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SPECIAL TRANS-SASAKIAN MANIFOLDS AND CURVATURE CONDITIONS

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Abstract: In this paper a special type of trans-Sasakian manifolds called (ε, δ)-trans-Sasakian manifolds are studied. We present some general results for sectional curvatures of higher dimensional(ε, δ)-trans-Sasakian manifolds and establish relations among the sectional curvatures. We also pay a special attention to 3-dimensional manifolds.

AMS Subject Classification: 53D10, 53D15
Key Words: trans-Sasakian manifold, totally real bi-sectional curvature, ϕ-sectional curvature, η-Einstein

1. Introduction

The concept of (ε)-Sasakian manifolds was introduced by A. Bejancu and K.L. Duggal [1] and further investigation was taken up by Xufend and Xiaoli [6] and Rakesh Kumar et al.[3]. In [1], the authors obtained Riemannian curvature tensor of (ε)-Sasakian manifolds and established relations among different curvatures. H.G.Nagaraja et al [4] introduced and studied (ε, δ)-

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trans-Sasakian structures which generalize both (ε)-Sasakian manifolds and (ε)-Kenmotsu manifolds[2]. In this paper we study the curvature conditions of (ε,δ)-trans-Sasakian structures. After prelimanaries, In Section 3, we prove in a three dimensional (ε,δ)-trans-Sasakian manifold the characteristic vector field ξ belongs to the k-nullity distribution. Further the ξ-sectional curvature and the ϕ-sectional curvature of a (ε,δ)-trans-Sasakian manifold are computed. In Section 4, we establish relations among sectional curvature, ϕ-sectional curvature and totally real bisectional curvature. Last section is devoted to 3-dimensional (ε,δ)-trans-Sasakian manifolds of positive and negative curvature.

2. Prelimanaries

In this section some special notions, results and more background material are presented which we will use later. Let M be a (2n +1) dimensional differentiable manifold with almost contact structure (ϕ,η,ξ,g), where ϕ is a (1, 1) tensor field, ξ is a vector field and η is a 1-form on M satisfying

$$\eta(\xi)=1,\phi^2(X)=-X+\eta(X)\xi. \tag{2.1}$$

If there exists a semi-Riemannian metric g satisfying

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X and Y, where $\varepsilon=\pm1$, then the structure (ϕ,η,ξ,g) is called an (ε)-almost contact metric structure and M is called an (ε)-contact manifold. For an (ε)-contact manifold, we have

$$\eta(X)=\varepsilon g(X,\xi), \tag{2.3}$$

$$\varepsilon=g(\xi,\xi). \tag{2.4}$$

If $\varepsilon=-1$ and index of g is odd then M is a time like Sasakian manifold. If $\varepsilon=-1$ and index of g is even then M is a space like Sasakian manifold. Further M is the usual Sasakian manifold for $\varepsilon=1$ and index of g as 0 and M is a Lorentz Sasakian manifold if $\varepsilon=1$ and index of g as 1.

In [4], J. A. Oubino introduced the notion of a tran-Sasakian manifold. An almost contact metric manifold M is a trans-Sasakian manifold if there exist two functions α and β on M such that

$$(\nabla_X\phi)Y=\alpha(g(X,Y)\xi-\eta(Y)X)+\beta(g(\phi X,Y)\xi-\eta(Y)\phi X), \tag{2.5}$$

for all vector fields X and Y on M.

Throughout this paper we study a class of almost contact metric manifolds called (ε,δ)-trans-Sasakian manifolds. An almost contact metric manifold M is a (ε,δ)-trans-Sasakian manifold [4]if

$$(\nabla_X\phi)\,Y=\alpha\,(g\,(X,Y)\,\xi-\varepsilon\eta\,(Y)\,X)+\beta\,(g\,(\phi X,Y)\,\xi-\delta\eta\,(Y)\,\phi X)\,, \quad (2.6)$$

for all vector fields X and Y on M, where α and β are some functions on M and $\varepsilon=\pm1,\delta=\pm1$.

If $\beta=0$ and $\alpha=1$(or $\beta=1$ and $\alpha=0$), then the manifold reduces to $(\varepsilon)-$ Sasakian (or (δ)-Kenmotsu manifold).

