International Journal of Pure and Applied Mathematics
Volume 81 No. 2 2012, 313-333
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

POSTULATION OF FINITE SETS IN $\mathbb{P}^3$ WITH $h^1(\mathcal{I}_S(m)) > 0$ AND $\sharp(S) \sim 4m$

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Fix an integer $m \ge 23$. Here we give the list of all finite sets $S \subset \mathbb{P}^3$ such that $\sharp(S) \le 4m$ and $h^1(\mathcal{I}_S(m)) > 0$. Fix an integer $\epsilon \ge 0$. We prove that if $m \gg \epsilon$, then there is no finite set $A \subset \mathbb{P}^3$ such that $\sharp(A) \le 4m+\epsilon$, $h^0(\mathcal{I}_A(2)) = 0$, $h^1(\mathcal{I}_A(m)) > 0$ and $h^1(\mathcal{I}_{A'}(m)) = 0$ for any $A' \subsetneq A$.

AMS Subject Classification: 14N05
Key Words: postulation, sets in a projective space, sets in a quadric surface

1. Introduction

In this paper we prove the following result.

Theorem 1. *Fix an integer $m \ge 23$ and a finite set $S \subset \mathbb{P}^3$ such that $\sharp(S) \le 4m$ and $\sharp(S \cap H) \le 4m-5$ for all planes $H \subset \mathbb{P}^3$. We have $h^1(\mathcal{I}_S(m)) = 0$ and $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$ if and only if S is as described in one of the following cases:*

(a) *$\sharp(S) = m+2$ and S is contained in a line;*

(b) *$\sharp(S) = 2m+2$ and S is contained in a reduced conic; if the conic is singular, say $L_1 \cup L_2$ with L_1 and L_2 lines, then $L_1 \cap L_2 \notin S$ and $\sharp(S \cap L_1) = \sharp(S \cap L_2) = m+1$;*

Received: June 22, 2012

(c) $\sharp(S) = 3m$ *and* S *is the complete intersection of a degree* 3 *plane curve and a degree* m *surface;*

(d) $\sharp(S) \geq 3m + 1$ *and* S *is contained in a degree* 3 *plane curve;*

(e) $\sharp(S) = 3m + 2$ *and* S *is contained in a reduced and connected degree* 3 *curve spanning* $\mathbb{P}^3$*;*

(f) $\sharp(S) = 4m$*, there are a plane* H*, a line* L *with* $L \cap H$ *a point,* P*, a degree* 3 *reduced curve* $T \subset H$ *such that* $P \in T \setminus S \cap T$*,* $\sharp(L \cap S) = m + 1$*,* $\sharp(S \cap T) = 3m - 1$ *and* $\{P\} \cup (S \cap T)$ *is the complete intersection of* T *and a degree* m *curve* $C \subset H$*.*

The assumption " $\sharp(S \cap H) \leq 4m - 5$ for all planes $H \subset \mathbb{P}^3$ " is just made to avoid a long (and easy) list (see [1], Proposition 1, for a proof that such a set S is contained in a plane and Lemma 1 for the list of all cases). Case (f) is the main result of this paper (it shows that a guess that we made in the introduction of [1] and then in [2] is wrong). See Example 1 for case (f) and Example 2 for a similar example with $\sharp(S) = 4m + 1$.

We also prove the following result.

Theorem 2. *Fix an integer* $\epsilon \geq 0$*. There is an integer* $\alpha(\epsilon) > 0$ *such that for all integers* $m \geq \alpha(\epsilon)$ *there is no* $S \subset \mathbb{P}^3$ *such that* $\sharp(S) \leq 4m + \epsilon$*,* $h^1(\mathcal{I}_S(m)) > 0$*,* $h^1(\mathcal{I}_{S'}(m)) = 0$ *for all* $S' \subsetneq S$ *and* $h^0(\mathcal{I}_S(2)) = 0$*.*

Question 1. Are Theorems 1 and/or 2 true for zero-dimensional schemes which are not reduced ?

2. The Proofs

Example 1. Let $H \subset \mathbb{P}^3$ be a plane. Fix a reduced degree 3 curve $T \subset H$. Fix $P \in T_{reg}$ and take a degree m curve $T_1 \subset H$ containing P and intersecting T outside P at $3m-1$ further points. Since $m \geq 3$, the linear system $|\mathcal{I}_{\{P\}}(m)|$ is very ample outside P and it separates the tangent vectors at P. Hence we may take as T_1 a general element of $|\mathcal{I}_{\{P\}}(m)|$ (use that $P \notin \mathrm{Sing}(T)$). Set $S_1 := T \cap T_1 \setminus \{P\}$. Take any line $R \subset \mathbb{P}^3$ such that $R \cap H = \{P\}$ and any $S_2 \subset R \setminus \{P\}$ such that $\sharp(S_2) = m+1$. Set $S := S_1 \cup S_2$. We have $\sharp(S) = 4m$.

Claim 1. $h^1(\mathcal{I}_S(m)) > 0$.

Proof of Claim 1. Fix $Q \in S_2$. Since $P \in R$, P is in the base locus of $|\mathcal{I}_{S_2}(m)|$. Hence $|\mathcal{I}_{S \cup \{P\}}(m)| = |\mathcal{I}_S(m)|$. Since $Q \in |\mathcal{I}_{(S_2 \setminus \{Q\} \cup \{P\}}|$, we have

$h^0(\mathcal{I}_S(m)) = h^0(\mathcal{I}_{(S\setminus\{Q\})\cup\{P\}}(m))$. Since $\sharp(S) = \sharp(S \setminus \{Q\}) \cup \{P\})$, we get $h^1(\mathcal{I}_S(m)) = h^1(\mathcal{I}_{(S\setminus\{Q\})\cup\{P\}}(m))$. Since $h^1(H, \mathcal{I}_{S_1\cup\{P\}}(m)) > 0$, we get Claim 1.

Claim 2. *For each $S' \subsetneq S$ we have $h^1(\mathcal{I}_{S'}(m)) = 0$.*

Proof of Claim 2. It is sufficient to prove Claim 2 for all $S' \subset S$ such that $\sharp(S') = \sharp(S) - 1$. Fix $Q \in S_2$ and set $S' := S \setminus \{Q\}$. Since $\sharp(S_2 \setminus \{Q\}) = m$, we have $h^1(\mathcal{I}_{S_2\setminus\{Q\}}(m-1)) = 0$. Since $h^1(H, \mathcal{I}_{S_1}(m)) = 0$, it is sufficient to apply [1], Remark 1 (often called the Horace lemma or the Castelnuovo's exact sequence). Now fix $Q' \in S_1$ and set $S'' := S \setminus \{Q\}$. We have $h^1(\mathcal{I}_{S_2}(m)) = 0$. Hence to prove $h^1(\mathcal{I}_{S''}(m)) = 0$ (and hence to conclude the proof of Claim 2) it is sufficient to prove $h^1(T, \mathcal{I}_{S_1\setminus\{Q'),T}(m)) = 0$. Since $S_1 \cup \{P\}$ is a complete intersection and $\sharp(S_1 \cup \{P\}) = 3m$, $S_1 \cup \{P\} \subset T_{reg} \cap (T_1)_{reg}$. Hence $S_1 \setminus \{Q\} \subset (T_1)_{reg}$. If T is integral, then it is sufficient to use that $\deg(\mathcal{I}_{S_1\setminus\{Q'),T}(m)) = 2 > 0$. If T_1 is reducible, we also need to check that no line $L \subset T_1$ contains $m+1$ points of $S_1 \setminus \{Q\}$ and no conic $L' \subset T$ contains $2m+1$ points of $S_1 \setminus \{Q\}$. No line of T_1 contains at least $m+1$ points of S_1 and no conic of T contains at least $2m+1$ points of S_1, because S_1 is contained in the complete intersection of T_1 and a degree m curve. Notice that if a set A of $3m$ points of T is the complete intersection of T and a degree m curves, then $A \subset T_{reg}$. In particular the assumption $\{P\} \in T_{reg}$ is necessary for this example and we always have $S_1 \subset T_{reg}$.

In the same way we prove the following result.

Example 2. Let $H \subset \mathbb{P}^3$ be a plane. Fix a reduced degree 3 curve $T \subset H$. Fix $P \in T$ and a set $S_1 \subset T \setminus \{P\}$ such that $\sharp(S_1) = 3m$ and S_1 is not the complete intersection of T and a degree m curve. Take any line $R \subset \mathbb{P}^3$ such that $R \cap H = \{P\}$ and any $S_2 \subset R \setminus \{P\}$ such that $\sharp(S_2) = m+1$. Set $S := S_1 \cup S_2$. We have $\sharp(S) = 4m+1$, $h^1(\mathcal{I}_S(m)) > 0$ and $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$.

Lemma 1. *Let $H, M \subset \mathbb{P}^3$ be distinct planes. Fix an integer $m \geq 5$ and a finite set $S \subset H \cup M$ such that $h^1(\mathcal{I}_S(m)) > 0$ and $\sharp(S) \leq 4m$. Then one of the following cases occurs:*

(i) there is a line D contained in one of the planes H, M such that $\sharp(D \cap S) \geq m+2$;

(ii) there is a conic $D' \subset H \cup M$ such that $\sharp(D' \cap S) \geq 2m+2$; $S \cap D'$ contains a set as described in case (b) of Theorem 1;

(iii) *there is a cubic curve D'' contained in one of the planes H, M (say H) such that $S \cap D$ is the complete intersection of D'' and a degree m subcurve of H;*

(iv) *there is a cubic curve D_2 contained in one of the planes H, M and $\sharp(S \cap D_2) \geq 3m+1$;*

(v) *there is a quartic curve T contained in one of the planes H, M such that $\sharp(T \cap S) \geq 4m-4$ and $h^1(T, \mathcal{I}_{T\cap S,T}(4)) > 0$;*

(vi) *S is the complete intersection of two quadric surfaces and a surface of degree m;*

(vii) *$\sharp(S) = 4m$ and S is as in case (f) of Theorem 1.*

Proof. We may assume $\sharp(S \cap H) \geq \sharp(S \cap M)$. Since $\sharp(S) \leq 4m$, we have $\sharp(M \cap S) \leq 2m$ and hence $\sharp(S \setminus S \cap H) \leq 2m$. If $h^1(H, \mathcal{I}_{S\cap H}(m)) > 0$, then $S \subset H$: use [4], Corollaire 2. Now assume $h^1(\mathcal{I}_{H\cap S}(m)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus H\cap S}(m-1)) > 0$ (see [1], Remark 1). Hence $h^1(M, \mathcal{I}_{S\setminus S\cap H}(m-1)) > 0$. Since $\sharp(S \setminus S \cap H) \leq 2m$, either there is a line $D \subset M$ such that $\sharp(D \cap (S \setminus S \cap M)) \geq m+1$ or there is a conic $D' \subset M$ such that $\sharp(D' \cap (S \setminus S \cap M)) \geq 2m-1$. Since the lemma is true if $h^1(\mathcal{I}_{S\cap M}(m)) > 0$, we may also assume $h^1(\mathcal{I}_{S\cap M}(m)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap M}(m-1)) > 0$. Hence $h^1(H, \mathcal{I}_{S\setminus S\cap M}(m-1)) > 0$. Since $\sharp(S \setminus S \cap M) \leq 4m-m-1 = 3m-1$, either there is a line $R \subset H$ such that $\sharp(R \cap (S \setminus S \cap H)) \geq m+1$ or there is a conic $R' \subset H$ such that $\sharp(R' \cap (S \setminus S \cap H)) \geq 2m-1$ or there is a cubic $R_1 \subset H$ such that $R_1 \cap (S \setminus S \cap H)$ is the complete intersection of R_1 and a plane curve of degree $m-1$ (the latter case does not occur if there is the conic D').

