotheses. In a recent paper, B. P. Rynne [8] extended and developed the results in [4] by dispensing with some of the technical hypotheses. He first showed that the spectral properties of the linearization of BVP(1.1) are similar to the well-known properties of the standard Sturm-Liouville problem with separated boundary conditions (with a modification to deal with the multi-point boundary condition). These spectral properties are then used to obtain the existence of nodal solutions. Recently using the method developed in [8], An and Ma [1] studied the global behavior of the components of nodal solutions for the eigenvalue problems (1.1) when there exist two constants $s_2 < 0 < s_1$ such that $f(s_1) = f(s_2) = f(0) = 0$. It should be pointed out that most of above mentioned papers use results on bifurcation that come from the line of trivial solutions ([9], Theorem A) and we note no use was made of global results on bifurcation from infinity.

To conclude this section we give some notation and state three lemmas, which will be used in Section 3. Following the notation of Rabinowitz, let E be a real Banach space and $L: E \to E$ be a compact linear map. If there exists $\mu \in \mathbf{R} = [0, +\infty)$ and $0 \neq v \in E$ such that $v = \mu L v$, μ is said to be a real characteristic value of L. The set of real characteristic values of L will be denoted by $\sigma(L)$. The multiplicity of $\mu \in \sigma(L)$ is

$$dim \bigcup_{j=1}^{\infty} N((I - \mu L)^j)$$

where $N(A)$ denotes the null space of A. Suppose that $H : \mathbf{R} \times E \to E$ is compact and $H(\lambda, u) = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals.

Then

$$u = \lambda Lu + H(\lambda, u) \tag{1.2}$$

possesses the line of trivial solutions $\Theta = \{(\lambda, 0)\big|\lambda \in \mathbf{R}\}$. It is well known that if $\mu \in \mathbf{R}$, a necessary condition for $(\mu, 0)$ to be a bifurcation point of (1.2) with respect to Θ is that $\mu \in \sigma(L)$. If μ is a simple characteristic value of L, let v denote the eigenvector of L corresponding to μ normalized so $\|v\| = 1$. By Σ we denote the closure of the set of nontrivial solutions of (1.2). A component of Σ is a maximal closed connected subset. It was shown in [Rabinowitz, [6], Theorem 1.3, 1.25, 1.27], that

Lemma 1.1. *If $\mu \in \sigma(L)$ is simple, then Σ contains a component C_μ which can be decomposed into two subcontinua C_μ^+, C_μ^- such that for some neighborhood B of $(\mu, 0)$,*

$$(\lambda, u) \in C_\mu^+(C_\mu^-) \cap B, \quad \text{and} \quad (\lambda, u) \neq (\mu, 0)$$

implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1)$, $\|w\| = o(|\alpha|)$ at $\alpha = 0$.

Moreover, each of C_μ^+, C_μ^- either

(i) meets infinity in Σ, or

(ii) meets $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in \sigma(L)$, or

(iii) contains a pair of points (λ, u), $(\lambda, -u)$, $u \neq 0$.

The following are global results for (1.2) on bifurcation from infinity [see, Rabinowitz, [7], Theorem 1.6 and Corollary 1.8].

Lemma 1.2. *Suppose L is compact and linear, $H(\lambda, u)$ is continuous on $\mathbf{R} \times E$, $H(\lambda, u) = o(\|u\|)$ at $u = \infty$ uniformly on bounded λ intervals, and $\|u\|^2 H(\lambda, \frac{u}{\|u\|^2})$ is compact. If $\mu \in \sigma(L)$ is of odd multiplicity, then Σ possesses an unbounded component D_μ which meets (μ, ∞). Moreover if $\Lambda \subset \mathbf{R}$ is an interval such that $\Lambda \cap \sigma(L) = \{\mu\}$ and $\wp$ is a neighborhood of (μ, ∞) whose projection on $\mathbf{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

(i) $D_\mu \backslash \wp$ is bounded in $\mathbf{R} \times E$ in which case $D_\mu \backslash \wp$ meets $\Theta = \{(\lambda, 0)\big|\lambda \in \mathbf{R}\}$ or

(ii) $D_\mu \setminus \wp$ is unbounded.

If (ii) occurs and $D_\mu \setminus \wp$ has a bounded projection on $\mathbf{R}$, then $D_\mu \setminus \wp$ meets $(\hat{\mu}, \infty)$ where $\mu \neq \hat{\mu} \in \sigma(L)$.

