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CHROMATIC EQUIVALENCE OF K_4-HOMEOMORPHS WITH GIRTH 9

R. Hasni[1 §], A. Ahmad[3], F. Mustapha[3]

[1]Department of Mathematics
Faculty of Science and Technology
University Malaysia Terengganu
21030, Kuala Terengganu, Terengganu, MALAYSIA

[2]School of Mathematical Sciences
Universiti Sains Malaysia
11800, USM Penang, MALAYSIA

Abstract: For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. In this paper, we discuss a pair of chromatically equivalent of K_4-homeomorphs with girth 9, that is, $K_4(1, 3, 5, d, e, f)$ and $K_4(1, 3, 5, d', e', f')$. As a result, we obtain two infinite chromatically equivalent non-isomorphic K_4-homeomorphs.

AMS Subject Classification: 05C15
Key Words: chromatic polynomial, chromatic equivalence, K_4-homeomorphs

1. Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically

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[§]Correspondence author

equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G.

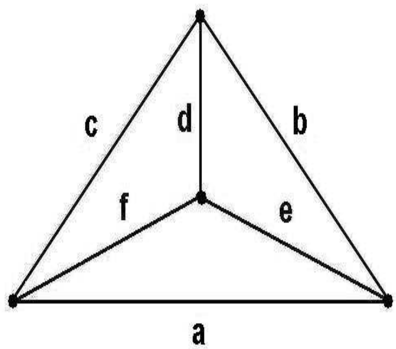

Figure 1: $K_4(a,b,c,d,e,f)$

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a,b,c,d,e,f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 7$ has been studied by many authors (see [2,5,6,7]). The chromaticity of K_4-homeomorphs with girth 8 or 9 is still remains open. For some results on chromatic equivalence of K_4-homeomorphs with girth 8, the reader is referred to [3,8,9].

In this paper, we shall discuss a chromatically equivalence pair of K_4-homeomorphs, $K_4(1,3,5,d,e,f)$ (as in Figure 2) and $K_4(1,3,5,d',e',f')$. We obtain two infinite chromatically equivalent non-isomorphic K_4-homeomorphs. This result can be extended to the study of chromatic equivalence classes of $K_4(1,3,5,d,e,f)$ and chromatic uniqueness of K_4-homeomorphs with girth 9.

2. Preliminary Results

In this section, we give some known results used in the sequel.

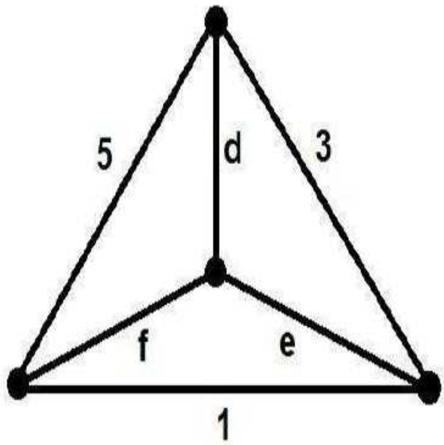

Figure 2: $K_4(1,3,5,d,e,f)$

Lemma 2.1. *Assume that G and H are $\chi-$equivalent. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see [4]);*

(2) *G and H has the same girth and same number of cycles with length equal to their girth (see [11]);*

(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph (see [1]);*

(4) *Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$, then*

   (i) *min (a,b,c,d,e,f) = min (a',b',c',d',e',f') and the number of times that this minimum occurs in the list $\{a,b,c,d,e,f\}$ is equal to the number of times that this minimum occurs in the list $\{a',b',c',d',e',f'\}$ (see[10]);*

   (ii) *if $\{a,b,c,d,e,f\} = \{a',b',c',d',e',f'\}$ as multisets, then $H \cong G$ (see [5]).*

Theorem 2.1. *(YanIing Peng [8]) Let $K_4(1,2,5,d,e,f)$ and $K_4(1,2,5,d',e',f')$ be chromatically equivalent, then*

$$K_4(1,2,5,i,i+6,i+1) \sim K_4(1,2,5,i+2,i,i+5),$$

$$K_4(1,2,5,i,i+1,i+6) \sim K_4(1,2,5,i+5,i,i+2),$$
$$K_4(1,2,5,i,i+1,i+3) \sim K_4(1,2,5,i+2,i+2,i).$$

Theorem 2.2. *(YanIing Peng [9]) Let* $K_4(2,3,3,d,e,f)$ *and* $K_4(2,3,3,d', e', f')$ *be chromatically equivalent, then* $K_4(2,2,3,d,e,f)$ *is isomorphic to* $K_4(2, 2,3,d',e',f')$.