(a) In this step we assume the existence of R. Since $R \cap (S \setminus H \cap S) \neq \emptyset$, we have $R \neq H \cap M$. Hence $R \cap H$ is a point, P. If $P \in S$, then $\sharp(R \cap S) \geq m+2$, because $\sharp(R \cap (S \setminus S \cap H)) \geq m+1$. Hence we may assume $P \notin S$.

(a1) Assume for the moment $h^1(\mathcal{I}_{\{P\}\cup(S\cap H)}(m)) > 0$. Since $\sharp(S \cap H) \cup \{P\}) \leq 4m-5$, there is a curve $T \subset H$ such that $\deg(T) \leq 3$ and

$$h^1(T, \mathcal{I}_{(S\cup\{P\})\cap T}(m)) > 0$$

(see [4], Corollaire 2). We also take T with minimal degree among such curves (it is the one coming from the proof of [4], Corollaire 2). Since $h^1(\mathcal{I}_{S\cap T}(m)) = 0$, we have $P \in T$. Hence $T \cup R$ is a connected curve of degree $\deg(T)+1$ containing $\sharp(S \cap T) + m + 1$ points of S. We get that we are in case (b) (resp. case (e)) of

Theorem 1 if $\deg(T) = 1$ (resp. $\deg(T) = 2$). Now assume $\deg(T) = 3$. In this case $\{P\} \cup S \cap H$ is as in case (f) of Theorem 1, because $\sharp(S \cap H) \le 3m - 1$.

(a2) From now on we assume $h^1(H, \mathcal{I}_{\{P\}\cup(S\cap H)}(m)) = 0$. Let $M' \subset \mathbb{P}^3$ be a general plane containing R. Since S is finite, we have $S \cap M' = S \cap R$.

Claim 1. $h^1(\mathcal{I}_{S\cap(H\cup M')}(m)) = 0$.

Proof of Claim 1. Claim 1 is equivalent to $h^0(\mathcal{I}_{S\cap(H\cup M')}(m)) = \binom{m+3}{3} - m - 1 - \sharp(S\cap H)$. Fix $P' \in S\cap R$ and write $S' := (S\cup H)\cup\{P\}\cup((S\cap R)\setminus\{P'\})$. Notice that $H^0(\mathcal{I}_{S\cap(H\cup M'}(m)) = H^0(\mathcal{I}_{S'}(m))$. Hence to prove the Claim it is sufficient to prove $h^1(\mathcal{I}_{S'}(m)) = 0$. Since $(S \cup H) \cup \{P\} = S' \cap H$, we have $h^1(H, \mathcal{I}_{S'\cap H}(m)) = 0$. Since $\sharp(S'\setminus S'\cap H) = m$, we have $h^1(\mathcal{I}_{S'\setminus S'\cap H}(m-1)) = 0$. Hence $h^1(\mathcal{I}_{S\cap(H\cup M')}(m)) = 0$ (see [1], Remark 1).

Since $\sharp(S\setminus S\cap(H\cup M')) \le 2m-m-1$, we have $h^1(\mathcal{I}_{S\setminus S\cap(H\cup M')}(m-2)) = 0$. Since $\deg(H \cup M') = 2$, the Claim and [1], Remark 1, give $h^1(\mathcal{I}_S(m)) = 0$, a contradiction.

(b) Now assume the existence of a conic $R' \subset M$ such that $\sharp(R' \cap (S \setminus S \cap H)) \ge 2m$. Since $\sharp(S \cap H) \ge \sharp(S \cap M)$, we get $\sharp(S) = 4m$, $\sharp(S \cap H) = \sharp(S \cap M) = 2m$, $S \cap M \subset R'$ and $S \cap H \cap M = \emptyset$.

(b1) First assume the existence of a line $D \subset H$ such that $\sharp(D\cap(S\setminus S\cap M)) \ge m + 1$. In particular $D \ne H \cap M$. Hence $D \cap H \cap M$ is a point (call it P_1). Since $P_1 \in H \cap M$, then $P_1 \notin S$. Excluding case (a) of Theorem 1 we may assume $\sharp(S \cap D) = m + 1$. Assume for the moment $D \cap R' \ne \emptyset$. In this case $D \cup R'$ is a connected degree 3 curve. If $\sharp(S \cap R') = 2m + 1$, then we are in case (e) of Theorem 1. Hence we may assume $\sharp(S \cap R) = 2m$. Since D is in the base locus of $\mathcal{I}_{S\cup D}(2)$, we see that $|\mathcal{I}_{S\cap(D\cup R')}(m)| = |\mathcal{I}_{D\cup R'}(m)|$. Hence we get $h^1(\mathcal{I}_{S\cap(D\cup R')}(m)) = 0$. In all cases (even when $D \cup R'$ is a cone with vertex P_1), the sheaf $\mathcal{I}_{D\cup R'}(2)$ is spanned. Hence there is a quadric $Q_2 \supset D \cup R'$ such that $Q_2 \cap S = Q_2 \cap (D \cup R') \cap S$. Since $h^1(\mathcal{I}_{S\cap(D\cup R')}(m)) = 0$, we have $h^1(Q_2, \mathcal{I}_{S\cap Q_2}(m)) = 0$. Since $\sharp(S) - \sharp(S \cap Q_2) \le 4m - 3m - 1 = m - 1$, we have $h^1(\mathcal{I}_{S\setminus S\cap Q_2}(m - 2)) = 0$. Hence $h^1(\mathcal{I}_S(m)) = 0$ (see [1], Remark 1), a contradiction. Now assume $D \cap R' = \emptyset$. The restriction map $\rho : H^0(D \cup R', \mathcal{O}_{D\cup R'}(m)) \to H^0(S\cap(D\cup R'), \mathcal{O}_{S\cap(D\cup R')}(m))$ is surjective. Since $\mathcal{I}_{D\cup R'}(3)$ is spanned, there is $T \in |\mathcal{I}_{D\cup R'}(3)|$ such that $T \cap S = S \cap (D \cup R')$. Since ρ is surjective, we have $h^1(\mathcal{I}_{S\cap(D\cup R')}(m)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap(D\cup R')}(m - 3)) > 0$. Since $\sharp(S\setminus S\cap(D\cup R')) = m-1$, there is a line $J \subset H$ such that $S\cap H \subset D\cup J$. Fix $P_3, P_4 \in D\cap S$ with $P_3 \ne P_4$. Since $P_1 \notin R'$, we have $h^0(M, \mathcal{I}_{\{P_1\}\cup R'}(2)) = 0$. Hence every quadric surface containing $R'\cup D$ is the union of M and a plane through D. Hence $h^0(\mathcal{I}_{D\cup R'}(2)) < h^0(\mathcal{I}_{\{P_3,P_4\}\cup R'}(2))$. Hence there is a quadric surface $Q_3 \subset \mathbb{P}^3$ such that $Q_3 \supset R'$ and $Q_3 \cap D = \{P_3, P_4\}$

Claim 2. *We have* $h^1(\mathcal{I}_{S\cap Q_3}(m)) = 0$.

Proof of Claim 2. We have $h^1(\mathcal{I}_{S\cap R'}(m)) = 0$, because $S \cap R' \neq S$. By [1], Remark 1, to prove Claim 2 it is sufficient to prove $h^1(\mathcal{I}_{S\cap Q_3\setminus S\cap R'}(m-1)) = 0$. This is true, because $P_3 \notin J$ and $S \cap Q_3 \setminus S \cap R'$ is the union of P_3, P_4 and at most $m-1$ points of the line J.

By Claim 2 we have $h^1(Q_3, \mathcal{I}_{S\cap Q_3}(m)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap Q_3}(m-2)) > 0$ (see [1], Remark 1). Since $S \setminus S \cap Q_3$ is the union of $m-1$ points on D and at most $m-1$ points on J, we have $h^1(\mathcal{I}_{S\setminus S\cap Q_3}(m-2)) = 0$ (see [3], Theorem 3.8), a contradiction.

(b2) Now assume the existence of a conic $D' \subset H$ such that $\sharp(D' \cap (S \setminus S \cap M)) \geq 2m$. Since $\sharp(S\cap H) = 2m$, we get $\sharp(S\cap D') = 2m$ and $S\cap H \subset D'$. Notice that $S \cap D' \cap R' = \emptyset$. First assume $\deg(D' \cap R') \leq 1$. Since $S \cap D' \cap R' = \emptyset$, the Mayer-Vietoris exact sequence

$$0 \to \mathcal{I}_S(m) \to \mathcal{I}_{S\cap D'}(m) \oplus \mathcal{I}_{S\cap R'}(m) \to \mathcal{O}_{D'\cap R'}(m) \to 0 \tag{1}$$

gives $h^1(D' \cup R', \mathcal{I}_{S,D'\cup R'}(m)) = 0$ (even if $D' \cap R' \neq \emptyset$, because $\sharp(S \cap D') = \sharp(S\cap R') = 2m$). Since $m \geq 3$, we have $h^1(\mathcal{I}_{D'\cup R'}(m)) = 0$. Hence $h^1(\mathcal{I}_S(m)) = 0$, a contradiction. Now assume $\deg(D' \cap R') = 2$ (as schemes). This is equivalent to $p_a(D' \cup R') = 1$. Since $\omega_{D'\cup R'} \cong \mathcal{O}_{D'\cup R'}$, Riemann-Roch gives $h^0(D' \cup R', \mathcal{O}_{D'\cup R'}(2)) = 8$. Hence $h^0(\mathcal{I}_{D'\cup R'}(2)) \geq 2$. It is easy to check that $D' \cup R'$ is the complete intersection of two quadric surfaces. Hence $\omega_{D'\cup R'} \cong \mathcal{O}_{D'\cup R'}$, $h^1(\mathcal{I}_{D'\cup R'}(m)) = 0$ and $h^1(\mathcal{I}_S(m)) = h^1(D' \cup R', \mathcal{I}_{S,D'\cup R'}(m))$. The sheaf $\mathcal{I}_{S,D'\cup R'}(m)$ is a rank 1 torsion free sheaf of degree 0. Since $\omega_{D'\cup R'} \cong \mathcal{O}_{D'\cup R'}$, we have $h^1(D' \cup R', \mathcal{I}_{S,D'\cup R'}(m)) = h^0(D' \cup R', \mathcal{I}_{S,D'\cup R'}(m))$. Fix $\sigma \in H^0(D' \cup R', \mathcal{I}_{S,D'\cup R'}(m))$ such that $\sigma \neq 0$. If σ does not vanishes on one of the irreducible components of $D' \cup R'$, then σ shows that S is the complete intersection of two quadrics (giving $D' \cup R'$) and a degree m surface, because the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(D' \cup R', \mathcal{O}_{D'\cup R'}(m))$ is surjective (remember that $D' \cup R'$ is a complete intersection). Now assume that σ vanishes identically on one of the components of $D'\cup R'$, say in a component of D'. Since $\deg(S \cap T) \leq m+1$ for every line $T \subset \mathbb{P}^3$, we see that $\sigma|D' \equiv 0$ even if D' is reducible. Hence σ vanishes at P_1. Since $\sharp((\{P_1\} \cup S \cap R') \geq 2m+1$, we get that σ vanishes identically on a component of R'. As above we get $\sigma|R' \equiv 0$. Hence $\sigma = 0$, a contradiction. □

Lemma 2. *Assume* $m \geq 23$. *There is no finite set* $S \subset \mathbb{P}^3$ *such that* $\sharp(S) \leq 4m$, $h^1(\mathcal{I}_S(m)) > 0$, $h^1(\mathcal{I}_{S'}(m)) = 0$ *for all* $S' \subsetneq S$ *and* $h^0(\mathcal{I}_S(2)) = 0$.

