Abstract: For a graph $G$, let $P(G, \lambda)$ denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is chromatically unique (or simply $\chi-$unique) if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e., $H$ is isomorphic to $G$. A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. In this paper, we discuss a pair of chromatically equivalent of $K_4$-homeomorphs with girth 9, that is, $K_4(1,3,5,d,e,f)$ and $K_4(1,3,5,d',e',f')$. As a result, we obtain two infinite chromatically equivalent non-isomorphic $K_4$-homeomorphs.

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1. Introduction

All graphs considered here are simple graphs. For such a graph $G$, let $P(G, \lambda)$ denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is chromatically unique if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e., $H$ is isomorphic to $G$. A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. In this paper, we discuss a pair of chromatically equivalent of $K_4$-homeomorphs with girth 9, that is, $K_4(1,3,5,d,e,f)$ and $K_4(1,3,5,d',e',f')$. As a result, we obtain two infinite chromatically equivalent non-isomorphic $K_4$-homeomorphs.

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equivalent (or simply $\chi$–equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is chromatically unique (or simply $\chi$–unique) if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e, $H$ is isomorphic to $G$.

A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ if the six edges of $K_4$ are replaced by the six paths of length $a, b, c, d, e, f$, respectively, as shown in Figure 1. So far, the chromaticity of $K_4$-homeomorphs with girth $g$, where $3 \leq g \leq 7$ has been studied by many authors (see [2,5,6,7]). The chromaticity of $K_4$-homeomorphs with girth 8 or 9 is still remains open. For some results on chromatic equivalence of $K_4$-homeomorphs with girth 8, the reader is referred to [3,8,9].

In this paper, we shall discuss a chromatically equivalence pair of $K_4$-homeomorphs, $K_4(1, 3, 5, d, e, f)$ (as in Figure 2) and $K_4(1, 3, 5, d', e', f')$. We obtain two infinite chromatically equivalent non-isomorphic $K_4$-homeomorphs. This result can be extended to the study of chromatic equivalence classes of $K_4(1, 3, 5, d, e, f)$ and chromatic uniqueness of $K_4$-homeomorphs with girth 9.

2. Preliminary Results

In this section, we give some known results used in the sequel.
Lemma 2.1. Assume that $G$ and $H$ are $\chi-$equivalent. Then

1. $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (see [4]);

2. $G$ and $H$ have the same girth and same number of cycles with length equal to their girth (see [11]);

3. If $G$ is a $K_4$-homeomorph, then $H$ must itself be a $K_4$-homeomorph (see [1]);

4. Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$, then
   
   (i) $\min (a, b, c, d, e, f) = \min (a', b', c', d', e', f')$ and the number of times that this minimum occurs in the list $\{a, b, c, d, e, f\}$ is equal to the number of times that this minimum occurs in the list $\{a', b', c', d', e', f'\}$ (see [10]);
   
   (ii) if $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets, then $H \cong G$ (see [5]).

Theorem 2.1. (Yanling Peng [8]) Let $K_4(1, 2, 5, d, e, f)$ and $K_4(1, 2, 5, d', e', f')$ be chromatically equivalent, then

$$K_4(1, 2, 5, i, i + 6, i + 1) \sim K_4(1, 2, 5, i + 2, i, i + 5),$$
In this section, we present our main results.

\[ K_4(1, 2, 5, i, i + 1, i + 6) \sim K_4(1, 2, 5, i + 5, i, i + 2), \]
\[ K_4(1, 2, 5, i, i + 1, i + 3) \sim K_4(1, 2, 5, i + 2, i + 2, i). \]

**Theorem 2.2.** (Yanling Peng [9]) Let \( K_4(2, 3, 3, d, e, f) \) and \( K_4(2, 3, 3, d', e', f') \) be chromatically equivalent, then \( K_4(2, 2, 3, d, e, f) \) is isomorphic to \( K_4(2, 2, 3, d', e', f') \).

**Theorem 2.3.** (Roslan et al. [3]) Let \( K_4(1, 3, 4, d, e, f) \) and \( K_4(1, 3, 4, d', e', f') \) be chromatically equivalent, then

\[ K_4(1, 3, 4, i, i + 5, i + 1) \sim K_4(1, 3, 4, i + 2, i, i + 4), \]
\[ K_4(1, 3, 4, i, i + 1, i + 4) \sim K_4(1, 3, 4, i + 2, i + 3, i). \]

where \( i \geq 1. \)

3. Main Results

In this section, we present our main results.