Theorem 2.3. *(Roslan et al. [3]) Let* $K_4(1,3,4,d,e,f)$ *and* $K_4(1,3,4,d',e', f')$ *be chromatically equivalent, then*

$$K_4(1,3,4,i,i+5,i+1) \sim K_4(1,3,4,i+2,i,i+4),$$
$$K_4(1,3,4,i,i+1,i+4) \sim K_4(1,3,4,i+2,i+3,i).$$

where $i \geq 1$.

3. Main Results

In this section, we present our main results.

Lemma 3.1. *Assume that* $G \cong K_4(1,3,5,d,e,f)$ *and* $H \cong K_4(1,3,5,d',e', f')$, *then*

(1) $P(G) = (-1)^{x-1}[s/(s-1)^2][-s^{x-1} - s^6 - s^5 - s^4 - s^3 + 2s + 2 + R(G)]$, *where* $R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8} + s^{d+e+f}$, $s = 1-\lambda$, x *is the number of the edges of* G.

(2) *If* $P(G) = P(H)$, *then* $R(G) = R(H)$.

Proof. (1) Let $s = 1-\lambda$. From [10], we have the chromatic polynomial of K_4-homeomorphs $K_4(a,b,c,d,e,f)$ is as follows:

$$\begin{aligned} P(K_4(a,b,c,d,e,f) &= (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2) - \\ &\quad (s+1)(s^a + s^b + s^c + s^d + s^e + s^f) + \\ &\quad (s^{a+d} + s^{b+f} + s^{c+e} + s^{a+b+e} + \\ &\quad s^{b+d+c} + s^{a+c+f} + s^{d+e+f} - s^{x-1})] \end{aligned}$$

So, when $a = 1, b = 3$ and $c = 5$, we have

$$\begin{aligned} P(K_4(1,3,5,d,e,f) &= (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2) - \\ &\quad (s+1)(s + s^3 + s^5 + s^d + s^e + s^f) + \end{aligned}$$

$$(s^{1+d} + s^{3+f} + s^{5+e} + s^{4+e} +$$
$$s^{8+d} + s^{6+f} + s^{d+e+f} - s^{x-1})]$$
$$= (-1)^{x-1}[s/(s-1)^2][-s^{x-1} - s^6$$
$$-s^5 - s^4 - s^3 + 2s + 2 + R(G)]$$

where $R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+4} + s^{f+6} + s^{d+8} + s^{d+e+f}$ as required.

(2) If $P(G) = P(H)$, then we can easily see that $R(G) = R(H)$. □

Theorem 3.1. *Let K_4-homeomorphs $K_4(1,3,5,d,e,f)$ and $K_4(1,3,5,d',e',f')$ be chromatically equivalent, then we have*

$$K_4(1,3,5,i,i+6,i+1) \sim K_4(1,3,5,i+2,i,i+5),$$
$$K_4(1,3,5,i,i+1,i+4) \sim K_4(1,3,5,i+2,i+3,i).$$

where $i \geq 1$.

Proof. Assume that $G \cong K_4(1,3,5,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$. We now solve for the equation $R(G) = R(H)$ to find G and H which are not isomorphic. From Lemma 3.1, we have

$R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8} + s^{d+e+f}$,

$R(H) = -s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{f'+3} + s^{e'+4} + s^{e'+5} + s^{f'+6} + s^{d'+8} + s^{d'+e'+f'}$.

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), $d+e+f = d'+e'+f'$. We obtain the following after simplification. Note that our assumption in the following steps of the proof is $R_j(G) = R_j(H)$, where $1 \leq j \leq 19$.

$R_1(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}$,

$R_1(H) = -s^{d'} - s^{e'} - s^{f'} - s^{e'+1} - s^{f'+1} + s^{f'+3} + s^{e'+4} + s^{e'+5} + s^{f'+6} + s^{d'+8}$.