Proof. Assume the existence of $S \subset \mathbb{P}^3$ such that $\sharp(S) \le 4m$, $h^1(\mathcal{I}_S(m)) > 0$, $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$ and $h^0(\mathcal{I}_S(2)) = 0$. Set $S_0 := S$. Let $E_1 \subset \mathbb{P}^3$ be a quadric surface such that $s_1 := \sharp(S \cap E_1)$ is maximal. Set $S_1 := S \setminus S \cap E_1$. For every integer $i \ge 2$ define recursively the quadric surface E_i, the integer s_i and the set $S_i \subseteq S_{i-1}$ in the following way. Let E_i be a quadric surface such that $s_i := \sharp(E_i \cap S_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap E_i$. Since S is reduced, we may take each E_i with the additional condition that it is reduced. By [1], Remark 1, there is an integer $t \ge 1$ such that $h^1(E_t, \mathcal{I}_{E_t \cap S_{t-1}}(m+2-2t)) > 0$. Call e the minimal such an integer t. Since S is not contained in a quadric surface, we have $E_1 \cap S \ne S$. Hence $h^1(E_1, \mathcal{I}_{S \cap E_1, E_1}(m)) = h^1(\mathcal{I}_{S \cap E_1}(m)) = 0$. Hence $e \ne 1$. Hence from now on we assume $e \ge 2$. Since $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$, either $s_i \ge 9$ or $s_{i+1} = 0$. Hence $e - 1 \le 4m/9$ and $4m \ge s_e + 9(e-1)$.

(a) Assume $e = 2$. Since $s_1 \ge s_2$ and $s_1 + s_2 \le 4m$, we have $s_2 \le 2m$. We claim that if E_2 is reducible, say $E_2 = H \cup M$ with H, M planes, then there is no plane cubic $T \subset H$, such that $\sharp(T \cap E_1) \ge 3(m-2)$. Assume the existence of such a curve T. Taking a plane $H' \ne H$ containing 3 points of $S_0 \setminus S_1$ we get $s_1 \ge 3m - 3$. Hence $4m \ge s_1 + s_2 \ge 6m - 9$. Since $m \ge 5$, this is absurd. Since E_2 contains no plane cubic T with $\sharp(T \cap S_1) \ge 3(m-2)$ and $2m \le 3(m-2)+1$, either there is a line $D \subset E_1$ such that $\sharp(S_1 \cap D) \ge m$ or there is a plane conic F such that $\sharp(S_1 \cap F) \ge 2m - 2$ (see [3], Theorem 3.8).

We first assume the existence of F as above. Let $\langle F \rangle$ be the plane spanned by F. If $h^1(\mathcal{I}_{\langle F \rangle \cap S}(m)) > 0$, then the minimality of S gives $S \subset \langle F \rangle$. Hence $h^0(\mathcal{I}_S(2)) \ge h^0(\mathcal{I}_S(1)) > 0$, a contradiction. Now assume $h^1(\mathcal{I}_{\langle F \rangle \cap S}(m)) = 0$. Hence $h^1(\mathcal{I}_{S \setminus S \cap \langle F \rangle}(m-1)) > 0$ (see [1], Remark 1). Since $\sharp(S \setminus S \cap \langle F \rangle) \le 2m + 2 \le 3(m-1)$, there is a plane curve T containing at least $m+1$ points of $S \setminus S \cap \langle F \rangle$. Since $T \cup F$ is contained in a reducible quadric, we get $s_1 \ge 3m - 1$. Hence $s_2 \le m + 1 < 2m - 2$, a contradiction.

Hence there is a line D such that $\sharp(S_1 \cap D) \ge m$. Let $V_1 \subset \mathbb{P}^3$ be a quadric surface containing D and such that $\sharp(V_1 \cap S)$ is maximal. Set $Z_1 := S \setminus S \cap V_1$. Define recursively the quadric surface V_i, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \ge 2$, in the following way. Let V_i be any quadric surface such that $z_i := \sharp(V_i \cap Z_{i-1})$ is maximal. Set $Z_i := Z_{i-1} \setminus Z_{i-1} \cap V_i$. The sequence $\{z_i\}_{i \ge 1}$ is non-decreasing. Since S is a reduced scheme, without losing generality we may find a sequence $\{V_i\}_{i \ge 1}$ with each V_i reduced. Since $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$, either $z_i \ge 9$ or $z_{i+1} = 0$. By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such that $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then the minimality property of S gives $S \subset V_1$, a contradiction. For the case $c \ge 3$, see steps (b) and (c) below. Here we assume $c = 2$. Since $z_1 \ge z_2$, we have $z_2 \le 2m$. As before we exclude the existence of a plane cubic T with $\sharp(T\cap Z_1) \ge 3(m-2)$. As above we exclude the existence of a plane conic $F \subset V_2$ such that $\sharp(F \cap (S \setminus S \cap V_1)) \ge 2m-2$. Hence there is a line $R \subset V_2$ such that $\sharp(R \cap Z_1) \ge m$. There is a quadric surface U containing $D \cup R$ and at least 3 points of $S \setminus S \cap (D \cup R)$. We have $h^1(U, \mathcal{I}_{U\cap S}(m)) = 0$, because $S \cap U \ne S$ since $h^0(\mathcal{I}_S(2)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap U}(m-2)) > 0$ (see [1], Remark 1). As above we get the existence of a line D' such that $\sharp(D' \cap (S \setminus S\cap U)) \ge m$. Hence $\sharp(S \cap (D \cup D' \cup R)) \ge 3m$. Since $D \cup D' \cup R$ is contained in a quadric surface, we have $s_1 \ge 3m$. Hence $s_2 \le m$. Since $h^1(\mathcal{I}_{S_1}(m-2)) = 0$, we get $s_1 = 3m$, $\sharp(S) = 4m$, $\sharp(S \setminus (D \cup D' \cup R)) = m$ and that $S \setminus (D \cup D' \cup R)$ is contained in a line D''. Hence $\sharp(S) = 4m$, $S \subset D \cup D' \cup R \cup D''$ and $\sharp(S \cap T) = m$ for each irreducible component T of $D \cup D' \cup R \cup D''$. Hence S contains no singular point of $D \cup D' \cup R \cup D''$. First assume that $D \cup D' \cup R \cup D''$ is not smooth. Just to fix the notation we assume $R \cap D' \ne \emptyset$. Hence there is a quadric surface containing $D' \cup R \cup D''$ and a point of $S \cap D$. Hence $s_1 \ge 3m+1$, a contradiction. Hence we may assume that $D \cup D' \cup R \cup D''$ is a disjoint union of 4 lines. Let H be a plane containing D and a point of $S \cap D'$. Obviously $h^1(H, \mathcal{I}_{S\cap H}(m)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap H}(m-1)) > 0$. Let M be a plane containing R and a point of $S \cap D''$. Obviously $h^1(\mathcal{I}_{S\cap M}(m-1)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap (H\cup M)}(m-2)) = 0$. The set $S \setminus S \cap (H \cup M)$ is formed by $m-1$ points on D' and $m-1$ points of D''. Since $D' \cap D'' = \emptyset$, we have $h^1(\mathcal{I}_{S\setminus S\cap (H\cup M)}(m-2)) = 0$, a contradiction.

(b) Assume $e = 3$. Since $s_1 \ge s_2 \ge s_3$, $s_1 + s_2 + s_3 \le 4m$ and $m \ge 8$, we have $s_3 < 3(m-4)$. Since $h^1(E_3, \mathcal{I}_{S_2\cap E_3, E_3}(m-4)) > 0$, either there is a line $J \subset \mathbb{P}^3$ such that $\sharp(J \cap S_2 \cap E_3) \ge m-2$ or there is a conic $J_1 \subset \mathbb{P}^3$ such that $\sharp(J_1 \cap S_2 \cap E_3) \ge 2m-6$ (see [3], Theorem 3.8). In the latter case we get $s_2 \ge 2m-2$, because $h^0(\mathcal{I}_{J_1}(2)) = 5$. Hence in the latter case we get $4m \ge s_1+s_2+s_3 \ge 2m-2+2m-2+2m-6$, a contradiction (because $m \ge 6$). Now assume the existence of a line $J \subset \mathbb{P}^3$ such that $\sharp(J\cap S_2\cap E_3) \ge m-2$. Let $V_1 \subset \mathbb{P}^3$ be a quadric surface containing J and such that $\sharp(V_1 \cap S)$ is maximal. Set $Z_1 := S \setminus S \cap V_1$. Define recursively the quadric surface V_i, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \ge 2$, as in step (a). By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t\cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such $h^1(V_t, \mathcal{I}_{V_t\cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then the minimality of S gives $S \subset V_1$, a contradiction. If $c = 2$, then we conclude as in step (a). If $c \ge 4$, then see step (c). Now assume $c = 3$. As above we exclude the existence of a conic J_1' such that $\sharp(J_1' \cap Z_2) \ge 2m-6$. Hence there is a line R_1 such that $\sharp(R_1 \cap Z_2) \ge m-2$. Since $h^0(\mathcal{I}_{R_1}(2)) = 7$, we get $z_2 \ge m-2+6$. Since $h^0(\mathcal{I}_{J\cup R_1}(2)) \ge 4$, we get $z_1 \ge 2(m-2)+3$. Hence $4m \ge (2m-1)+(m+4)+(m-2)$, a contradiction.

(c) Assume $e \geq 4$. For each $i < e$, we have $s_e \leq 4m - 9(e-1)$ and in particular $e < (m+2)/2$. Hence $s_e > 0$. By [3], Theorem 3.8, either $s_e \geq 3(m+2-2e)$ or there is a conic D' such that $\sharp(D' \cap S_{e-1}) \geq 2(m+2-2e)+2$ or there is a line L such that $\sharp(L \cap S_{e-1}) \geq m+4-2e$.

(c1) First assume $s_e \geq 3(m+2-2e)$. Hence $4m \geq 3e(m+2-2e)$. For any $t \in \mathbb{R}$ set $\psi(t) := t(m+2-2t)$. The function $\psi : \mathbb{R} \to \mathbb{R}$ increases if $0 \leq t \leq (m+2)/4$ and decreases if $t > (m+2)/4$. Assume for the moment $e \leq 4m/9$. Taking first $t = 4$ and then $t = 4m/9$ we get that either $4m \geq 12(m-6)$ or $4m \geq 4m(m+18)/27$. Both inequalities are false, because $m \geq 10$. Now assume $e > 4m/9$. Since $s_e \leq 4m-9(e-1)$, we get $s_e < 9$. Since $s_e \geq 3(m+2-2e) > 0$, we get $m+2-2e \in \{1,2\}$. First assume $e = (m+1)/2$. In this case we have $s_e \geq 3$. We get $4m \geq 9(m-1)/2+3$, absurd. If $e = m/2$, then $s_e \geq 6$ and hence $4m \geq 6+9m/2-9$, a contradiction (since $m \geq 10$).