Lemma 1.3. *Suppose the assumptions of Lemma 1.2 hold. If $\mu \in \sigma(L)$ is simple, then D_μ can be decomposed into two subcontinua D_μ^+, D_μ^- and there exists a neighborhood $\Im \subset \wp$ of (μ, ∞) such that $(\lambda, u) \in D_\mu^+(D_\mu^-) \cap \Im$ and $(\lambda, u) \neq (\mu, \infty)$ implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1)$, $\|w\| = o(|\alpha|)$ at $|\alpha| = \infty$.*

2. Preliminaries

Let $X = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$, $Y = \{u \in C^1[0,1] : \ u(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$ with the norm $\|u\|_1 = \max\{\|u\|, \ \|u'\|\}$, $Z = \{u \in C^2[0,1] : \ u(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)\}$ with the norm $\|u\|_2 = \max\{\|u\|, \ \|u'\|, \|u''\|\}$. Then X, Y, and Z are Banach spaces.

We remark here that the conditions at 0 and 1 were inadvertently left out of Y in [4, 9]. Unfortunately in [4, 9] the sets $S_k^\pm$ are not open in Y. However the results there are easily corrected if one uses the sets $T_k^\pm$ below instead of $S_k^\pm$.

For any C^1 function u, if $u(t_0) = 0$, then t_0 is a simple zero of u if $u'(t_0) \neq 0$. For any integer $k \in \mathbf{N}$ and any $\nu \in \{\pm\}$, as in [8], define sets $T_k^\nu \subset Z$ consisting of the set of functions $u \in Z$ satisfying the following conditions:

(i) $u(0) = 0, \ \nu u'(0) > 0$ and $u'(1) \neq 0$;

(ii) u' has only simple zeros in $(0,1)$, and has exactly k such zeros;

(iii) u has a zero strictly between each two consecutive zeros of u'.

Note $T_k^- = -T_k^+$ and let $T_k = T_k^- \cup T_k^+$. It is easy to see that the sets T_k^- and T_k^+ are disjoint and open in Z. Moreover, if $u \in T_k^\nu$, then u has at least $k-1$ zeros in $(0, \ 1)$, and at most k zeros in $(0, \ 1]$.

Let $E = \mathbf{R} \times Y$ under the product topology. As in [7], we add the points $\{(\lambda, \ \infty) : \ \lambda \in \mathbf{R}\}$ to the space E. Let $\Phi_k^+ = \mathbf{R} \times T_k^+$, $\Phi_k^- = \mathbf{R} \times T_k^-$, and $\Phi_k = \mathbf{R} \times T_k$.

To obtain our results we need some information on the spectrum structure of the linear multi-point boundary value problem corresponding to (1.1)

$$\begin{cases} u''(t) + \lambda u(t) = 0, \quad t \in (0,1); \\ u(0) = 0, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases} \tag{2.1$_\lambda$}$$

Define an operator L on Y by

$$\begin{aligned}(Lu)(t) &= \frac{t}{1-\sum\limits_{i=1}^{m-2}\alpha_i\eta_i}\int_0^1(1-s)u(s)\mathrm{d}s-\int_0^t(t-s)u(s)\mathrm{d}s\\ &\quad -\frac{t}{1-\sum\limits_{i=1}^{m-2}\alpha_i\eta_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\eta_i}(\eta_i-s)u(s)\mathrm{d}s,\quad t\in[0,1].\end{aligned} \tag{2.2}$$

Then the following Lemma can be easily proved.

Lemma 2.1. *The linear operator $L : Y \to Y$ is completely continuous. Moreover, $(\lambda, u) \in (0, +\infty) \times C^2[0, 1]$ is a solution of (2.1)$_\lambda$ if and only if $(\lambda, u) \in E$ is a solution of the operator equation $u = \lambda L u$.*

Let the function $\Gamma(s)$ be defined by

$$\Gamma(s) = \sin s - \sum_{i=1}^{m-2}\alpha_i \sin \eta_i s, \quad s \in \mathbf{R}.$$

From Lemma 3.3, Lemma 3.8 (see also Theorem 4.2) and Theorem 3.1 of [8] we have the following lemma.

Lemma 2.2. *(i) For each $k \geq 1$, $\Gamma(s)$ has exactly one zero $s_k \in I_k := \Big(\Big(k - \frac{1}{2}\Big)\pi,\ \Big(k + \frac{1}{2}\Big)\pi\Big)$, so*

$$s_1 < s_2 < \cdots < s_k \to +\infty\ (k \to +\infty);$$

(ii) the characteristic value of L is exactly given by $\lambda_k = s_k^2$, $k = 1, 2, \dots,$ and the eigenfunction ϕ_k corresponding to λ_k is $\phi_k(t) = \sin s_k t$;

(iii) the algebraic multiplicity of each characteristic value λ_k of L is 1;

(iv) $\phi_k \in T_k^+$ for $k = 1, 2, 3, \cdots$, and ϕ_1 is strictly positive on $(0,\ 1)$.