Let us consider the h.r.p. in $R_1(G)$ and the h.r.p. in $R_1(H)$. We have max $\{e+5, f+6, d+8\}$= max $\{e'+5, f'+6, d'+8\}$. Without loss of generality, we will consider only the following six cases.

Case 1. If max $\{e+5, f+6, d+8\} = e+5$ and max $\{e'+5, f'+6, d'+8\} = e'+5$, then $e = e'$. Thus, we can cancel the following pairs of terms in the equations : $-s^e$ with $-s^{e'}$, $-s^{e+1}$ with $-s^{e'+1}$, s^{e+4} with $s^{e'+4}$ and s^{e+5} with $s^{e'+5}$. Therefore, the l.r.p. in $R_1(G)$ is d or f and the l.r.p. in $R_1(H)$ is d' or f'. So, $d = f'$ or $d = d'$ or $f = f'$ or $f = d'$. We have $e = e'$ and $d+e+f = d'+e'+f'$. So, we know that $\{d,e,f\} = \{d',e',f'\}$ as multisets. From Lemma 2.1 (4(ii)), $G \cong H$.

Case 2. If $\max\{e+5, f+6, d+8\} = f+5$ and $\max\{e'+5, f'+6, d'+8\} = f'+5$, then $f = f'$. We can deal with this case in the same way as Case 1, thus, $G \cong H$.

Case 3. If $\max\{e+5, f+6, d+8\} = d+7$ and $\max\{e'+5, f'+6, d'+8\} = d'+7$, then we can deal with this case in the same way as Case 1. So, we have $G \cong H$.

Case 4. If $\max\{e+5, f+6, d+8\} = e+5$ and $\max\{e'+5, f'+6, d'+8\} = f'+6$, then $e+5 = f'+6$, that is

$$f' = e - 1 \tag{1}$$

from $d+e+f = d'+e'+f'$, we have

$$d + f = d' + e' - 1. \tag{2}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. From Lemma 2.1(4(i)), $\min\{d, e, f\} = \min\{d', e', f'\}$. Without loss of generality, let $\min\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 4.1. If $\min\{d, e, f\} = d$ and $\min\{d', e', f'\} = d'$, then $d = d'$. Thus, we can consider this case the same way as Case 1. So, $G \cong H$.

Subcase 4.2. If $\min\{d, e, f\} = d$ and $\min\{d', e', f'\} = e'$, then $d = e'$. From Equation (2), we have $d' = f + 1$. Note that $f' = e - 1$ by Equation (1). We can write $R_1(G)$ and $R_1(H)$ as follows:

$R_2(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}$,

$R_2(H) = -s^{f+1} - s^d - s^{e-1} - s^{d+1} - s^e + s^{e+2} + s^{d+4} + s^{d+5} + s^{e+5} + s^{f+9}$.

After simplifying $R_2(G)$ and $R_2(H)$, we have:

$R_3(G) = -s^f - s^{e+1} + s^{f+3} + s^{e+4} + s^{f+6} + s^{d+8}$,

$R_3(H) = -s^{e-1} - s^{d+1} + s^{e+2} + s^{d+4} + s^{d+5} + s^{f+9}$.

Consider the term $-s^{d+1}$ in $R_3(H)$. Since the $\min\{d, e, f\} = d$, $-s^{d+1}$ in $R_3(H)$ cannot be cancelled by any of the positive terms in $R_3(H)$. Thus, $-s^{d+1}$ must be equal to $-s^f$ or $-s^{e+1}$ in $R_3(G)$. Note that $\max\{e+5, f+6, d+8\} = e+5$, so $e+5 \geq d+8$, that is, $e+1 \geq d+4 > d+1$. Thus, $-s^{e+1} \neq -s^{d+1}$.

If $-s^{d+1} = -s^f$, then $d+1 = f$. Thus, $R_3(G)$ and $R_3(H)$ can be written as follows:

$R_4(G) = -s^{d+1} - s^{e+1} + s^{d+4} + s^{e+4} + s^{d+7} + s^{d+8}$,

$R_4(H) = -s^{e-1} - s^{d+1} + s^{e+2} + s^{d+4} + s^{d+5} + s^{d+10}$.