(c2) Now assume the existence of a conic D' such that $\sharp(D' \cap S_{e-1}) \geq 2(m+2-2e)+2 \geq 3$. Since $h^0(\mathcal{I}_{D'}(2)) = 5$, we get $s_{e-1} \geq 2(m+2-2e)+6$. Hence $4m \geq 2e(m+2-2e)+6e-4$, i.e. $2m \geq e(m+5-2e)-2$. Set $\phi(t) = t(m+5-2t)-2$. The function $\phi(t)$ is increasing in the interval $0 \leq t \leq (m+5)/4$ and decreasing if $t \geq (m+5)/4$. We have $\phi(4) = 4m-14 > 2m$ if $m \geq 8$ and $\phi(4m/9) = 4m^2/81 + 20m/9 - 2 > 2m$ for all $m \geq 9$. Now assume $e > 4m/9$. Since $s_e + 9(e-1) \leq 4m$, we have $s_e \leq 8$. Hence $2(m+2-2e)+2 \leq 8$, i.e. $2e \geq m-1$. Since $4m \geq 3+9(e-1) \geq 3+9(m-3)/2$, this case cannot occur if $m \geq 3$.

(c3) From now on we assume the existence of a line L such that $\sharp(L \cap S_{e-1}) \geq m+4-2e$. Let $U_1 \subset \mathbb{P}^3$ be a quadric surface containing L and such that $w_1 := \sharp(U_1 \cap S)$ is maximal. Set $W_1 := S \setminus S \cap U_1$. Define recursively the quadric surface U_i, the non-negative integer w_i and the set $W_i \subseteq W_{i-1}$, $i \geq 2$, as in step (a). By [1], Remark 1, there is an integer $t > 0$ such that $h^1(W_t, \mathcal{I}_{W_t \cap U_{t-1}}(m+2-2t)) > 0$. Let c' be the minimal such an integer. We have $c' > 1$ by the minimality of S and the assumption $h^0(\mathcal{I}_S(2)) = 0$. Using c' instead of e we get $4m \geq s_{c'} + 9(c'-1)$. If $c' = 2$, then we do as in step (a). If $c' = 3$, then we do as in step (b). If $c' \geq 4$, we repeat the case $e \geq 4$ with c' instead of e; we get the existence of a line M such that $\sharp(M \cap U_{c'-1}) \geq m+4-2c'$. Since $h^0(\mathcal{I}_{L \cup M}(2)) \geq 4$, we get $z_1 \geq 3 + (m+4-2c') + (m+4-2e)$. Since $h^0(\mathcal{I}_R(2)) = 7$, we also get $z_i \geq m+10-2c'$ for all $i \in \{2, \dots, c'-1\}$. Hence $4m \geq c'(m+4-2c') + 6(c'-1) + 3 + m + 4 - 2e$, i.e. $3m + 2e - 1 \geq c'(m+10-2c')$. Recall that $2e \leq m+2$ and that $2c' \leq m+2$. Hence the last equation is false if $c' = 4$. Hence we may assume $c' \geq 5$. The function $\eta(t) := t(m+10-2t)$ is increasing if $0 \leq t \leq (m+10)/4$ and decreasing if $t \geq (m+10)/4$. We have

$\eta(5) = 5m > 4m+1$. We have $\eta((m+2)/2)) = 4(m+2) > 4m+1$. Hence even this case cannot occur. $\square$

Lemma 3. *Let $Q \subset \mathbb{P}^3$ be a smooth quadric. Fix an integer $m \geq 11$ and a finite set $S \subset Q$ such that $\sharp(S) \leq 4m$, $h^1(\mathcal{I}_S(m)) > 0$ and $h^1(\mathcal{I}_{S'}(m)) = 0$ for every $S' \subsetneq S$. We have $h^1(\mathcal{I}_S(m)) > 0$ if and only if S is as in one of the following cases:*

(a) $\sharp(S) = m+2$ and W is contained in a line;

(b) $\sharp(S) = 2m+2$ and S is contained in a plane conic;

(c) $\sharp(S) = 3m+2$, $m \geq 2$, and S is contained in a reduced element of $|\mathcal{O}_Q(2,1)|$ or $|\mathcal{O}_Q(1,2)|$.

Proof. We often silently assume $\sharp(S) \geq 3m+1$ (e.g., to say $s_1 \geq 8$ below we need $\sharp(S) \geq 8$).

Fix $E_1 \in |\mathcal{O}_Q(2,2)|$ such that $\sharp(S \cap E_1)$ is maximal. Set $S_0 := S$, $s_1 := \sharp(S \cap E_1)$ and $S_2 := S \setminus S \cap E_1$. Define recursively the curve $E_i \in |\mathcal{O}_Q(2,2)|$, the non-negative integer s_i and the set $S_i \subseteq S_{i-1}$ in the following way. Let E_i be any element of $|\mathcal{O}_Q(2,2)|$ such that $s_i := \sharp(E_i \cap S_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus E_i \cap S_{i-1}$. The sequence $\{s_i\}_{i\geq 1}$ is non-decreasing. Since S is a reduced scheme, without losing generality we may find a sequence $\{E_i\}_{i\geq 1}$ with each E_i reduced. Since $h^0(Q, \mathcal{O}_Q(2,2)) = 9$, either $s_i \geq 8$ or $s_{i+1} = 0$. By [1], Remark 1, there is an integer $t > 0$ such that $h^1(E_t, \mathcal{I}_{E_t \cap S_{t-1}}(m+2-2t)) > 0$. Let e be the minimal integer $t > 0$ such that $h^1(E_t, \mathcal{I}_{E_t \cap S_{t-1}}(m+2-2t)) > 0$, if either m is even or there is such an integer $t \leq \lfloor (m+2)/2 \rfloor$. We will handle the case in which m is odd and there is no such integer $\leq (m+1)/2$ in step (iv). If $e = 1$, then we may apply [1], Lemma 9. Hence we may assume $e \neq 1$. We need to find a contradiction, because each S as in the statement of Lemma 3 is contained in some $E \in |\mathcal{O}_Q(2,2)|$.

In steps (i) and (ii) we assume $e = 2$. Since $s_1 \geq s_2$ and $s_1 + s_2 \leq 4m$, we have $s_2 \leq 2m < 3(m-1)$. Hence either there is $D \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ such that $\sharp(S_1 \cap D) \geq m$ or there is $F \in |\mathcal{O}_Q(1,1)|$ such that $\sharp(S_1 \cap F) \geq 2m-2$ (e.g. by [1], Lemma 9, applied to the integer $m-2$). In both cases we have $s_2 \geq m$.

(i) Assume the existence of F as above. Hence $s_2 \geq 2m-2$. Take $U_2 \in |\mathcal{O}_Q(2,2)|$ containing F and such that $\sharp(U_2 \cap S)$ is maximal.

Since $h^0(Q, \mathcal{O}_Q(2,2)(-F)) = 4$, we have $\sharp(U_2 \cap S) \geq 2m+1$. Fix any $E' \in |\mathcal{O}_Q(2,2)|$ containing F and at least 3 points, say P_1, P_2, P_3, of $S \setminus S \cap F$. We get $\sharp(E' \cap S) \geq 2m+1$. Since the case $e = 1$ was excluded for each choice of the curve $E_1 \in |\mathcal{O}_Q(2,2)|$, we get $h^1(Q, \mathcal{I}_{S\setminus S\cap E'}(m-2)) > 0$. Hence either there is a line L' with $\sharp(L' \cap (S \setminus S \cap E')) \geq m$ or there is $F' \in |\mathcal{O}_Q(1,1)|$ with $\sharp(F' \cap (S \setminus S \cap E')) \geq 2m-2$. Since $F' \cup F$ is contained in a reducible quadric and $\sharp(S \cap (F \cup F')) \geq 4m-4$, we have $s_1 \geq 4m-4$. Hence $s_2 \leq 4 < m$, a contradiction. Hence there is a line L' such that $\sharp(L' \cap (S \setminus S \cap E')) \geq m$. Since $F \cup L'$ is contained in a reducible quadric surface, we get $s_1 \geq 3m-2$. Hence $s_2 \leq m+2$. Hence F' cannot exist.

(ii) From now on we assume the non-existence of F. Hence there is $D \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ such that $\sharp((S \setminus S \cap E_1) \cap D) \geq m$. Fix $V_1 \in |\mathcal{O}_Q(2,2)|$ containing D and such that $\sharp(V_1 \cap S)$ is maximal. Set $Z_1 := S \setminus S \cap U_1$. Define recursively the curve $V_i \in |\mathcal{O}_Q(2,2)|$, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \geq 2$, in the following way. Let V_i be any element of $|\mathcal{O}_Q(2,2)|$ such that $z_i := \sharp(V_i \cap Z_{i-1})$ is maximal. Set $Z_i := Z_{i-1} \setminus Z_{i-1} \cap V_i$. The sequence $\{z_i\}_{i\geq 1}$ is non-decreasing. Since S is a reduced scheme, without losing generality we may find a sequence $\{V_i\}_{i\geq 1}$ with each V_i reduced. Since $h^0(Q, \mathcal{O}_Q(2,2)) = 9$, either $z_i \geq 8$ or $z_{i+1} = 0$. By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t\cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such $h^1(V_t, \mathcal{I}_{V_t\cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then we may apply [1], Lemma 9. For the case $c \geq 3$, see from step (iii) on.

(ii.1) Here we assume $c = 2$. Since $z_1 \geq z_2$ and $z_1 + z_2 \leq 4m$, as in step (i) we get the existence of a line $D' \subset V_2$ such that $\sharp(D' \cap Z_1) \geq m$. Since $\sharp(S \cap D' \setminus (D \cap D')) \geq m$, we have $\sharp(S \cap (D \cup D')) \geq 2m$. Take $E(1) \in |\mathcal{O}_Q(2,2)|$ containing $D \cup D'$ and with maximal $s(1) := \sharp(E(1) \cap S)$. Set $S(0) := S$ and $S(2) := S \setminus S \cap E(1)$. Define recursively the curve $E(i) \in |\mathcal{O}_Q(2,2)|$, the non-negative integer $s(i)$ and the set $S(i) \subseteq S(i-1)$ in the following way. Let $E(i)$ be any element of $|\mathcal{O}_Q(2,2)|$ such that $s(i) := \sharp(E(i) \cap S(i-1))$ is maximal. Set $S(i) := S(i-1) \setminus S(i-1) \cap E(i)$. Since S is reduced, we may find a sequence $\{E(i)\}$ with each $E(i)$ reduced. The sequence $\{s(i)\}$ is non-decreasing. By [1], Remark 1, there is an integer $t \geq 1$ such that $h^1(E(t), \mathcal{I}_{E(t)\cap S(t-1)}(m+2-2t)) > 0$. Let $e(1)$ be the minimal integer t such that $h^1(E(t), \mathcal{I}_{E(t)\cap S(t-1)}(m+2-2t)) > 0$. If $e(1) = 1$, then we may apply [1], Lemma 9. See step (v) below for the case $e(1) \geq 3$. Now assume $e(1) = 2$. Since $s(1) \geq s(2)$ and $s(1) + s(2) \leq 4m$, as in step (i) we get the existence of a line $D_1 \subset E(2)$ such that $\sharp(D_1 \cap S(2)) \geq m$. Since $D_1 \cap S_2 \neq \emptyset$, we have $D_1 \neq D$ and $D_1 \neq D'$. If the lines D, D' and D_1 are not disjoint, then either $D \cup D' \cup D_1 \in |\mathcal{O}_Q(2,1)|$ or $D \cup D' \cup D_1 \in |\mathcal{O}_Q(1,2)|$. In both cases there is $E \in |\mathcal{O}_Q(2,2)|$ containing $D \cup D' \cup D_1$ and a point of $S \setminus S \cap (D \cup D' \cup D_1)$. Hence $s_1 \geq 3m+1$. Hence $s_2 \leq m-1$, a contradiction. Hence D, D' and D_1 are lines in the same system of lines, say $|\mathcal{O}_Q(1,0)|$. Since each of these lines contains at most $m+1$ points of S, we have $h^1(D\cup D'\cup D_1,\mathcal{I}_{S\cap(D\cup D'\cup D_1)}(m))=0$. Hence $h^1(Q,\mathcal{I}_{S\setminus S\cap(D\cup D'\cup D_1)}(m-3,m))>0$ (see [1], Remark 1). Since $\sharp(S\setminus S\cap(D\cup D'\cup D_1))\le m$, we get the existence of a line $L'\in|\mathcal{O}_Q(0,1)|$ such that $S\subset L'\cup D\cup D'\cup D_1$, $\sharp(L'\cap(S\setminus S\cap(D\cup D'\cup D_1))\ge m-1$ and $S\setminus S\cap(L'\cup D\cup D'\cup D_1)$ is at most one point. We have $m\le\sharp(T\cap S)\le m+1$ for all $T\in\{D,D',D_1\}$ and $\sharp(T\cap S)=m+1$ for at most one component T of $D\cup D'\cup D_1$. Just to fix the notation we assume $\sharp(D_1\cap S)=m$. Fix $P\in S_1$ such that $P\notin L'$. Let E'' be the unique element of $|\mathcal{O}_Q(2,2)|$ containing $D\cup D'\cup L'\cup\{P\}$. Since in the proof that $e\ge 2$ we may use E'' instead of E, we have $h^1(E'',\mathcal{I}_{E''\cap S}(m,m))=0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap E''}(m-2))>0$ (see [1], Remark 1). Since $\sharp(S\setminus S\cap E'')\le m$ there is a line $J\supset S\setminus S\cap E''$ and $\sharp(S\setminus S\cap E'')=m$. Since D_1 contains $m-1\ge 4$ points of $S\setminus S\cap E''$, we have $J=D_1$. Since $\sharp(D_1\cap(S\setminus S\cap E''))=m-1$, we obtained a contradiction.