**Lemma 3.1.** Assume that \( G \cong K_4(1, 3, 5, d, e, f) \) and \( H \cong K_4(1, 3, 5, d', e', f') \), then

1. \( P(G) = (-1)^{x-1}[s/(s - 1)^2][-s^x - 4 - s^4 - s^3 + 2s + 2 + R(G)], \)
   where
   \[ R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{d+2} + s^{e+4} + s^{e+5} + s^{d+6} + s^{d+e+f}, \]
   \( s = 1 - \lambda, x \) is the number of the edges of \( G. \)
2. If \( P(G) = P(H) \), then \( R(G) = R(H). \)

**Proof.** (1) Let \( s = 1 - \lambda. \) From [10], we have the chromatic polynomial of \( K_4 \)-homeomorphs \( K_4(a, b, c, d, e, f) \) is as follows:

\[ P(K_4(a, b, c, d, e, f)) = (-1)^{x-1}[s/(s - 1)^2][s^2 + 3s + 2 - (s + 1)(s^a + s^b + s^c + s^d + s^e + s^f) + (s^{a+d} + s^{b+f} + s^{c+e} + s^{a+b+e} + s^{b+d+c} + s^{a+c+f} + s^{d+e+f} - s^{x-1})] \]

So, when \( a = 1, b = 3 \) and \( c = 5 \), we have

\[ P(K_4(1, 3, 5, d, e, f)) = (-1)^{x-1}[s/(s - 1)^2][s^2 + 3s + 2 - (s + 1)(s^3 + s^5 + s^d + s^e + s^f) + \]
isomorphic. From Lemma 3.1, we have
\[ R = \text{as required.} \]

We now solve for the equation \( R \) with \( d \) by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), the following steps of the proof is:
\[ R = -(s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8} + s^{d+e+f}) \]
as required.

(2) If \( P(G) = P(H) \), then we can easily see that \( R(G) = R(H) \).

\[ \text{Theorem 3.1. Let } K_4 \text{-homeomorphs } K_4(1, 3, 5, d, e, f) \text{ and } K_4(1, 3, 5, d', e', f') \text{ be chromatically equivalent, then we have} \]
\[ K_4(1, 3, 5, i, i + 6, i + 1) \sim K_4(1, 3, 5, i + 2, i, i + 5), \]
\[ K_4(1, 3, 5, i, i + 1, i + 4) \sim K_4(1, 3, 5, i + 2, i + 3, i). \]

where \( i \geq 1 \).

\[ \text{Proof. Assume that } G \cong K_4(1, 3, 5, d, e, f) \text{ and } H \cong K_4(1, 3, 5, d', e', f'). \]

We now solve for the equation \( R(G) = R(H) \) to find \( G \) and \( H \) which are not isomorphic. From Lemma 3.1, we have
\[ R(G) = -(s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8} + s^{d+e+f}) \]
\[ R(H) = -(s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{f'+3} + s^{e'+4} + s^{e'+5} + s^{f'+6} + s^{d'+8} + s^{d'+e+f'}). \]

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), \( d + e + f = d' + e' + f' \). We obtain the following after simplification. Note that our assumption in the following steps of the proof is \( R_j(G) = R_j(H) \), where \( 1 \leq j \leq 19 \).
\[ R_1(G) = -(s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}) \]
\[ R_1(H) = -(s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{f'+3} + s^{e'+4} + s^{e'+5} + s^{f'+6} + s^{d'+8}). \]

Let us consider the h.r.p. in \( R_1(G) \) and the h.r.p. in \( R_1(H) \). We have max \( \{e + 5, f + 6, d + 8\} \) = max \( \{e' + 5, f + 6, d' + 8\} \). Without loss of generality, we will consider only the following six cases.