After simplifying $R_4(G)$ and $R_4(H)$, we have

$$-s^{e+1} + s^{e+4} + s^{d+7} + s^{d+8} = -s^{e-1} + s^{e+2} + s^{d+5} + s^{d+10}.$$

Therefore, we have $e = d + 6$. At this point, we acquire the following equations : $e = d + 6$, $f' = e - 1 = d + 5$, $f = d + 1$, $d' = f + 1 = d + 2$ and $e' = d$. Let $d = i$. Therefore, we obtain the solution where G is isomorphic to $K_4(1, 3, 5, i, i + 6, i + 1)$ and H is isomorphic to $K_4(1, 3, 5, i + 2, i, i + 5)$.

Subcase 4.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. Note that max $\{e' + 5, f' + 6, d' + 8\} = f' + 6$. So, $f' + 6 \geq d' + 8$, that is, $f' \geq d' + 2 > d'$. This contradicts with the fact that min $\{d', e', f'\} = f'$.

Case 5. If max $\{e + 5, f + 6, d + 8\} = f + 6$ and max $\{e' + 5, f' + 6, d' + 8\} = d' + 8$, then $f + 6 = d' + 8$, that is,

$$d' = f - 2 \tag{3}$$

from $d + e + f = d' + e' + f'$, we have

$$d + e = e' + f' - 2. \tag{4}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$, where min $\{d, e, f\}$ = min $\{d', e', f'\}$ by Lemma 2.1(4(i)). Without loss of generality, let min $\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 5.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then we deal with this case the same way with Case 1. So, we get $G \cong H$.

Subcase 5.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From Equation (4), we have $f' = e + 2$. Note that $d' = f - 2$ by Equation (3). Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$R_5(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}$,

$R_5(H) = -s^{f-2} - s^d - s^{e+2} - s^{d+1} - s^{e+3} + s^{e+5} + s^{d+4} + s^{d+5} + s^{e+8} + s^{f+6}$.

Consider the term $-s^{d+1}$ in $R_6(H)$. Since min $(d, e, f) = d$, then $-s^{d+1}$ cannot be cancelled by any positive terms in $R_5(H)$. Note that max $\{e + 5, f + 6, d + 8\} = f + 6$, so $f + 6 \geq d + 8$, that is $f + 1 \geq d + 3 > d + 1$, thus $f + 1 \neq d + 1$, i.e, $-s^{d+1} \neq -s^{f+1}$. Moreover $f \geq d + 2 > d + 1$, thus $f \neq d + 1$, i.e, $-s^{d+1} \neq -s^f$. Therefore the term $-s^{d+1}$ in $R_5(H)$ must be cancelled by the term $-s^e$ or $-s^{e+1}$ in $R_5(G)$.

If $-s^{d+1} = -s^e$, then $e = d + 1$. Thus, $R_5(G)$ and $R_5(H)$ can be written as follows:

$R_6(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} + s^{d+6} + s^{f+6} + s^{d+8}$,

$R_6(H) = -s^{f-2} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+6} + s^{d+4} + s^{d+5} + s^{d+9} + s^{f+6}$.

After simplifying, we obtain

$$-s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+8} = -s^{f-2} - s^{d+3} + s^{d+9} + s^{f+6}.$$

The resulting equations contradict $R_6(G) = R_6(H)$.

If $-s^{d+1} = -s^{e+1}$, then $e = d$. Thus, $R_5(G)$ and $R_5(H)$ can be written as follows:

$R_7(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{f+6} + s^{d+8}$,

$R_7(H) = -s^{f-2} - s^{d+2} - s^{d+1} - s^{d+3} + s^{d+5} + s^{d+4} + s^{d+5} + s^{d+8} + s^{f+6}$.

After simplifying, we obtain

$$-s^d - s^f - s^{f+1} + s^{f+3} = -s^{f-2} - s^{d+2} - s^{d+3} + s^{d+5}.$$

Therefore, we have $f - 2 = d$. But $e = d$, so $e = f - 2$. From Equa. (3), $d' = d = e$. Since $d = e'$, we have $d' = d = e = e'$. From $d+e+f = d'+e'+f'$, we have $f = f'$. Therefore, $G \cong H$.