(iii) In this step we handle all cases with $m/2-2\le e\le m/2$. By [1], lemma 9, either $S_{e-1}\cap E_e$ contains at least $m+4-2e$ collinear points or $S_{e-1}\cap E_e$ contains at least $2m+6-4e$ points on a conic or $\sharp(S_{e-1}\cap E_e)\ge 3(m+4-2e)$. In the first (resp. second) case we get $s_{e-1}\ge 5+(m+4-2e)$ (resp. $s_{e-1}\ge 3+(2m+6-4e)$). Hence $\sharp(S)\ge 5(e-1)+e(m+4-2e)=e(m+9-2e)-5$. The function $\phi(t):=t(m+9-2t)$ is decreasing if $t\ge(m+9)/4$. Hence $\sharp(S)\ge 9(m/2)-5>4m$ (here we use $m\ge 11$). In the third case we have $s_e\ge 12$ and hence $\sharp(S)\ge 12(m/2-2)$, a contradiction. In the same way we handle the cases with $m/2-2\le c\le m/2$ and $m/2-2\le e(1)\le m/2$.

(iv) In this step we assume m even and $e=(m+2)/2$. Hence $m+2-2e=0$. Since $h^1(\mathcal{I}_{S_{m/2}\cap E_{(m+2)/2}}(m+2-2e))>0$, we get $\sharp(S_{m/2}\cap E_{(m+2)/2})\ge 2$. Since $s_i\ge 8$ if $s_{i+1}>0$, we get $\sharp(S)\ge 4m+2$, a contradiction. In the same way we handle the cases $c=(m+2)/2$ and $e(1)=(m+2)/2$.

(v) In this step we assume m odd and $h^1(E_t,\mathcal{I}_{E_t\cap S_{t-1}}(m+2-2t))=0$ for all $t\le(m+1)/2$. Applying $(m-1)/2$ times [1], Remark 1, we get $h^1(\mathcal{I}_{S_{(m-1)/2}}(1))>0$. Hence either $\sharp(S_{(m-1)/2})\ge 5$ or $S_{(m-1)/2}$ contains 4 coplanar points or $S_{(m-1)/2}$ contains 3 collinear points. First assume $\sharp(S_{(m-1)/2})\ge 5$. Since $s_i\ge 8$ for all $i\le(m-1)/2$, we get $\sharp(S)\ge 8(m-1)/2+5=4m+1$, a contradiction. Now assume that $S_{(m-1)/2}$ contains 4 coplanar points. Since $s_i\ge 8$ for all $i\le(m-1)/2$, we get $\sharp(S)=4m$, $s_1=8$ and that $S_{(m-1)/2}$ is formed by 4 coplanar points. Since $s_1=8$, we have $h^0(Q,\mathcal{I}_{S'}(2,2))=\max\{0,9-\sharp(S')\}$ for each $S'\subset S$. Hence we may take as $E_1\cap S_0$ any 8 points of S, as $E_2\cap S_1$ any 8 points of S_1, and so on. Since $h^0(Q,\mathcal{I}_S(1,1))=0$, at the end we may obtain as $S_{(m-1)/2}$ 4 non-coplanar points, a contradiction. Now assume that $S_{(m-1)/2}$ contains 3 collinear points. Since $s_i\ge 8$ for all $i\le(m-1)/2$ and $\sharp(S)\le 4m$, we get $\sharp(S_{(m-1)/2}) \le 4$ and $s_i = 8$ for all $i \in \{2, \dots, (m-1)/2\}$. Since $s_2 = 8$, $h^0(Q, \mathcal{I}_{S'}(2,2)) = \max\{0, 9 - \sharp(S')\}$ for each $S' \subset S_1$. Hence we may take as $E_2 \cap S_1$ any 8 points of S_1. Since $(m-1)/2 \ge 4$, at the end we may obtain a new $S_{(m-1)/2}$ with cardinality 3 or 4 and which does not contain 3 collinear points.

(vi) In this step we assume $3 \le e \le (m-5)/2$. The last inequality implies $m+2-2e > 0$. Q contains no plane curve of degree ≥ 3. First assume the non-existence of a plane curve F of degree ≤ 2 such that $h^1(F, \mathcal{I}_{S_{e-1} \cap F}(m+2-2e)) > 0$. By [3], Theorem 3.8, we get $\sharp(F \cap S_{e-1}) \ge 3(m+2-2e)+1$ and hence $s_e \ge 3(m+2-2e)+1$. Therefore $\sharp(S) \ge e(3m+7-6e)$. Since $6e < 3m-12$ and $e \ge 3$, we get $\sharp(S) \ge 3(3m-11)$. Hence $9m - 33 \le 4m$, i.e. $m \le 6$, a contradiction. Now assume that such a curve F exists, but it is not a line. We get $s_e \ge 2m+6-4e$; since $h^0(Q, \mathcal{O}_Q(1,1)) = 4$, there is a curve $A \in |\mathcal{O}_Q(2,2)|$ containing F and at least 3 further points of S_{e-1} (unless $\sharp(S_{e-1} \cap F) \ge \sharp(S_{e-1}) + 2$, but this inequality and $h^0(Q, \mathcal{O}_Q(1,1)) = 4$ would imply $s_e = 0$, absurd). We get $s_{e-1} \ge 3 + 2m + 6 - 4e$. Hence $\sharp(S) \ge 3(e-1) + e(2m+6-4e)$; since $2m+6-4e \ge 3$, we get $\sharp(S) \ge 6m-12$; hence $m \le 6$, absurd. From now on in this step we assume that such a curve F has degree 1, i.e. we assume the existence of $D_0 \in (|\mathcal{O}_Q(1,0)| \cup |\mathcal{O}_Q(0,1)|)$ such that $\sharp(S_e \cap D_0) \ge m+4-2e$. Fix $E[1] \in |\mathcal{O}_Q(2,2)|$ containing D_0 and with maximal $s[1] := \sharp(E[1] \cap S)$. Define recursively the curve $E[i] \in |\mathcal{O}_Q(2,2)|$, the non-negative integer $s[i]$ and the set $S[i] \subseteq S[i-1]$ in the following way. Let $E[i]$ be any element of $|\mathcal{O}_Q(2,2)|$ such that $s[i] := \sharp(E[i] \cap S[i-1])$ is maximal. Set $S[i] := S[i-1] \setminus S[i-1] \cap E[i]$. Since S is reduced, we may find a sequence $\{E[i]\}$ with each $E[i]$ reduced. The sequence $\{s[i]\}$ is non-decreasing. By [1], Remark 1, there is an integer $t \ge 1$ such that $h^1(E[t], \mathcal{I}_{E[t] \cap S[t-1]}(m+2-2t)) > 0$. Let $e[1]$ be the minimal integer t such that $h^1(E[t], \mathcal{I}_{E[t] \cap S[t-1])}(m+2-2t)) > 0$ (with again the need to look at step (v) if m is odd). If $e[1] = 1$, then we apply [1], Lemma 9. Now assume $e[1] = 2$. We get a line $D[1] \subset E[2]$ such that $\sharp(D[1] \cap S[2]) \ge m$. Since $D[1] \cap S[2] \ne \emptyset$, we have $D[1] \ne D_0$. Since $\sharp(S \cap (D_0 \cup D[1])) \ge 2m$ and there is an element of $|\mathcal{O}_Q(2,2)|$ containing $D_0 \cup D[1]$ and at least two other points of S (or 3 if $D_0 \cap D[1] \ne \emptyset$ (S is not contained in $D_0 \cup D[1]$, because we assumed $e \ne 1$), we get $s[1] \ge 2m+2$ with strict inequality if $D_0 \cap D[1] \ne \emptyset$. Hence we may assume $s_1 \ge 2m+2$. Since $s_i \ge 5 + (m+4-2e)$ for $i < e$, we get $\sharp(S) \ge 2m+2+5(e-2)+(e-1)(m+4-2e)$. Hence $3m+12 \ge e(m+11-2e)$. The function $\alpha(t) := t(m+11-2t)$ is increasing if $0 \le t \le (m+11)/4$ and decreasing if $t \ge (m+11)/4$. Since $\alpha(3) = 3m+15$ and $\alpha((m-5)/2) = 8(m-5) \ge 3m+13$ (here we use that $m \ge 11$), we get a contradiction. In the same way we get a contradiction if c (resp.$e(1)$, resp. $e[1]$) are defined and $3 \le c \le (m-5)/2$

(resp. $3 \le e(1) \le (m-5)/2$, resp. $3 \le e[1] \le (m-5)/2$). □

Remark 1. Let $T \subset \mathbb{P}^3$ be an irreducible quadric cone. Let O be the vertex of T. Let $\pi : Y \to T$ be a minimal resolution of T. The surface Y is isomorphic to the Hirzebruch surface F_2 and π is described in the following way. Set $h := \pi^{-1}(O)$. Let $u : Y \to \mathbb{P}^1$ denote the ruling of Y. Call f any fiber of u. We have $f \cong \mathbb{P}^1$ and $\pi(f)$ is a line. The abelian group $\mathrm{Pic}(Y)$ is freely generated by h and f . We have $f^2 = 0$, $h \cdot f = 1$ and $h^2 = -2$. The morphism π is induced by the complete linear system $|\mathcal{O}_Y(h+2f)|$ and $\pi : Y \setminus h \to T \setminus \{O\}$ is an isomorphism. We have $h^1(Y, \mathcal{O}_Y(ah+bf)) = 0$ if either $a = -1$ or $a \ge 0$ and $b \ge 2a-1$. If $a \ge 0$ and $b \ge 2a$ (resp. $b > 2a > 0$), then $\mathcal{O}_Y(ah+bf)$ is spanned (resp. very ample) and $h^0(Y, \mathcal{O}_Y(ah+bf)) = \sum_{i=0}^{a}(b+1-2i) = (a+1)(b+1) - a(a+1) = (a+1)(b+1-a)$. Hence $h^0(\mathcal{O}_Y(ah+2af)) = (a+1)^2$ for all $a \ge 0$.