The linear existence theory for BVP (2.1)$_\lambda$ can be stated as: for each integer $k \in \mathbf{N}$ and each $\nu = +$, or $-$, there exists one half line of solutions of BVP (2.1)$_\lambda$ in Φ_k^ν. They are of the form $(\lambda_k,\ \alpha\phi_k)$, $\alpha \in \mathbf{R}^\nu$. The half line joins $(\lambda_k,\ 0)$ to infinity in E (Here $\mathbf{R}^\nu = \{\lambda \in \mathbf{R}\big|\ 0 \leq \nu\lambda < +\infty\}$, $\nu = +$, or $-$).

3. Main Results

Now we list the following hypotheses for convenience.

(H1) There exists $f_0 \in (0, +\infty)$ such that

$$f(u) = f_0 u + o(|u|), \quad \text{as } |u| \to 0.$$

(H2) There exists $f_\infty \in (0, +\infty)$ satisfying

$$f(u) = f_\infty u + o(|u|), \quad \text{as } |u| \to +\infty.$$

(H3) There exists $R > 0$ such that

$$|f(u)| < \frac{R}{M}, \quad |u| \leq R,$$

where $M = \dfrac{3}{2(1 - \sum_{i=1}^{m-2} \alpha_i \eta_i)}$.

Now we are ready to give our main results.

Theorem 3.1. *Suppose (H1)-(H2) hold. Suppose there exist two positive integers $k_1 \leq k_2$ such that either*

$$\frac{\lambda_{k_2}}{f_0} < 1 < \frac{\lambda_{k_1}}{f_\infty}$$

or

$$\frac{\lambda_{k_2}}{f_\infty} < 1 < \frac{\lambda_{k_1}}{f_0}$$

holds. Then BVP(1.1) has at least $2(k_2 - k_1) + 2$ nontrivial solutions.

Theorem 3.2. *Suppose (H1)-(H3) hold. Suppose there exist two positive integers i_0 and j_0 such that $\lambda_{i_0} < f_0$ and $\lambda_{j_0} < f_\infty$. Then BVP(1.1) has at least $2(i_0 + j_0)$ solutions.*

To set it up using a bifurcation technique, we first consider the following nonlinear eigenvalue problem corresponding to BVP (1.1)

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0, 1); \\ u(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases} \tag{3.1$_\lambda$}$$

Define two operators F and A on Y by

$(Fu)(t) = f(u(t))$ and $(Au)(t) = (LFu)(t)$ for $t \in [0,1]$, respectively, where the operator L is defined as in (2.2).

A similar analysis as in Lemma 2.1 in this paper and Proposition 4.1 in [8] yield the following results.

Lemma 3.1. *The operator $A: Y \to Y$ is completely continuous and $(\lambda, u) \in (0,+\infty) \times C^2[0,1]$ is a solution of (3.1)$_\lambda$ if and only if $(\lambda, u) \in E$ is a solution of the operator equation*

$$u = \lambda A u. \tag{3.2}$$

Lemma 3.2. *Suppose that (H1) holds and (λ, u) is a solution of (3.1)$_\lambda$ and $u \not\equiv 0$. Then $u \in \cup_{i=1}^{\infty} T_i$.*

Under the condition (H1), (3.2) can be rewritten as

$$u = \lambda f_0 L u + H(\lambda, u), \tag{3.3}$$

here $H(\lambda, u) = \lambda A u - \lambda f_0 L u$ and L is defined as in (2.2). Obviously, by (H1) and the definition of the operator L, it can be seen that $H(\lambda, u)$ is $o(\|u\|_1)$ for u near 0 uniformly on bounded λ intervals. Notice that L is a compact linear map on Y. A solution of $(3.1)_\lambda$ is a pair $(\lambda, u) \in E$. By (H1), the known curve of solutions $\{(\lambda, 0)\big|\lambda \in \mathbf{R}\}$ will henceforth be referred to as the trivial solutions. The closure of the set on nontrivial solutions of $(3.1)_\lambda$ will be denoted by Σ as in Lemma 1.1.