Subcase 5.3. If $\min\{d, e, f\} = d$ and $\min\{d', e', f'\} = f'$, then $d = f'$. From Equation (4), $e' = e + 2$. Note that from Equation (3), we have $d' = f - 2$. We can write $R_1(G)$ and $R_1(H)$ as follows:

$R_8(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}$,

$R_8(H) = -s^{f-2} - s^{e+2} - s^d - s^{e+3} - s^{d+1} + s^{d+3} + s^{e+6} + s^{e+7} + s^{d+6} + s^{f+6}$.

After simplifying $R_8(G)$ and $R_8(H)$, we have

$R_9(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{d+8}$,

$R_9(H) = -s^{f-2} - s^{e+2} - s^{e+3} - s^{d+1} + s^{d+3} + s^{e+6} + s^{e+7} + s^{d+6}$.

For the same reason stated by Subcase 5.2, $-s^{d+1}$ in $R_9(H)$ must be equal to $-s^e$ or $-s^{e+1}$ in $R_9(G)$. If $-s^{d+1} = -s^e$, then $e = d + 1$. We can write $R_9(G)$ and $R_9(H)$ as follows:

$R_{10}(G) = -s^{d+1} - s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} + s^{d+6} + s^{d+8}$,

$R_{10}(H) = -s^{f-2} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + s^{d+7} + s^{d+8} + s^{d+6}$.

After simplifying, we have

$$-s^f - s^{d+2} - s^{f+1} + s^{f+3} + s^{d+5} = -s^{f-2} - s^{d+4} + s^{d+7}.$$

So, we get $f = d + 4$. We also have $e = d + 1$, $d' = f - 2 = d + 2$, $f' = d$ and $e' = e + 2 = d + 3$. Let $d = i$. Therefore, we obtain the solution where $G \cong K_4(1, 3, 5, i, i + 1, i + 4)$ and $H \cong K_4(1, 3, 5, i + 2, i + 3, i)$.

If $-s^{d+1} = -s^{e+1}$, we have $e = d$. Thus, we have the following:

$R_{11}(G) = -s^d - s^f - s^{d+1} - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{d+8}$,

$R_{11}(H) = -s^{f-2} - s^{d+2} - s^{d+3} - s^{d+1} + s^{d+3} + s^{d+6} + s^{d+7} + s^{d+6}$.

After simplifying, we have

$$-s^d - s^f - s^{f+1} + s^{f+3} + s^{d+4} + s^{d+5} + s^{d+8} = -s^{f-2} - s^{d+2} + s^{d+6} + s^{d+7} + s^{d+6}.$$

The resulting equation contradicts $R_{11}(G) = R_{11}(H)$.

Case 6. If max $\{e+5, f+6, d+8\} = e+5$ and max $\{e'+5, f'+6, d'+8\} = d'+8$, then $e+5 = d'+8$, that is,

$$d' = e - 3 \tag{5}$$

from $d+e+f = d'+e'+f'$, we have

$$d + f = e' + f' - 3. \tag{6}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have min $\{d, e, f\}$ = min $\{d', e', f'\}$ by Lemma 2.1(4(i)). Without loss of generality, let min $\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 6.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then we deal with this case the same way with Case 1. So, we get $G \cong H$.

Subcase 6.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From Equation (6), we have $f' = f + 3$. Note that $d' = e - 3$ by Equation (5). Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$R_{12}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}$,

$R_{12}(H) = -s^{e-3} - s^d - s^{f+3} - s^{d+1} - s^{f+4} + s^{f+6} + s^{d+4} + s^{d+5} + s^{f+9} + s^{e+5}$.

After simplifying, we have

$R_{13}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{d+8}$,

$R_{13}(H) = -s^{e-3} - s^{f+3} - s^{d+1} - s^{f+4} + s^{d+4} + s^{d+5} + s^{f+9}$.

Consider the term $-s^{d+1}$ in $R_{13}(H)$. Since min $\{d, e, f\} = d$, $-s^{d+1}$ cannot be cancelled by any positive term in $R_{13}(H)$. From max $\{e+5, f+6, d+8\} = e+5$, we have $e+5 \geq d+8$, i.e., $e+1 \geq d+4 > d+1$. So, $-s^{d+1} \neq -s^{e+1}$. Moreover, $e \geq d+3 > d+1$, thus, $e \neq d+1$, i.e., $-s^e \neq -s^{d+1}$. So, $-s^{d+1}$ in $R_{13}(H)$ must be equal to $-s^f$ or $-s^{f+1}$ in $R_{13}(G)$.