Lemma 4. *Let $T \subset \mathbb{P}^3$ be an irreducible quadric cone. Call O the vertex of T. Assume $m \ge 11$. Fix $S \subset Y$ such that $\sharp(S) \le 4m$ and $\sharp(S \cap h) \le 1$. Set $B := \pi(S)$. We have $h^1(Y, \mathcal{I}_S(mh + (2m)f)) = h^1(\mathbb{P}^3, \mathcal{I}_B(m))$. Assume $h^1(\mathcal{I}_B(m)) > 0$ and $h^1(\mathcal{I}_{B'}(m)) = 0$ for all $B' \subsetneq B$. Then S is as in one of the following cases:*

(a) $\sharp(B) = m+2$ and B is contained in a line of T;

(b) $\sharp(S) = 2m+2$ and S is contained in an integral element of $|\mathcal{O}_Y(h+2f)|$ (equivalently, B is contained in a smooth plane section of T);

(c) there are $F_1, F_2 \in |\mathcal{O}_Y(f)|$, such that $S \subset (F_1 \cup F_2)$, $F_1 \ne F_2$ and $\sharp(S \cap F_1) = \sharp(S \cap F_2) = m+1$ (equivalently, $O \notin B$, $m+1$ of its points are on a line through O and the other $m+1$ ones are on another line though O);

(d) $\sharp(S) = 3m+2$ and S is contained in an element of $|\mathcal{O}_Y(h+3f)|$ (equivalently, B is contained in a reduced degree 3 subcurve of T; this curve is connected and with arithmetic genus 0).

Proof. Notice that $h^0(Y, \mathcal{I}_S(mh + (2m)f)) = h^0(\mathbb{P}^3, \mathcal{I}_B(m))$. Since $\sharp(S \cap h) \le 1$, we have $\sharp(S) = \sharp(B)$. Hence $h^1(Y, \mathcal{I}_S(mh + (2m)f)) = h^1(\mathbb{P}^3, \mathcal{I}_B(m))$.

Notice that each line (resp. irreducible conic, resp. irreducible cubic) containing at least 3 (resp. 5, resp. 7) points of T is contained in T. Each line (resp. irreducible conic) contained in T is the image of an element of $|\mathcal{O}_Y(f)|$ (resp. $|\mathcal{O}_Y(h+2f)|$. Every reducible conic of T is the image of an element of $|\mathcal{O}_Y(2f)|$; it is also the image of a unique element of $|\mathcal{O}_Y(h+2f)|$ with h as an irreducible component. Let $C \subset T$ be a reduced curve of degree 3, which is not the union of 3 lines. Then C is the image of a unique element of $|\mathcal{O}_Y(h+3f)|$. Any union of 3 lines of T is the image of a unique element of $|\mathcal{O}_Y(h+3f)|$ with 4 irreducible components.

Set $S_0 := S$. Fix any $E_1 \in |\mathcal{O}_Y(2h+4f)|$ such that $s_1 := \sharp(E_1 \cap S)$ is maximal. Set $S_1 := S \setminus S \cap E_1$. Define recursively for all $i \geq 2$ the curve $E_i \in |\mathcal{O}_Y(2h+4f)|$, the integer s_i and the set S_i in the following way. Fix any $E_i \in |\mathcal{O}_Y(2h+4f)|$ such that $s_i := \sharp(S_{i-1} \cap E_i)$ is maximal and set $S_i := S_{i-1} \setminus S_{i-1} \cap E_i$. There is an integer $t > 0$ such that $h^1(E_i, \mathcal{I}_{S_{i-1} \cap E_i}((m+2-2t)h+(2m+4-4t)f)) > 0$ (see [1], Remark 1) and we call e the minimal such an integer. Notice that the sequence $\{s_i\}$ is non-decreasing. Since $h^0(Y, \mathcal{O}_Y(2h+4f)) = 9$, either $s_i \geq 8$ or $s_{i+1} = 0$. Since S is reduced, we may assume that either E_i is reduced or $E_i = 2h + F_1 + F_2$ with $F_i \in |\mathcal{O}_Y(f)|$ and $F_1 \neq F_2$. We repeat the proof of Lemma 3 with the following modifications. We exclude the cases $e = 1$, $c = 1$ and $e[1] = 1$ by quoting Lemma 1 (case E_1 reducible) and [1], Lemma 9 (case E_1 smooth) and [1], Lemma 10 (case E_1 an integral cone).

First assume $e = 2$. Since $s_1 \geq s_2$ and $s_1 + s_2 \leq 4m$, we have $s_2 \leq 2m < 3(m-1)$. Either there is $D \in |\mathcal{O}_Y(f)|$ such that $\sharp(S_1 \cap D) \geq m$ or there is $F \in |\mathcal{O}_Y(h+2f)|$ such that $\sharp(S_1 \cap F) \geq 2m-2$ (e.g. by [3], Theorem 3.2 applied to $\pi(S_1)$ or by [1], Lemma 9, applied to the integer $m-2$). In both cases we have $s_2 \geq m$.

(i) Assume the existence of F as above. Hence $s_2 \geq 2m-2$. Take $U_2 \in |\mathcal{O}_Y(2h+4f)|$ containing F and such that $\sharp(U_2 \cap S)$ is maximal.

Since $h^0(Y, \mathcal{O}_Y(2h+4f)(-F)) = 4$, we have $\sharp(U_2 \cap S) \geq 2m+1$. Fix any $E' \in |\mathcal{O}_Y(2h+4f)|$ containing F and at least 3 points, say P_1, P_2, P_3, of $S \setminus S \cap F$. We get $\sharp(E' \cap S) \geq 2m+1$. Since the case $e = 1$ was excluded for each choice of the curve $E_1 \in |\mathcal{O}_Y(2h+4f)|$, we get $h^1(Y, \mathcal{I}_{S \setminus S \cap E'}((m-2)h+(2m-4)f)) > 0$. Hence either there is $L' \in |\mathcal{O}_Y(f)|$ with $\sharp(L' \cap (S \setminus S \cap E')) \geq m$ or there is $F' \in |\mathcal{O}_Y(h+2f)|$ with $\sharp(F' \cap (S \setminus S \cap E')) \geq 2m-2$. Since $F' \cup F$ is contained in a reducible element of $|\mathcal{O}_Y(2h+4f)|$ and $\sharp(S \cap (F \cup F')) \geq 4m-4$, we have $s_1 \geq 4m-4$. Hence $s_2 \leq 4 < m$, a contradiction. Hence there is a $L' \in |\mathcal{O}_Y(f)|$ such that $\sharp(L' \cap (S \setminus S \cap E')) \geq m$. Since $F \cup L'$ is contained in a reducible element of $|\mathcal{O}_Y(2h+4f)|$, we get $s_1 \geq 3m-2$. Hence $s_2 \leq m+2$. Hence F cannot exist.

(ii) Since F does not exist, there is $D \in |\mathcal{O}_Y(f)|$ such that $\sharp((S \setminus S \cap E_1) \cap D) \geq m$. Fix $V_1 \in |\mathcal{O}_Y(2h+4f)|$ containing D and such that $\sharp(V_1 \cap S)$ is maximal. Set $Z_1 := S \setminus S \cap U_1$. Define recursively the curve $V_i \in |\mathcal{O}_Q(2,2)|$, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \geq 2$, in the following way. Let V_i be any element of $|\mathcal{O}_Y(2h+4f)|$ such that $z_i := \sharp(V_i \cap Z_{i-1})$ is maximal.

Set $Z_i := Z_{i-1} \setminus Z_{i-1} \cap V_i$. The sequence $\{z_i\}_{i\ge 1}$ is non-decreasing. Since S is a reduced scheme, without losing generality we may find a sequence $\{V_i\}_{i\ge 1}$ with each V_i reduced. Since $h^0(Y, \mathcal{O}_Y(2h+4f)) = 9$, either $z_i \ge 8$ or $z_{i+1} = 0$. By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then we may apply [1], Lemmas 9 and 10. The cases with $c \ge 3$ are done as in the proof of Lemma 3. Hence from now on we assume $c = 2$. Since $z_1 \ge z_2$ and $z_1 + z_2 \le 4m$, as in step (i) we get the existence of $D' \in |\mathcal{O}_Y(f)|$ such that $D' \subset V_2$ and $\sharp(D' \cap Z_1) \ge m$. Since $\sharp(S \cap D' \setminus (D \cap D')) \ge m$, we have $\sharp(S \cap (D \cup D')) \ge 2m$. Take $E(1) \in |\mathcal{O}_Y(2h+4f)|$ containing $D \cup D'$ and with maximal $s(1) := \sharp(E(1) \cap S)$. Set $S(0) := S$ and $S(2) := S \setminus S \cap E(1)$. Define recursively the curve $E(i) \in |\mathcal{O}_Y(2h+4f)|$, the non-negative integer $s(i)$ and the set $S(i) \subseteq S(i-1)$ in the following way. Let $E(i)$ be any element of $|\mathcal{O}_Y(2h+4f)|$ such that $s(i) := \sharp(E(i) \cap S(i-1))$ is maximal. Set $S(i) := S(i-1) \setminus S(i-1) \cap E(i)$. Since S is reduced, we may find a sequence $\{E(i)\}$ with each $E(i)$ reduced. The sequence $\{s(i)\}$ is non-decreasing. By [1], Remark 1, there is an integer $t \ge 1$ such that $h^1(E(t), \mathcal{I}_{E(t) \cap S(t-1)}(m+2-2t)) > 0$. Let $e(1)$ be the minimal integer t such that $h^1(E(t), \mathcal{I}_{E(t) \cap S(t-1)}(m+2-2t)) > 0$. If $e(1) = 1$, then we may apply [1], Lemmas 9 and 10. Since the case $e(1) \ge 3$ is done as in the proof of Lemma 3, we assume $e(1) = 2$. Since $s(1) \ge s(2)$ and $s(1) + s(2) \le 4m$, as in step (i) we get the existence of $D_1 \in |\mathcal{O}_Y(f)|$ such that $D_1 \subset E(2)$ and $\sharp(D_1 \cap S(2)) \ge m$. Since $D_1 \cap S_2 \ne \emptyset$, we have $D_1 \ne D$ and $D_1 \ne D'$. Since $h^0(Y, \mathcal{O}_Y(2h+4f)(-D-D'-D_1)) = 2$, there is $E \in |\mathcal{O}_Y(2h+4f)|$ containing $D \cup D' \cup D_1$ and a point of $S \setminus S \cap (D \cup D' \cup D_1)$. Hence $s_1 \ge 3m+1$. Hence $s_2 \le m-1$, a contradiction. □

Proof of Theorem 1. The “ if ” part of cases (a), (b), (c), (d), (e) was proved in [1]. See Example 1 for a proof of the “ if ” part of case (f). The “ only if ” part is true by Lemma 2 (case $h^0(\mathcal{I}_S(2)) = 0$), Lemma 3 (case in which S is contained in a smooth quadric surface), Lemma 4 (case in which S is contained in an irreducible quadric cone) and Lemma 1 (case in which S is contained in a reducible quadric surface). □