If $H(\lambda, u) \equiv 0$, then (3.3) become a linear system

$$u = \lambda f_0 L u. \tag{3.4}$$

By Lemma 2.2, (3.4) possesses an increasing sequence of simple eigenvalues $0 < \dfrac{\lambda_1}{f_0} < \dfrac{\lambda_2}{f_0} < \cdots < \dfrac{\lambda_k}{f_0} < \dots$ with $\dfrac{\lambda_k}{f_0} \to +\infty$ as $k \to +\infty$. Any eigenfunction $\phi_k(t) = \sin s_k t$ corresponding to $\dfrac{\lambda_k}{f_0}$ is in T_k^+.

Lemma 3.3. *Assume (H1) holds. Then for each integer $k \in \mathbf{N}$ and each $\nu = +$, or $-$, there exists a continua C_k^ν of solutions of (3.1)$_\lambda$ in $\Phi_k^\nu \cup \{(\dfrac{\lambda_k}{f_0},\ 0)\}$, which meets $\{(\dfrac{\lambda_k}{f_0},\ 0)\}$ and ∞ in Σ.*

Proof. Consider (3.3) as a bifurcation problem from the trivial solution. From Lemma 1.1 and condition (H1) it follows that for each positive integer $k \in \mathbf{N}$, Σ contains a component $C_k \subseteq E = \mathbf{R} \times Y$ which can be decomposed into two subcontinua C_k^+, C_k^- such that for some neighborhood B of $(\frac{\lambda_k}{f_0},\ 0)$,

$$(\lambda, u) \in C_k^+(C_k^-) \cap B, \quad \text{and} \quad (\lambda, u) \neq (\frac{\lambda_k}{f_0},\ 0)$$

imply $(\lambda, u) = (\lambda, \alpha\phi_k + w)$, where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \frac{\lambda_k}{f_0}| = o(1)$, $\|w\|_1 = o(|\alpha|)$ at $\alpha = 0$.

By (3.2) and the continuity of the operator $A: \ Y \to Z$, the set C_k^ν lies in $\mathbf{R} \times Z$ and the injection $C_k^\nu \to \mathbf{R} \times Z$ is continuous. Thus, C_k^ν is also a continuum in $\mathbf{R} \times Z$, and the above properties hold in $\mathbf{R} \times Z$.

Since T_k is open in Z and $\phi_k \in T_k^+$, we know

$$\frac{u}{\alpha} = \phi_k + \frac{w}{\alpha} \in T_k^+$$

for $0 \neq \alpha$ sufficiently small. Then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have

$$(\lambda, u) \in \Phi_k, \quad \text{and} \quad (C_k \setminus \{(\frac{\lambda_k}{f_0},\ 0)\}) \cap B_\varepsilon \subset \Phi_k,$$

where B_ε is an open ball in $\mathbf{R} \times Z$ of radius ε centered at $(\frac{\lambda_k}{f_0},\ 0)$. It follows from the proof of [[8], Proposition 4.1] that

$$(\lambda, u) \in C_k \cap (\mathbf{R} \times \partial T_k) \Rightarrow u = 0,$$

which means $C_k \setminus \{(\frac{\lambda_k}{f_0},\ 0)\} \cap \partial\Phi_k = \emptyset$. Consequently, C_k lies in $\Phi_k \cup \{(\frac{\lambda_k}{f_0},\ 0)\}$.

Similarly we have that C_k^ν lies in $\Phi_k^\nu \cup \{(\frac{\lambda_k}{f_0},\ 0)\}$ $(\nu = +$, or $-)$.

Next we show alternative (ii) of Lemma 1.1 is impossible. If not, without loss of generality, assume that C_k^+ meets $(\frac{\lambda_i}{f_0}, 0)$ with $\lambda_k \neq \lambda_i \in \sigma(L)$. Then there exists a sequence $(\xi_m, z_m) \in C_k^+$ with $\xi_m \to \frac{\lambda_i}{f_0}$ and $z_m \to 0$ as $m \to +\infty$. Notice that

$$z_m = \xi_m f_0 L z_m + H(\xi_m, z_m). \tag{3.5}$$

Dividing this equation by $\|z_m\|_1$ and using Lemma 2.1 and $H(\xi_m, z_m) = o(\|z_m\|_1)$ as $m \to +\infty$, we may assume without loss of generality that $\frac{z_m}{\|z_m\|_1} \to \bar{z}$ as $m \to +\infty$. Thus from (3.5) it follows that

$$\bar{z} = \lambda_i L\bar{z}.$$

Since $\bar{z} \neq 0$, by Lemma 2.2, $\bar{z}$ belongs to T_i^+ or T_i^-. By (3.2) and the continuity of the operator $A: Y \to Z$, from $\|z_m - \bar{z}\|_1 \to 0$ it follows that $\|z_m - \bar{z}\|_2 \to 0$. Notice that T_i^+ and T_i^- are open in Z. Therefore, $z_m \in T_i^+$ or T_i^- for m sufficiently large, which is a contradiction with $z_m \in T_k^+$ $(m \geq 1)$, $i \neq k$. Hence alternative (ii) of Lemma 1.1 is not possible.