Case 1. If max \( \{e + 5, f + 6, d + 8\} = e + 5 \) and max \( \{e' + 5, f' + 6, d' + 8\} = e' + 5 \), then \( e = e' \). Thus, we can cancel the following pairs of terms in the equations: \(-s^e \) with \(-s^{e'} \), \(-s^{e+1} \) with \(-s^{e'+1} \), \( s^{e+4} \) with \( s^{e'+4} \) and \( s^{e+5} \) with \( s^{e'+5} \). Therefore, the l.r.p. in \( R_1(G) \) is \( d \) or \( f \) and the l.r.p. in \( R_1(H) \) is \( d' \) or \( f' \). So, \( d = f' \) or \( d = d' \) or \( f = f' \) or \( f = d' \). We have \( e = e' \) and \( d + e + f = d' + e' + f' \). So, we know that \( \{d, e, f\} = \{d', e', f'\} \) as multisets. From Lemma 2.1 (4(ii)), \( G \cong H \).
Case 2. If \( \max \{e + 5, f + 6, d + 8\} = f + 5 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = f' + 5 \), then \( f = f' \). We can deal with this case in the same way as Case 1, thus, \( G \cong H \).

Case 3. If \( \max \{e + 5, f + 6, d + 8\} = d + 7 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = d' + 7 \), then we can deal with this case in the same way as Case 1. So, we have \( G \cong H \).

Case 4. If \( \max \{e + 5, f + 6, d + 8\} = e + 5 \) and \( \max \{e' + 5, f' + 6, d' + 8\} = f' + 6 \), then \( e + 5 = f' + 6 \), that is

\[
\hat{f}' = e - 1 \tag{1}
\]

from \( d + e + f = d' + e' + \hat{f}' \), we have

\[
d + f = d' + e' - 1. \tag{2}
\]

Consider the l.r.p. in \( R_1(G) \) and the l.r.p. in \( R_1(H) \). From Lemma 2.1(4(i)), \( \min \{d, e, f\} = \min \{d, e', f'\} \). Without loss of generality, let \( \min \{d, e, f\} = d \).

The following subcases need to be considered.

Subcase 4.1. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = d' \), then \( d = d' \).

Thus, we can consider this case the same way as Case 1. So, \( G \cong H \).

Subcase 4.2. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = e' \), then \( d = e' \). From Equation (2), we have \( d = f + 1 \). Note that \( \hat{f}' = e - 1 \) by Equation (1). We can write \( R_1(G) \) and \( R_1(H) \) as follows:

\[
R_2(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},
\]

\[
R_2(H) = -s^{f+1} - s^d - s^{e-1} - s^{d+1} - s^e + s^{e+2} + s^{d+4} + s^{d+5} + s^{e+5} + s^{f+9}.
\]

After simplifying \( R_2(G) \) and \( R_2(H) \), we have:

\[
R_3(G) = -s^f - s^{e+1} + s^{f+3} + s^{e+4} + s^{f+6} + s^{d+8},
\]

\[
R_3(H) = -s^{e-1} - s^{d+1} + s^{e+2} + s^{d+4} + s^{d+5} + s^{f+9}.
\]

Consider the term \(-s^{d+1}\) in \( R_3(H) \). Since the min \( \{d, e, f\} = d \), \(-s^{d+1}\) in \( R_3(H) \) cannot be cancelled by any of the positive terms in \( R_3(H) \). Thus, \(-s^{d+1}\) must be equal to \(-s^f\) or \(-s^{e+1}\) in \( R_3(G) \). Note that max \( \{e + 5, f + 6, d + 8\} = e + 5 \), so \( e + 5 \geq d + 8 \), that is, \( e + 1 \geq d + 4 \geq d + 1 \). Thus, \(-s^{e+1} \neq -s^{d+1}\).

If \(-s^{d+1} = -s^f\), then \( d + 1 = f \). Thus, \( R_3(G) \) and \( R_3(H) \) can be written as follows:

\[
R_4(G) = -s^{d+1} - s^{e+1} + s^{d+4} + s^{e+4} + s^{d+7} + s^{d+8},
\]

\[
R_4(H) = -s^{e-1} - s^{d+1} + s^{e+2} + s^{d+4} + s^{d+5} + s^{d+10}.
\]

After simplifying \( R_4(G) \) and \( R_4(H) \), we have

\[
-s^{e+1} + s^{e+4} + s^{d+7} + s^{d+8} = -s^{e-1} + s^{e+2} + s^{d+5} + s^{d+10}.
\]
Therefore, we have \( e = d + 6 \). At this point, we acquire the following equations: \( e = d + 6, f = e - 1 = d + 5, f = d + 1, d = f + 1 = d + 2 \) and \( e' = d \). Let \( d = i \). Therefore, we obtain the solution where \( G \) is isomorphic to \( K_4(1, 3, 5, i, i + 6, i + 1) \) and \( H \) is isomorphic to \( K_4(1, 3, 5, i + 2, i, i + 5) \).