Lemma 5. *Fix a finite set $S \subset \mathbb{P}^3$ such that $h^1(\mathcal{I}_S(m)) > 0$ and $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$. Assume $h^0(\mathcal{I}_S(2)) = 0$. Let β be the maximal cardinality of a subset of S contained in a quadric surface (not necessarily smooth or irreducible). Then $\sharp(S) - \beta \ge m-1$.*

Proof. Fix a quadric surface $T \subset \mathbb{P}^3$ such that $\sharp(S \cap T) = \beta$. Assume $b := \sharp(S) - \beta \le m-2$ and make an ordering $P_1, \dots, P_b$ of the points of $S \setminus S \cap T$. Set

$S' := S \backslash \{P_b\}$. Since $S' \subsetneq S$, we have $h^1(\mathcal{I}_{S'}(m)) = 0$. Since S is finite, there are planes H_i, $1 \le i \le b-1$, such that $H_i \cap S_i := \{P_i\}$. Set $A := T \cup H_1 \cup \cdots \cup H_{b-1}$. Since $A \cap S = S'$, we have $h^0(\mathcal{I}_S(b+1)) < h^0(\mathcal{I}_{S'}(b+1))$. Hence $h^0(\mathcal{I}_S(m)) < h^0(\mathcal{I}_{S'}(m))$. Hence $h^1(\mathcal{I}_S(m)) = h^1(\mathcal{I}_{S'}(m)) = 0$, a contradiction. □

Lemma 6. *Fix a finite set $S \subset \mathbb{P}^3$ such that $h^1(\mathcal{I}_S(m)) > 0$ and $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$. Assume $h^0(\mathcal{I}_S(2)) = 1$ and call B the quadric surface containing S. Let β be the maximal cardinality of a subset of S contained in another quadric surface (not necessarily smooth or irreducible). Then $\sharp(S) - \beta \ge m - 1$.*

Proof. Fix a quadric surface $T \subset \mathbb{P}^3$ such that $T \ne B$ and $\sharp(S \cap T) = \beta$. Assume $b := \sharp(S) - \beta \le m - 2$ and make an ordering $P_1, \dots, P_b$ of the points of $S \setminus S \cap T$. Set $S' := S \setminus \{P_b\}$. Since $S' \subsetneq S$, we have $h^1(\mathcal{I}_{S'}(m)) = 0$. Since S is finite, there are planes H_i, $1 \le i \le b-1$, such that $H_i \cap S_i := \{P_i\}$. Set $A := T \cup H_1 \cup \cdots \cup H_{b-1}$. Since $A \cap S = S'$, we have $h^0(\mathcal{I}_S(b+1)) < h^0(\mathcal{I}_{S'}(b+1))$. Hence $h^0(\mathcal{I}_S(m)) < h^0(\mathcal{I}_{S'}(m))$. Hence $h^1(\mathcal{I}_S(m)) = h^1(\mathcal{I}_{S'}(m)) = 0$, a contradiction. □

Lemma 7. *Fix an integer $m \ge 4$, and 4 pairwise disjoint lines $L_i \subset \mathbb{P}^3$, $1 \le i \le 4$. Fix a finite set $A \subset \mathbb{P}^3$ such that $\sharp(A \cap L_i) \le m+1$ for all i and $\sharp(A \setminus A \cap (L_1 \cup L_2 \cup L_3 \cup L_4)) \le m - 4$. Then $h^1(\mathcal{I}_A(m)) = 0$.*

Proof. Adding points to A if necessary, we may assume $\sharp(A \cap L_i) = m+1$ for all i. Hence every degree m surface containing A contains $L_1 \cup L_2 \cup L_3 \cup L_4$. Set $A' := A \setminus A \cap (L_1 \cup L_2 \cup L_3 \cup L_4)$. Let $Q \subset \mathbb{P}^3$ be the only quadric containing $L_1 \cup L_2 \cup L_3$ (Q is smooth). First assume $L_4 \subset Q$. Call $|\mathcal{O}_Q(1,0)|$ the system of lines containing L_1. Since $L_i \cap L_1 = \emptyset$ for all $i \ne 1$, we have $L_i \in |\mathcal{O}_Q(1,0)|$. Since $\sharp(A \cap L_i) = m+1$ for all i, we have $h^i(Q, \mathcal{I}_{Q\cap A}(m,m)) = h^i(Q, \mathcal{I}_{A'\cap Q}(m-4,m))$, $i = 0,1$. Since $\sharp(A' \cap Q) \le \sharp(A') \le m-3$, we have $h^1(Q, \mathcal{I}_{A'\cap Q}(m-4,m)) = 0$. Since $\sharp(A' \setminus A' \cap Q) \le \sharp(A') \le m-1$, we have $h^1(\mathcal{I}_{A\setminus A\cap Q}(m-2)) = 0$. Hence $h^1(\mathcal{I}_A(m)) = 0$.

Now assume that Q does not contain L_4. Set $Z := Q \cap L_4$ (scheme-theoretic intersection). The scheme Z has degree 2 and $L_i \cap Z = \emptyset$ for all $i = 1,2,3$, because $L_i \cap L_4 = \emptyset$ if $i \ne 4$. Set $B := (A \cap Q) \cup Z$. Fix $P_1, P_2 \in L_4 \cap A$ such that $P_1 \ne P_2$ and $P_i \notin Q$ for all i. Set $W := (A \setminus \{P_1, P_2\}) \cup Z$. Since $\deg(L_4 \cap W) = \deg(L_4 \cap A) = m+1$ and $A \setminus A \cap L_4 = W \setminus W \cap L_4$, we have $H^0(\mathcal{I}_A(m)) = H^0(\mathcal{I}_W(m))$. Since $\deg(A) = \deg(W)$, we get $h^1(\mathcal{I}_A(m)) = h^1(\mathcal{I}_W(m))$. Set $W' := W \setminus W \cap (L_1 \cup L_2 \cup L_3)$. Since $\deg(W \cap L_i) = m+1$ for $i = 1,2,3$, we have $h^i(Q, \mathcal{I}_{Q\cap W}(m,m)) = h^i(Q, \mathcal{I}_{W'}(m-3,m))$, $i = 0,1$. Since

$\deg(W') \le \deg(A') + \deg(Z) \le m-2$, we have $h^1(Q, \mathcal{I}_{W'}(m-3,3)) = 0$. Hence to prove $h^1(\mathcal{I}_W(m)) = 0$ (and hence to prove $h^1(\mathcal{I}_A(m)) = 0$), it is sufficient to prove $h^1(\mathcal{I}_{W \setminus W \cap Q}(m-2)) = 0$ (see [1], Remark 1). The set $W \setminus W \cap Q$ is the union of $m-1$ points of L_4 and at most $\sharp(A')$ points not on L_4. Use that $\sharp(A') \le m-2$. □

Lemma 8. *Fix an integer $m \ge 4$, and 4 lines $L_i \subset \mathbb{P}^3$, $1 \le i \le 4$, such that $L_1 \cap L_2$ is a point and $L_i \cap L_j = \emptyset$ for all $i \ne j$ with $\{i,j\} \ne \{1,2\}$. Fix a finite set $A \subset \mathbb{P}^3$ such that $\sharp(A \cap L_i) \le m+1$ for all i, $\sharp(A \cap (L_1 \cup L_2)) \le 2m+1$ and $\sharp(A \setminus A \cap (L_1 \cup L_2 \cup L_3 \cup L_4)) \le m-4$. Then $h^1(\mathcal{I}_A(m)) = 0$.*

Proof. Set $\{O\} := L_1 \cup L_2$. Adding points to A we reduce to the case $\sharp(A \cap L_3) = \sharp(A \cap L_4) = m+1$ and $\sharp(A \cap (L_1 \cup L_2)) = 2m+1$. Let Q be the only quadric containing L_2, L_3, L_4. The quadric Q is smooth and we call $|\mathcal{O}_Q(1,0)|$ the ruling of Q containing L_2, L_3 and L_4. Since $L_1 \cap L_3 = \emptyset$ and $L_1 \cap L_2 \ne \emptyset$, L_1 is not contained in Q. Hence the scheme $Z := L_1 \cap Q$ has degree 2 and it contains O. Fix any finite set $E_i \subset L_i \setminus \{O\}$, $i = 1,2$, such that $\sharp(E_1) = m-1$ and $\sharp(E_2) = m$. Set $W := Z \cup E_1 \cup E_2 \cup (A \setminus A \cap (L_1 \cup L_2))$. Notice that $\sharp(A) = \deg(W)$ and $H^0(\mathcal{I}_A(m)) = H^0(\mathcal{I}_W(m))$. Hence $h^1(\mathcal{I}_A(m)) = h^1(\mathcal{I}_W(m))$. Hence it is sufficient to prove $h^1(\mathcal{I}_W(m)) = 0$. Since $\deg(W \cap L_i) = m+1$ for $i = 2,3,4$, we have $h^i(Q, \mathcal{I}_{Q \cap W}(m,m)) = h^i(Q, \mathcal{I}_{Q \cap W \setminus W \cap (L_2 \cup L_3 \cup L_4)}(m-3,m))$. Since $\deg(Q \cap W \setminus W \cap (L_2 \cup L_3 \cup L_4)) \le \deg(Z) + \sharp(A \setminus A \cap (L_1 \cup L_2 \cup L_3 \cup L_4)) \le m-2$, we have $h^1 Q, \mathcal{I}_{Q \cap W \setminus W \cap (L_2 \cup L_3 \cup L_4)}(m-3,m)) = 0$. Hence it is sufficient to prove $h^1(\mathcal{I}_{W \setminus W \cap Q}(m-2)) = 0$ (see [1], Remark 1). Since $W \setminus W \cap Q$ is the union of E_1 (i.e. $m-1$ points of L_1) and at most $m-2$ points, none of them on L_1, we have $h^1(\mathcal{I}_{W \setminus W \cap Q}(m-2)) = 0$ (e.g., by [3], Theorem 3.8). □

Lemma 9. *Fix an integer $m \ge 6$ and 4 distinct lines $L_i \subset \mathbb{P}^3$, $1 \le i \le 4$, such that $X := L_1 \cup L_2 \cup L_3$ is a connected curve with arithmetic genus 0 and $L_i \cap L_4 = \emptyset$ for all $i \ne 4$. Fix a finite set $A \subset \mathbb{P}^3$ such that $\sharp(A \cap L_i) \le m+1$ for all i, $\sharp(A \cap X) \le 3m+1$, $\sharp(A \cap F) \le 2m+1$ for each connected degree 2 curve $F \subset X$ and $\sharp(A) - \sharp(A \cap (X \cup L_4)) \le m-6$. Then $h^1(\mathcal{I}_A(m)) = 0$.*

Proof. Set $A' := A \setminus A \cap (T \cup L_4)$. The assumptions made on the set $A \cap X$ imply the surjectivity of the restriction map $\rho : H^0(X, \mathcal{O}_X(m)) \to H^0(A \cap X, \mathcal{O}_{A \cap X}(m))$. Adding points to A if necessary we may assume $\sharp(A \cap L_4) = m+1$. There is an irreducible quadric surface U containing X (U is a cone if and only if $L_1 \cap L_2 \cap L_3 \ne \emptyset$). Since $X \cap L_4 = \emptyset$, U does not contain L_4. Hence the scheme $Z := L_4 \cap U$ has degree 2. Notice that if U is a cone, then Z does not contain the vertex of U. Fix $E \subset L_4 \setminus Z_{red}$ such that $\sharp(E) = m-1$

and set $W := E \cup Z \cup (A \setminus A \cap L_4)$. Since $\deg(W \cap L_4) = \deg(A \cap L_4)$, we have $H^0(\mathcal{I}_A(m)) = H^0(\mathcal{I}_W(m))$. Hence $h^1(\mathcal{I}_A(m)) = h^1(\mathcal{I}_W(m))$. Hence it is sufficient to prove that $h^1(\mathcal{I}_W(m)) = 0$. Since $h^1(X, \mathcal{I}_{W\cap X}(m)) = 0$ and $\deg(W \cap U) - \deg(W \cap X) \le m-4$, it is easy to check that $h^1(U, \mathcal{I}_{W\cap U}(m)) = 0$ (even if U is a cone). Hence it is sufficient to prove $h^1(\mathcal{I}_{W\setminus W\cap U}(m-2)) = 0$. $W \setminus W \cap U$ is the union of E (i.e. of $m-1$ points of L_4) and at most $m-6$ points, none of them on L_4. Hence $h^1(\mathcal{I}_{W\setminus W\cap U}(m-2)) = 0$. □