Finally it remains to show alternative (iii) of Lemma 1.1 is impossible. In fact, notice that $T_k^- = -T_k^+$, and $T_k^- \cap T_k^+ = \emptyset$. If $u \in T_k^+$, then $-u \in T_k^-$. This guarantees that C_k^ν does not contain a pair of points (λ, u), $(\lambda, -u)$, $u \neq 0$.

Therefore alternative (i) of Lemma 1.1 holds. This implies that for each integer $k \in \mathbf{N}$ and each $\nu = +$, or $-$, there exists a continua C_k^ν of solutions of $(3.1)_\lambda$ in $\Phi_k^\nu \cup \{(\frac{\lambda_k}{f_0},\ 0)\}$, which meets $\{(\frac{\lambda_k}{f_0},\ 0)\}$ and ∞ in Σ. □

Lemma 3.4. *Assume that (H1)-(H2) hold. Then for each integer $k \in \mathbf{N}$ and each $\nu = +$, or $-$, there exists a continua D_k^ν of solutions of $(3.1)_\lambda$ in $\Phi_k^\nu \cup \{(\frac{\lambda_k}{f_\infty},\ \infty)\}$ coming from $\{(\frac{\lambda_k}{f_\infty},\ \infty)\}$, which meets $(\frac{\lambda_k}{f_0}, 0)$ or has an unbounded projection on $\mathbf{R}$.*

Proof. Since (H2) is satisfied, (3.2) can be rewritten as

$$u = \lambda f_\infty \cdot Lu + \tilde{H}(\lambda, u), \tag{3.6}$$

here $\tilde{H}(\lambda, u) = \lambda Au - \lambda f_\infty \cdot Lu$ for $u \in Y$ and L is defined by (2.2).

Let $h(u) := f(u) - f_\infty u$. Then from (H2) it follows that $\lim\limits_{|u|\to\infty} \frac{h(u)}{|u|} = 0$. Define a function

$$\hat{h}(r) := \max\{|h(u)|:\ |u| \leq r\}.$$

Then $\hat{h}(r)$ is nondecreasing and

$$\lim_{r\to\infty} \frac{\hat{h}(r)}{r} = 0. \tag{3.7}$$

Consider (3.6) as a bifurcation problem from infinity. Obviously, by (3.7) and (2.2), it can be seen that $\tilde{H}(\lambda, u)$ is continuous on E and $\tilde{H}(\lambda, u) = o(\|u\|_1)$ for $u \in Y$ near ∞ uniformly on bounded λ intervals. Moreover a similar analysis as in the proof of Theorem 2.4 in [7] guarantees that $\|u\|_1^2\tilde{H}(\lambda, \frac{u}{\|u\|_1^2})$ is compact. From Lemma 2.2 we know λ_k is a simple characteristic value of L for each integer $k \in \mathbf{N}$. Thus by Lemma 1.2-1.3, Σ contains a component D_k which can be decomposed into two subcontinua D_k^+, D_k^- which meet $\{(\frac{\lambda_k}{f_\infty}, \infty)\}$.

Now we show that for a smaller neighborhood $\Im \subset \wp$ of $\{(\frac{\lambda_k}{f_\infty}, \infty)\}$, $(\lambda, u) \in D_k^+(D_k^-) \cap \Im$ and $(\lambda, u) \neq \{(\frac{\lambda_k}{f_\infty}, \infty)\}$ imply that $u \in T_k^+$ (T_k^-). In fact, by Lemma 1.3 we already know that there exists a neighborhood $\Im \subset \wp$ of $\{(\frac{\lambda_k}{f_\infty}, \infty)\}$ such that $(\lambda, u) \in D_k^+(D_k^-) \cap \Im$ and $(\lambda, u) \neq \{(\frac{\lambda_k}{f_\infty}, \infty)\}$ implies $(\lambda, u) = (\lambda, \alpha\phi_k + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \frac{\lambda_k}{f_\infty}| = o(1)$, $\|w\|_1 = o(|\alpha|)$ at $|\alpha| = \infty$.