Subcase 4.3. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = f' \), then \( d = f' \). Note that \( \max \{e' + 5, f' + 6, d + 8\} = f' + 6 \). So, \( f' + 6 \geq d + 8 \), that is, \( f' \geq d + 2 > d \). This contradicts with the fact that \( \min \{d', e', f'\} = f' \).

Case 5. If \( \max \{e + 5, f + 6, d + 8\} = f + 6 \) and \( \max \{e' + 5, f' + 6, d + 8\} = d' + 8 \), then \( f + 6 = d' + 8 \), that is,

\[
d' = f - 2 \tag{3}
\]

from \( d + e + f = d' + e' + f' \), we have

\[
d + e = e' + f' - 2 \tag{4}
\]

Consider the l.r.p. in \( R_1(G) \) and the l.r.p. in \( R_1(H) \), where \( \min \{d, e, f\} = \min \{d', e', f'\} \) by Lemma 2.1(4(i)). Without loss of generality, let \( \min \{d, e, f\} = d \). The following subcases need to be considered.

Subcase 5.1. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = d' \), then we deal with this case the same way with Case 1. So, we get \( G \cong H \).

Subcase 5.2. If \( \min \{d, e, f\} = d \) and \( \min \{d', e', f'\} = e' \), then \( d = e' \). From Equation (4), we have \( f' = e + 2 \). Note that \( d' = f - 2 \) by Equation (3). Thus, we can write \( R_1(G) \) and \( R_1(H) \) as follows:

\[
R_5(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{d+3} + s^{e+5} + s^{d+6} + s^{e+8},
\]

\[
R_5(H) = -s^{f-2} - s^d - s^{e+2} - s^{d+1} - s^{e+3} + s^{d+4} + s^{d+5} + s^{d+8} + s^{f+6}.
\]

Consider the term \(-s^{d+1}\) in \( R_6(H) \). Since \( \min (d, e, f) = d \), then \(-s^{d+1}\) cannot be cancelled by any positive terms in \( R_5(H) \). Note that \( \max \{e + 5, f + 6, d + 8\} = f + 6 \), so \( f + 6 \geq d + 8 \), that is \( f + 1 \geq d + 3 > d + 1 \), thus \( f + 1 \neq d + 1 \), i.e., \(-s^{d+1} \neq s^{f+1}\). Moreover \( f \geq d + 2 > d + 1 \), thus \( d \neq d + 1 \), i.e., \(-s^{d+1} \neq -s^{f} \). Therefore the term \(-s^{d+1}\) in \( R_5(H) \) must be cancelled by the term \(-s^e\) or \(-s^{e+1}\) in \( R_5(G) \).

If \(-s^{d+1} = -s^e\), then \( e = d + 1 \). Thus, \( R_5(G) \) and \( R_5(H) \) can be written as follows:

\[
R_6(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{d+3} + s^{d+5} + s^{d+6} + s^{f+6} + s^{d+8},
\]

\[
R_6(H) = -s^{f-2} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+6} + s^{d+9} + s^{f+6}.
\]

After simplifying, we obtain

\[
-s^f - s^{d+2} - s^{f+1} + s^{d+8} = -s^{f-2} - s^{d+3} + s^{d+9} + s^{f+6}.
\]
The resulting equations contradict $R_6(G) = R_6(H)$.

If $-s^{d+1} = -s^{e+1}$, then $e = d$. Thus, $R_5(G)$ and $R_5(H)$ can be written as follows:

$$R_7(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{f+6} + s^{d+8},$$
$$R_7(H) = -s^{f-2} - s^{d+2} - s^{d+1} - s^{d+3} + s^{d+5} + s^{d+4} + s^{d+8} + s^{f+6}.$$  

After simplifying, we obtain

$$-s^d - s^f - s^{f+1} + s^{f+3} = -s^{f-2} - s^{d+2} - s^{d+3} + s^{d+5}.$$

Therefore, we have $f - 2 = d$. But $e = d$, so $e = f - 2$. From Equa. (3), $d' = d = e$. Since $d = e$, we have $d = d = e = e'$. From $d + e + f = d + e + f'$, we have $f = f'$. Therefore, $G \cong H$.