If $-s^{d+1} = -s^{f+1}$, then $d = f$. So, we have

$R_{14}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+8}$,

$R_{14}(H) = -s^{e-3} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+4} + s^{d+5} + s^{f+9}$.

After simplifying, we have

$$-s^e - s^d - s^{e+1} + s^{d+3} + s^{e+4} + s^{d+8} = -s^{e-3} - s^{d+3} + s^{d+5} + s^{f+9}.$$

The resulting equation contradicts $R_{14}(G) = R_{14}(H)$.

If $-s^{d+1} = -s^f$, then $d+1 = f$. Thus, we have

$R_{15}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+8}$,

$R_{15}(H) = -s^{e-3} - s^{d+4} - s^{d+1} - s^{d+5} + s^{d+4} + s^{d+5} + s^{d+10}$.

After simplifying, we have

$$-s^e - s^{e+1} - s^{d+2} + s^{e+4} + s^{d+8} = -s^{e-3} - s^{d+4} + s^{d+10}.$$

The resulting equation contradicts $R_{15}(G) = R_{15}(H)$.

Subcase 6.3. If $\min\{d, e, f\} = d$ and $\min\{d', e', f'\} = f'$, then $d = f'$. From Equation (6), we have $e' = f + 3$. Note that $d' = e - 3$ by Equation (5). Thus, we have

$R_{16}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{e+5} + s^{f+6} + s^{d+8}$,

$R_{16}(H) = -s^{e-3} - s^{f+3} - s^d - s^{f+4} - s^{d+1} + s^{d+3} + s^{f+7} + s^{f+8} + s^{d+6} + s^{e+5}$.

After simplifying, we have

$R_{17}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+3} + s^{e+4} + s^{f+6} + s^{d+8}$,

$R_{17}(H) = -s^{e-3} - s^{f+3} - s^{f+4} - s^{d+1} + s^{d+3} + s^{f+7} + s^{f+8} + s^{d+6}$.

For the same reason stated in Subcase 6.2, $-s^{d+1}$ in $R_{17}(H)$ can only be equal to $-s^f$ or $-s^{f+1}$ in $R_{16}(G)$.

If $-s^{d+1} = -s^{f+1}$, then $d = f$. So, we have

$R_{18}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+3} + s^{e+4} + s^{d+6} + s^{d+8}$,

$R_{18}(H) = -s^{e-3} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+3} + s^{d+7} + s^{d+8} + s^{d+6}$.

After simplifying, we have

$$-s^e - s^d - s^{e+1} + s^{e+4} = -s^{e-3} - s^{d+3} - s^{d+4} + s^{d+7}.$$

Then, we know that the term s^{e-3} must be equal to s^d. So, we have $d = e-3$. Also we obtain $d = f' = f$ and $e' = f + 3 = f' + 3 = d + 3 = e$. From $d + e + f = d' + e' + f'$, we have $d = d'$. Therefore $G \cong H$.

If $-s^{d+1} = -s^f$, then $d + 1 = f$. So, we have

$R_{19}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+7} + s^{d+8}$,

$R_{19}(H) = -s^{e-3} - s^{d+4} - s^{d+5} - s^{d+1} + s^{d+3} + s^{d+8} + s^{d+9} + s^{d+6}$.

After simplifying, we obtain

$$-s^e - s^{e+1} - s^{d+2} + s^{d+4} + s^{e+4} + s^{d+7} = -s^{e-3} - s^{d+4} - s^{d+5} + s^{d+3} + s^{d+9} + s^{d+6}.$$

The resulting equation contradicts $R_{19}(G) = R_{19}(H)$.

At this point, we have solved the equation $R(G) = R(H)$ and the solution is as follows:

$K_4(1, 3, 5, i, i + 6, i + 1) \sim K_4(1, 3, 5, i + 2, i, i + 5)$,

$K_4(1, 3, 5, i, i + 1, i + 4) \sim K_4(1, 3, 5, i + 2, i + 3, i)$,

where $i \geq 1$. The proof is now completed. □

We close the paper with the following problem.

Problem. Study the chromatic uniqueness of the graph $K_4(1,3,5,d,e,f)$, where $d+e \geq 6$, $d+f \geq 4$ and $e+f \geq 8$.

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