As in the proof of Lemma 3.3, D_k^ν is also a continuum in $\mathbf{R} \times Z$, and the above properties hold in $\mathbf{R} \times Z$. Since T_k^ν is open in Z and $\frac{w}{\alpha}$ is smaller compared to $\phi_k \in T_k^+$ near $\alpha = +\infty$, $\phi_k + \frac{w}{\alpha}$ and therefore $u = \alpha\phi_k + w \in T_k^+$ for α near $+\infty$. Here and in the following the same argument works if $+$ is replaced by $-$.

Therefore, $D_k^+ \cap \Im \subset (\mathbf{R} \times T_k^+) \cup (\frac{\lambda_k}{f_\infty}, \infty)$. First suppose $D_k^+ \setminus \Im$ is bounded. Then there exists $(\lambda, u) \in \partial D_k^+$ with $u \in \partial T_k^+$. If $u \neq 0$, by Lemma 3.2 we know $u \in T_j^\nu$ for some positive integer $j \neq k$ and $\nu \in \{+, -\}$. As in the proof of Lemma 3.3, we get a contradiction, which means $u = 0$. Thus there exists a sequence $(\xi_m, z_m) \in D_k^+$ with $z_m \to u \equiv 0$ as $m \to +\infty$. This together with (H1) guarantees that (ξ_m, z_m) satisfies (3.5).

As in the proof of Lemma 3.3, we may assume without loss of generality that $\frac{z_m}{\|z_m\|_1} \to \bar{z}$ and $\xi_m \to \xi$ as $m \to +\infty$. Then we have

$$\bar{z} = \xi f_0 L \bar{z}. \tag{3.8}$$

Since $\bar{z} \neq 0$, $\xi f_0 \neq 0$ is an eigenvalue of operator L. From this and (3.8) it follows that $\xi f_0 = \lambda_j$ for some positive integer j. Then by Lemma 2.2, $\bar{z}$ belongs to T_j^+ or T_j^-. Notice that $\|z_m - \bar{z}\|_1 \to 0$ and so $\|z_m - \bar{z}\|_2 \to 0$ as in the proof of Lemma 3.3. Thus $z_m \in T_j^+$ or T_j^- for m sufficiently large since T_j^+ and T_j^- are open. This together with $z_m \in T_k^+$ $(m \geq 1)$ guarantees that $k = j$. This means D_k^+ meets $(\frac{\lambda_k}{f_0}, 0)$ if $D_k^+ \setminus \Im$ is bounded.

Next suppose $D_k^+ \setminus \Im$ is unbounded. In this case we show $D_k^+ \setminus \Im$ has an unbounded projection on $\mathbf{R}$. If not, then there exists a sequence $(\eta_m, y_m) \in D_k^+ \setminus \Im$ with $\eta_m \to \bar{\eta}$ and $\|y_m\|_1 \to +\infty$ as $m \to +\infty$. Let $x_m := \dfrac{y_m}{\|y_m\|_1}$, $m \geq 1$. From the fact that

$$y_m = \eta_m f_\infty L y_m + \tilde{H}(\eta_m, y_m)$$

it follows that

$$x_m = \eta_m f_\infty L x_m + \frac{\tilde{H}(\eta_m, y_m)}{\|y_m\|_1}. \tag{3.9}$$

Notice that $L: \; Y \to Y$ is completely continuous. We may assume that there exists $w \in Y$ with $\|w\|_1 = 1$ such that $\|x_m - w\|_1 \to 0$ as $m \to +\infty$.

Letting $m \to +\infty$ in (3.9) and noticing $\dfrac{\tilde{H}(\eta_m, y_m)}{\|y_m\|_1} \to 0$ as $m \to +\infty$ we have

$$w = \bar{\eta} f_\infty L w. \tag{3.10}$$

Since $w \neq 0$, $\bar{\eta} f_\infty \neq 0$ is an eigenvalue of operator L, that is, $\bar{\eta} = \dfrac{\lambda_j}{f_\infty}$ for some positive integer $j \neq k$. Then by Lemma 2.2, w belongs to T_j^+ or T_j^-. Recall also that $\|x_m - w\|_1 \to 0$ and so $\|x_m - w\|_2 \to 0$ as in the proof of Lemma 3.3. Thus $x_m \in T_j^+$ or T_j^- for m sufficiently large since T_j^+ and T_j^- are open. From $j \neq k$ we get a contradiction with $x_m \in T_k^+$ $(m \geq 1)$. Thus $D_k^+ \setminus \Im$ has an unbounded projection on $\mathbf{R}$. □