Subcase 5.3. If min $\{d,e,f\} = d$ and min $\{d',e',f\} = f'$, then $d = f'$. From Equation (4), $e' = e + 2$. Note that from Equation (3), we have $d = f - 2$. We can write $R_1(G)$ and $R_1(H)$ as follows:

$$R_8(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},$$
$$R_8(H) = -s^{f-2} - s^{e+2} - s^d - s^{e+3} - s^{d+1} + s^{d+3} + s^{e+6} + s^{e+7} + s^{d+6} + s^{f+6}.$$

After simplifying $R_8(G)$ and $R_8(H)$, we have

$$R_9(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{d+8},$$
$$R_9(H) = -s^{f-2} - s^{e+2} - s^{e+3} - s^{d+1} + s^{d+3} + s^{e+6} + s^{e+7} + s^{d+6}.$$

For the same reason stated by Subcase 5.2, $-s^{d+1}$ in $R_9(H)$ must be equal to $-s^e$ or $-s^{e+1}$ in $R_9(G)$. If $-s^{d+1} = -s^e$, then $e = d + 1$. We can write $R_9(G)$ and $R_9(H)$ as follows:

$$R_{10}(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} + s^{d+6} + s^{d+8},$$
$$R_{10}(H) = -s^{f-2} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + s^{d+7} + s^{d+8} + s^{d+6}.$$

After simplifying, we have

$$-s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} = -s^{f-2} - s^{d+4} + s^{d+7}.$$

So, we get $f = d + 4$. We also have $e = d + 1$, $d' = f - 2 = d + 2$, $f' = d$ and $e' = e + 2 = d + 3$. Let $d = i$. Therefore, we obtain the solution where $G \cong K_4(1,3,5,i,i+1,i+4)$ and $H \cong K_4(1,3,5,i+2,i+3,i)$.  

If $-s^{d+1} = -s^{e+1}$, we have $e = d$. Thus, we have the following:

$$R_{11}(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{d+8},$$
$$R_{11}(H) = -s^{f-2} - s^{d+2} - s^{d+3} - s^{d+1} + s^{d+3} + s^{d+6} + s^{d+7} + s^{d+6}.$$

After simplifying, we have

$$-s^d - s^f - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{d+8} = -s^{f-2} - s^{d+2} + s^{d+6} + s^{d+7} + s^{d+6}. $$
The resulting equation contradicts $R_{11}(G) = R_{11}(H)$.

**Case 6.** If $\max \{e + 5, f + 6, d + 8\} = e + 5$ and $\max \{e' + 5, f' + 6, d' + 8\} = d' + 8$, then \(e + 5 = d' + 8\), that is, 
\[
d' = e - 3
\]  
from \(d + e + f = d' + e' + f'\), we have 
\[
d + f = e' + f' - 3.
\]

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have \(\min \{d, e, f\} = \min \{d', e', f'\}\) by Lemma 2.1(4(i)). Without loss of generality, let \(\min \{d, e, f\} = d\). The following subcases need to be considered.

**Subcase 6.1.** If \(\min \{d, e, f\} = d\) and \(\min \{d', e', f'\} = d'\), then we deal with this case the same way with Case 1. So, we get $G \cong H$.

**Subcase 6.2.** If \(\min \{d, e, f\} = d\) and \(\min \{d', e', f'\} = e'\), then $d = e'$. From Equation (6), we have \(f = f + 3\). Note that \(d = e - 3\) by Equation (5). Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:
\[
R_{12}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},
\]
\[
R_{12}(H) = -s^{e-3} - s^d - s^{f+3} - s^{d+1} - s^{f+4} + s^{f+6} + s^{d+4} + s^{d+5} + s^{f+9} + s^{e+5}.
\]

After simplifying, we have
\[
R_{13}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{d+8},
\]
\[
R_{13}(H) = -s^{e-3} - s^{f+3} - s^{d+1} - s^{f+4} + s^{d+4} + s^{d+5} + s^{f+9}.
\]

Consider the term $-s^{d+1}$ in $R_{13}(H)$. Since \(\min \{d, e, f\} = d\), $-s^{d+1}$ cannot be cancelled by any positive term in $R_{13}(H)$. From \(\max \{e + 5, f + 6, d + 8\} = e + 5\), we have $e + 5 \geq d + 8$, i.e., $e + 1 \geq d + 4 > d + 1$. So, $-s^{d+1} \neq -s^{e+1}$. Moreover, $e \geq d + 3 > d + 1$, thus, \(e \neq d + 1\), i.e., $-s^e \neq -s^{d+1}$. So, $-s^{d+1}$ in $R_{13}(H)$ must be equal to $-s^f$ or $-s^{f+1}$ in $R_{13}(G)$.