Proof of Theorem 2. Assume the existence of a finite set $S \subset \mathbb{P}^3$ such that $\sharp(S) \le 4m+\epsilon$, $h^1(\mathcal{I}_S(m)) > 0$, $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$ and $h^0(\mathcal{I}_S(2)) = 0$. We need to find a contradiction if $m \gg \epsilon$. Set $S_0 := S$. Let $E_1 \subset \mathbb{P}^3$ be a quadric surface such that $s_1 := \sharp(S \cap E_1)$ is maximal. Set $S_1 := S_0 \setminus S \cap E_1$. For every integer $i \ge 2$ define recursively the quadric surface E_i, the integer s_i and the set $S_i \subseteq S_{i-1}$ in the following way. Let E_i be a quadric surface such that $s_i := \sharp(E_i \cap S_{i-1})$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap E_i$. Since S is reduced, we may take each E_i with the additional condition that it is reduced. By [1], Remark 1, there an integer $t \ge 1$ such that $h^1(E_t, \mathcal{I}_{E_t\cap S_{t-1}}(m+2-2t)) > 0$. Call e the minimal such an integer t. Since $h^0(\mathcal{I}_S(2)) = 0$ and $h^1(\mathcal{I}_{S'}(m)) = 0$ for all $S' \subsetneq S$, we have $e \ge 2$.

(a) Assume $e = 2$. Since $s_1 \ge s_2$ and $s_1 + s_2 \le 4m + \epsilon$, we have $s_2 \le 2m+\epsilon/2 < 3(m-1)$. Hence either there is a line $D \subset E_1$ such that $\sharp(S_1 \cap D) \ge m$ or there is a plane conic F such that $\sharp(S_1 \cap F) \ge 2m-2$ (see [3], Theorem 3.8).

We first assume the existence of F as above. Let $\langle F\rangle$ be the plane spanned by F. If $h^1(\mathcal{I}_{\langle F\rangle\cap S}(m)) > 0$, then the minimality of S gives $S \subset \langle F\rangle$. Hence $h^0(\mathcal{I}_S(2)) \ge h^0(\mathcal{I}_S(1)) > 0$, a contradiction. Now assume $h^1(\mathcal{I}_{\langle F\rangle\cap S}(m)) = 0$. Hence $h^1(\mathcal{I}_{S\setminus S\cap\langle F\rangle}(m-1)) > 0$ (see [1], Remark 1). Since $\sharp(S \setminus S \cap \langle F\rangle) \le 2m+2 \le 3(m-1)$, there is a plane curve T containing at least $m+1$ points of $S \setminus S \cap \langle F\rangle$. Since $T \cup F$ is contained in a reducible quadric, we get $s_1 \ge 3m-1$. Hence $s_2 \le m+1+\epsilon < 2m-2$, a contradiction.

Hence there is a line D such that $\sharp(S_1 \cap D) \ge m$. Let $V_1 \subset \mathbb{P}^3$ be a quadric surface containing D and such that $\sharp(V_1 \cap S)$ is maximal. Set $Z_1 := S \setminus S \cap V_1$. Define recursively the quadric surface V_i, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \ge 2$, in the following way. Let V_i be any quadric surface such that $z_i := \sharp(V_i \cap Z_{i-1})$ is maximal. Set $Z_i := Z_{i-1} \setminus Z_{i-1} \cap V_i$. The sequence $\{z_i\}_{i\ge1}$ is non-decreasing. Since S is a reduced scheme, without losing generality we may find a sequence $\{V_i\}_{i\ge1}$ with each V_i reduced. Since $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$, either $z_i \ge 9$ or $z_{i+1} = 0$. By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t\cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such that $h^1(V_t, \mathcal{I}_{V_t\cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then the minimality property of S gives

$S \subset V_1$, a contradiction. For the case $c \geq 3$, see steps (b) and (c) below. Here we assume $c = 2$. Since $z_1 \geq z_2$, we have $z_2 \leq 2m + \epsilon/2 < 3(m-1)$. As above we exclude the existence of a plane conic $F \subset V_2$ such that $\sharp(F \cap (S \setminus S \cap V_1)) \geq 2m - 2$. Hence there is a line $R \subset V_2$ such that $\sharp(R \cap Z_1) \geq m$. There is a quadric surface U containing $D \cup R$ and at least 3 points of $S \setminus S \cap (D \cup R)$. We have $h^1(U, \mathcal{I}_{U \cap S}(m)) = 0$, because $S \cap U \neq S$ since $h^0(\mathcal{I}_S(2)) = 0$. Hence $h^1(\mathcal{I}_{S \setminus S \cap U}(m-2)) > 0$ (see [1], Remark 1). As above we get the existence of a line D' such that $\sharp(D' \cap (S \setminus S \cap U)) \geq m$. Hence $\sharp(S \cap (D \cup D' \cup R)) \geq 3m$. Since $D \cup D' \cup R$ is contained in a quadric surface, we have $s_1 \geq 3m$. Hence $s_2 \leq m + \epsilon$. Since $\sharp(S \setminus S \cap (D \cup D' \cup D_1)) \leq m + \epsilon < 2(m-2) + 1$, there a line $L' \subset \mathbb{P}^3$ such that $\sharp(L' \cap (S \setminus S \cap (D \cup D' \cup D_1)) \geq m$. Set $M := D \cup D' \cup D' \cup L$. Notice that $\sharp(S \cap M) \geq 4m$ and hence $\sharp(S \setminus S \cap M) \leq \epsilon \leq m - 6$. Lemma 5 gives $h^0(\mathcal{I}_M(2)) = 0$. Hence M is not connected and it is not the union of two reducible conics. Hence we may order the lines L_1, L_2, L_3, L_4 of M so that they satisfy one of the set of assumptions of Lemmas 7, 8 or 9. We also know that $\sharp(S \cap J) \leq m+1$ for each of the irreducible components of M, $\sharp(M \cap J_2) \leq 2m+1$ for each connected degree 2 subcurve of M and $\sharp(S \cap J_3) \leq 3m+1$ for the degree 3 connected component of M (if there is such a connected component). Lemmas 7, 8 and 9 with $A := S$ give $h^1(\mathcal{I}_S(m)) = 0$.

(b) Assume $e = 3$. Since $s_1 \geq s_2 \geq s_3$, $s_1 + s_2 + s_3 \leq 4m$ and $m \geq 8$, we have $s_3 < 3(m-4)$. Since $h^1(E_3, \mathcal{I}_{S_2 \cap E_3, E_3}(m-4)) > 0$, either there is a line $J \subset \mathbb{P}^3$ such that $\sharp(J \cap S_2 \cap E_3) \geq m - 2$ or there is a conic $J_1 \subset \mathbb{P}^3$ such that $\sharp(J_1 \cap S_2 \cap E_3) \geq 2m - 6$. In the latter case we get $s_2 \geq 2m - 2$, because $h^0(\mathcal{I}_{J_1}(2)) = 5$. Hence in the latter case we get $4m + \epsilon \geq s_1 + s_2 + s_3 \geq 2m - 2 + 2m - 2 + 2m - 6$, a contradiction. Now assume the existence of a line $J \subset \mathbb{P}^3$ such that $\sharp(J \cap S_2 \cap E_3) \geq m - 2$. Let $V_1 \subset \mathbb{P}^3$ be a quadric surface containing J and such that $\sharp(V_1 \cap S)$ is maximal. Set $Z_1 := S \setminus S \cap V_1$. Define recursively the quadric surface V_i, the non-negative integer z_i and the set $Z_i \subseteq Z_{i-1}$, $i \geq 2$, as in step (a). By [1], Remark 1, there is an integer $t > 0$ such that $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. Let c be the minimal integer t such $h^1(V_t, \mathcal{I}_{V_t \cap Z_{t-1}}(m+2-2t)) > 0$. If $c = 1$, then the minimality of S gives $S \subset V_1$, a contradiction. If $c = 2$, then we conclude as in step (a). If $c \geq 4$, then see step (c). Now assume $c = 3$. As above we exclude the existence of a conic J_1' such that $\sharp(J_1' \cap Z_2) \geq 2m - 6$. Hence there is a line R_1 such that $\sharp(R_1 \cap Z_2) \geq m - 2$. Let M_1 be a quadric surface containing $J \cup R_1$ and with $w_i := \sharp(M_1 \cap S)$ maximal. Set $S_1' := S \setminus S \cap M_1$. For each integer $i \geq 2$ define recursively the quadric surface M_i, the integer w_i and the set S_i' in the following way. Let M_i be any quadric surface such that $w_i := \sharp(M_i \cap S_{i-1}')$ is maximal. Set $S_i' := S_{i-1}' \setminus S_{i-1}' \cap M_i$. There is an integer t such that $2 \leq t \leq (m+2)/2$ and

$h^1(M_t, \mathcal{I}_{M_t \cap S'_{t-1}}(m+2-2t)) > 0$ and we call μ the minimal such an integer. If $\mu = 2$, then we repeat the proof of step (a). If $\mu \geq 4$, then see step (c). Now assume $\mu = 3$. As in the proof of Lemma 2 we exclude the case $w_3 \geq 2(m-4)+2$. Since $w_3 \leq 2(m-4)+1$, there is a line T such that $\sharp(T \cap M_3 \cap S_2) \geq m-2$. Taking a quadric surface containing R, J and T we get $s_1 \geq 3m-2$. Since $s_3 \geq m-2$, we have $s_2 \geq m-2$. Hence $4m+\epsilon \geq (3m-3)+(m-2)+(m-2)$, a contradiction.

(c) Assume $e \geq 4$. We modify step (c) of the proof of Lemma 2 in the following way. We have $s_e + 9(e-1) \leq 4m+\epsilon$. Hence for $m \gg \epsilon$ we have $2e \leq m-4$. For instance, to adapt part (c3) we just use that $\eta((m-4)/2) = 7(m-4) > 4m+\epsilon$ if $m \gg \epsilon$. In the same way we conclude if either $e(1) \geq 4$ or $c \geq 4$ or $\mu \geq 4$. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Finite subsets of projective spaces with bad postulation in a fixed degree, *Beitr. Algebra Geom.*, DOI 10.1007/s13366-012-0104-8.

[2] E. Ballico, Finite sets and postulation: further remarks, *Journal of Pure and Applied Math.*, To Appear.

[3] A. Couvreur, The dual minimum distance of arbitrary dimensional algebraic-geometric codes, *J. Algebra*, **350**, No. 1 (2012), 84-107.

[4] Ph. Ellia, Ch. Peskine, Groupes de points de $\mathbf{P}^2$: caractère et position uniforme, *Algebraic Geometry*, L'Aquila (1988), 111-116; *Lecture Notes in Math.*, **1417**, Springer, Berlin (1990).