Proof of Theorem 3.1 Suppose first that

$$\frac{\lambda_{k_2}}{f_0} < 1 < \frac{\lambda_{k_1}}{f_\infty}.$$

Then from Lemma 2.2 we know

$$\frac{\lambda_{k_1}}{f_0} < \frac{\lambda_{k_1+1}}{f_0} < \cdots < \frac{\lambda_{k_2}}{f_0} < 1 < \frac{\lambda_{k_1}}{f_\infty} < \frac{\lambda_{k_1+1}}{f_\infty} < \cdots < \frac{\lambda_{k_2}}{f_\infty}.$$

Consider Eq.(3.3) as a bifurcation problem from the trivial solution. We need only show that

$$C^\nu_{k_1+j} \bigcap \Big(\{1\} \times Y\Big) \neq \emptyset, \quad j = 0, 1, 2, \cdots, k_2 - k_1; \; \nu = +, \; -. \tag{3.11}$$

Suppose, on the contrary and without loss of generality, that

$$C^+_{k_1+i} \bigcap \Big(\{1\} \times Y\Big) = \emptyset, \quad \text{for some } i, 0 \leq i \leq k_2 - k_1. \tag{3.12}$$

By Lemma 3.3 we know that $C^+_{k_1+i}$ joins $(\frac{\lambda_{k_1+i}}{f_0}, 0)$ to infinity in Σ and $(\lambda, u) = (0, 0)$ is the unique solution of Eq.$(3.2)_{\lambda=0}$ in E. This together with $\frac{\lambda_{k_1+i}}{f_0} < 1$ guarantees that there exists a sequence $\{(\eta_m,\ y_m)\} \subset C^+_{k_1+i}$ such that $\eta_m \in (0, 1)$ and $\|y_m\|_1 \to \infty$ as $m \to +\infty$. We may assume that $\eta_m \to \bar{\eta} \in [0, 1]$ as $m \to +\infty$. Let $x_m := \frac{y_m}{\|y_m\|_1}$, $m \geq 1$. Then (3.9) holds. Similarly, we may assume that there exists $w \in Y$ with $\|w\|_1 = 1$ such that $\|x_m - w\|_1 \to 0$ as $m \to +\infty$ and so (3.10) holds. From the proof of Lemma 3.4 one can see $\bar{\eta} = \frac{\lambda_{k_1+i}}{f_\infty}$, which contradicts $\frac{\lambda_{k_1+i}}{f_\infty} > 1$. Thus (3.12) is not true, which means (3.11) holds.

Next suppose that

$$\frac{\lambda_{k_2}}{f_\infty} < 1 < \frac{\lambda_{k_1}}{f_0}.$$

Then from Lemma 2.2 we know

$$\frac{\lambda_{k_1}}{f_\infty} < \frac{\lambda_{k_1+1}}{f_\infty} < \cdots < \frac{\lambda_{k_2}}{f_\infty} < 1 < \frac{\lambda_{k_1}}{f_0} < \frac{\lambda_{k_1+1}}{f_0} < \cdots < \frac{\lambda_{k_2}}{f_0}.$$

Consider Eq.(3.6) as a bifurcation problem from infinity. As above we need only prove that

$$D^\nu_{k_1+j} \bigcap \Big(\{1\} \times Y\Big) \neq \emptyset, \quad j = 0, 1, 2, \cdots, k_2 - k_1;\ \nu = +,\ -. \tag{3.13}$$

From Lemma 3.4, we know that $D^\nu_{k_1+j}$ comes from $\{(\frac{\lambda_{k_1+j}}{f_\infty}, \infty)\}$, meets $(\frac{\lambda_{k_1+j}}{f_0}, 0)$ or has an unbounded projection on $\mathbf{R}$. If it meets $(\frac{\lambda_{k_1+j}}{f_0}, 0)$, then the connectedness of $D^\nu_{k_1+j}$ and $\frac{\lambda_{k_1+j}}{f_0} > 1$ guarantee that (3.13) is satisfied. On the other hand, if $D^\nu_{k_1+j}$ has an unbounded projection on $\mathbf{R}$, notice that $(\lambda, u) = (0, 0)$ is the unique solution of Eq.$(3.2)_{\lambda=0}$ in E, so (3.13) also holds. □

Proof of Theorem 3.2 From (H3) we know there exists $\varepsilon > 0$ such that

$$(1 + \varepsilon)|f(u)| < \frac{R}{M}, \quad |u| \leq R. \tag{3.14}$$

Now we show

$$\Sigma \cap ([0, 1 + \varepsilon] \times \partial \overline{B}_R) = \emptyset \tag{3.15}$$

where $B_R = \{v \in Y \mid \|v\|_1 < R\}$ and $\overline{B}_R = \{v \in Y \mid \|v\|_1 \leq R\}$.