If $-s^{d+1} = -s^f$, then $d = f$. So, we have
\[
R_{14}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+8},
\]
\[
R_{14}(H) = -s^{e-3} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+4} + s^{d+5} + s^{f+9}.
\]

After simplifying, we have
\[
-s^e - s^d - s^{e+1} + s^{d+3} + s^{e+4} + s^{d+8} = -s^{e-3} - s^{d+3} + s^{d+5} + s^{f+9}.
\]

The resulting equation contradicts $R_{14}(G) = R_{14}(H)$.

If $-s^{d+1} = -s^f$, then $d + 1 = f$. Thus, we have
\[
R_{15}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+8},
\]
\[
R_{15}(H) = -s^{e-3} - s^{d+4} - s^{d+1} - s^{d+5} + s^{d+4} + s^{d+5} + s^{d+10}.
\]
After simplifying, we have
\[-s^e - s^{e+1} - s^{d+2} + s^{e+4} + s^{d+8} = -s^{e-3} - s^{d+4} + s^{d+10}.\]

The resulting equation contradicts \(R_{15}(G) = R_{15}(H)\).

Subcase 6.3. If \(\min \{d, e, f\} = d\) and \(\min \{d', e', f'\} = f'\), then \(d = f'\).

From Equation (6), we have \(e = f + 3\). Note that \(d' = e - 3\) by Equation (5).

Thus, we have
\[
R_{16}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8},
\]
\[
R_{16}(H) = -s^{e-3} - s^{f+3} - s^d - s^{f+4} - s^{d+1} + s^{d+3} + s^{f+7} + s^{f+8} + s^{d+6} + s^{e+5}.
\]

After simplifying, we have
\[
R_{17}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{f+6} + s^{d+8},
\]
\[
R_{17}(H) = -s^{e-3} - s^{f+3} - s^{f+4} - s^{d+1} + s^{d+3} + s^{f+7} + s^{f+8} + s^{d+6}.
\]

For the same reason stated in Subcase 6.2, \(-s^{d+1}\) in \(R_{17}(H)\) can only be equal to \(-s^f\) or \(-s^{f+1}\) in \(R_{16}(G)\).

If \(-s^{d+1} = -s^f\), then \(d = f\). So, we have
\[
R_{18}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+6} + s^{d+8},
\]
\[
R_{18}(H) = -s^{e-3} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + s^{d+7} + s^{d+8} + s^{d+6}.
\]

After simplifying, we have
\[-s^e - s^d - s^{e+1} + s^{e+4} = -s^{e-3} - s^{d+3} - s^{d+4} + s^{d+7}.\]

Then, we know that the term \(s^{e-3}\) must be equal to \(s^d\). So, we have \(d = e - 3\).

Also we obtain \(d = f' = f\) and \(e' = f + 3 = f + 3 = d + 3 = e\). From \(d + e + f = d' + e' + f'\), we have \(d = d'\). Therefore \(G \cong H\).

If \(-s^{d+1} = -s^f\), then \(d + 1 = f\). So, we have
\[
R_{19}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+7} + s^{d+8},
\]
\[
R_{19}(H) = -s^{e-3} - s^{d+4} - s^{d+5} - s^{d+1} + s^{d+3} + s^{d+8} + s^{d+9} + s^{d+6}.
\]

After simplifying, we obtain
\[-s^e - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+7} = -s^{e-3} - s^{d+4} - s^{d+5} + s^{d+3} + s^{d+9} + s^{d+6}.\]

The resulting equation contradicts \(R_{19}(G) = R_{19}(H)\).

At this point, we have solved the equation \(R(G) = R(H)\) and the solution is as follows:
\[
K_4(1, 3, 5, i, i + 6, i + 1) \sim K_4(1, 3, 5, i + 2, i, i + 5),
\]
\[
K_4(1, 3, 5, i, i + 1, i + 4) \sim K_4(1, 3, 5, i + 2, i + 3, i),
\]

where \(i \geq 1\). The proof is now completed.

We close the paper with the following problem.
Problem. Study the chromatic uniqueness of the graph $K_4(1,3,5,d,e,f)$, where $d+e \geq 6$, $d+f \geq 4$ and $e+f \geq 8$.

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References