Suppose, on the contrary, (λ, u) is a solution of $(3.1)_\lambda$ such that $\lambda \in [0, 1+\varepsilon]$ and $\|u\|_1 = R$. Notice that

$$\sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s)\mathrm{d}s = \frac{1}{2}\sum_{i=1}^{m-2} \alpha_i \eta_i^2, \quad \text{and} \quad \sum_{i=1}^{m-2} \alpha_i \eta^2 < \sum_{i=1}^{m-2} \alpha_i \eta.$$

This together with (2.2), (3.2), and (3.14) guarantees that

$$\begin{aligned}
\|u\|_1 \leq &\max\Big\{\lambda \max_{t\in[0,1]} \Big\{\frac{t}{1-\sum\limits_{i=1}^{m-2}\alpha_i\eta_i} \int_0^1 (1-s)|f(u(s))|\mathrm{d}s + \int_0^t (t-s)|f(u(s))|\mathrm{d}s \\
&+ \frac{t}{1-\sum\limits_{i=1}^{m-2}\alpha_i\eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s)|f(u(s))|\mathrm{d}s\Big\}, \\
&\lambda \max_{t\in[0,1]} \Big\{\frac{1}{1-\sum\limits_{i=1}^{m-2}\alpha_i\eta_i} \int_0^1 (1-s)|f(u(s))|\mathrm{d}s + \int_0^t |f(u(s))|\mathrm{d}s \\
&+ \frac{1}{1-\sum\limits_{i=1}^{m-2}\alpha_i\eta_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\eta_i} (\eta_i - s)|f(u(s))|\mathrm{d}s\Big\}\Big\} \\
< &R,
\end{aligned}$$

which implies (3.15) holds.

On the other hand, from $\lambda_{i_0} < f_0$ and $\lambda_{j_0} < f_\infty$ we know

$$\frac{\lambda_1}{f_0} < \frac{\lambda_2}{f_0} < \frac{\lambda_3}{f_0} < \cdots < \frac{\lambda_{i_0}}{f_0} < 1 \tag{3.16}$$

and

$$\frac{\lambda_1}{f_\infty} < \frac{\lambda_2}{f_\infty} < \frac{\lambda_3}{f_\infty} < \cdots < \frac{\lambda_{j_0}}{f_\infty} < 1. \tag{3.17}$$

Using Lemma 3.3-3.4 and (3.15)-(3.17) we obtain

$$C_i^\nu \cap ([0, 1+\varepsilon] \times \overline{B}_R) \subset [0, 1+\varepsilon] \times B_R, \quad i = 1, 2, \cdots, i_0 \tag{3.18}$$

and

$$D_j^\nu \cap ([0, 1+\varepsilon] \times \partial\overline{B}_R) = \emptyset, \quad j = 1, 2, \cdots, j_0, \tag{3.19}$$

where $B_R = \{v \in Y \mid \|v\|_1 < R\}$ and $\overline{B}_R = \{v \in Y \mid \|v\|_1 \leq R\}$, C_k^ν and D_j^ν are obtained from Lemma 3.3 and Lemma 3.4, respectively.

Since C_i^ν is an unbounded component of solutions of $(3.2)_\lambda$ joining $\{(\frac{\lambda_i}{f_0}, 0)\}$ in E, it follows from (3.15), (3.16) and (3.18) that C_i^ν crosses the hyperplane $\{1\} \times Y(\nu = +$ or $-$, $i = 1, 2, \cdots, i_0)$. This means BVP(1.1) has $2i_0$ nontrivial solutions $\{u_i^\nu\}_1^{i_0}$ in which u_1^+ and u_1^- are positive and negative solutions, respectively.

On the other hand, by (3.15), (3.17), (3.19), and Lemma 3.4 one can obtain

$$D_j^\nu \cap \big(\{1\} \times (Y \backslash \overline{B}_R)\big) \neq \emptyset, \quad j = 1, 2, \cdots, j_0.$$

This means BVP(1.1) has $2j_0$ nontrivial solutions $\{y_i^\nu\}_1^{j_0}$ in which y_1^+ and y_1^- are positive and negative solutions, respectively. $\square$

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