The sectional curvature of the plane section spanned by vectors X and Y is given by

$$K\,(X,Y)=\frac{R\,(X,Y,X,Y)}{g\,(X,X)\,g\,(Y,Y)-g\,(X,Y)^2}. \quad (2.7)$$

Totally real bisectional curvature of totally real section $\{X,Y\}$, where X and Y are orthonormal to ξ and $\phi\,\{X,Y\}\perp\{X,Y\}$ is given by

$$B\,(X,Y)=R\,(X,\phi X,Y,\phi Y)\,. \quad (2.8)$$

If X is orthonormal to ξ then the plane section $\{X,\phi X\}$ is called ϕ-section and the curvature associated with this section is called ϕ-sectional curvature which is given by

$$H\,(X)=K\,(X,\phi X)=R\,(X,\phi X,X,\phi X)\,. \quad (2.9)$$

3. (ε,δ)-Trans-Sasakian Manifolds

Let M be a (ε,δ)-trans-Sasakian manifold. Then from (2.6), it is easy to see that

$$(\nabla_X\xi)=-\varepsilon\alpha\phi X-\beta\delta\phi^2X \quad (3.1)$$

and

$$(\nabla_X\eta)\,Y=-\alpha g\,(Y,\phi X)+\varepsilon\delta\beta g\,(\phi X,\phi Y)\,. \quad (3.2)$$

Also we have

$$\xi\alpha+2\alpha\beta=0. \quad (3.3)$$

By (3.1), we get

$$\begin{aligned} R(X,Y)\xi =& \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\ =& \varepsilon\left((Y\alpha)\phi X - (X\alpha)\phi Y\right) \\ &+ \left(\alpha^2-\beta^2\right)\left(\eta(Y)X - \eta(X)Y\right) \\ &- \delta\left((X\beta)\phi^2 Y - (Y\beta)\phi^2 X\right) \\ &+ 2\alpha\beta\left(\eta(Y)\phi X - \eta(X)\phi Y\right), \end{aligned} \tag{3.4}$$

provided $\varepsilon\delta = 1$.

For constants α and β, the above equation reduces to

$$\begin{aligned} &R(X,Y)\xi \\ &\quad = \left(\alpha^2-\beta^2\right)\left(\eta(Y)X - \eta(X)Y\right) + 2\alpha\beta\left(\eta(Y)\phi X - \eta(X)\phi Y\right), \end{aligned} \tag{3.5}$$

$$S(X,\xi) = 2n\left(\alpha^2-\beta^2\right)\eta(X). \tag{3.6}$$

From (2.6) we can derive that

$$\begin{aligned} R(\xi,X,Y,\xi) =& \varepsilon\left(\alpha^2-\beta^2\right)\left(g(X,Y) - \eta(X)\eta(Y)\right) \\ &- \varepsilon\left(\xi\alpha + 2\alpha\beta\right)g(\phi X,Y) + \delta(\xi\beta)g(\phi X,\phi Y). \end{aligned} \tag{3.7}$$

For constants α and β the above equation becomes

$$R(\xi,X,Y,\xi) = \varepsilon\left(\alpha^2-\beta^2\right)\left(g(X,Y) - \eta(X)\eta(Y)\right). \tag{3.8}$$

We recall that the curvature tensor in a 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y,Z,W) =& S(Y,Z)g(X,W) - g(X,Z)S(Y,W) + g(Y,Z)S(X,W) \\ &-g(Y,W)S(X,Z) - \frac{r}{2}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)]. \end{aligned} \tag{3.9}$$

Taking $X = W = \xi$ in (3.9), in view of (3.6) and (3.8), we obtain

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.10}$$

where

$$a = \left(\frac{r}{2} - \left(\alpha^2-\beta^2\right)\right), b = \left(3\left(\alpha^2-\beta^2\right) - \varepsilon\frac{r}{2}\right). \tag{3.11}$$

Thus we have the following: An (ε,δ)-trans-Sasakian manifold for constants α and β is η-Einstein.

From (3.9) and (3.10), we have

$$\begin{aligned} R(X,Y)Z =&(2a-\frac{r}{2})(g(Y,Z)X-g(X,Z)Y)+b(g(Y,Z)\eta(X) \\ &-g(X,Z)\eta(Y))\xi+b\eta(Z)(\eta(Y)X-\eta(X)Y). \end{aligned} \tag{3.12}$$

Taking $Z=\xi$ in (3.12), we get

$$R(X,Y)\xi = k(\eta(Y)X-\eta(X)Y), \tag{3.13}$$

where

$$k=\varepsilon\left(\left(\alpha^2-\beta^2\right)+\frac{2-\varepsilon}{2}r\right). \tag{3.14}$$

Thus we have

Theorem 1. *In a three dimensional (ε,δ)-trans-Sasakian manifold, the characteristic vector field ξ belongs to the k-nullity distribution.*

The ξ-sectional curvature and the ϕ-sectional curvature of a (ε,δ)-trans-Sasakian manifold are computed in the following.

From (3.13), we have

$$R(X,Y,Z,\xi)=k(\eta(X)g(Y,Z)-\eta(Y)g(X,Z)).$$

Taking $Y=\xi$ and $Z=X$, we get $K(X,\xi)=R(X,\xi,X,\xi)=-k$ or

$$K(X,\xi)=\varepsilon\left(\frac{\varepsilon-2}{2}r-\left(\alpha^2-\beta^2\right)\right). \tag{3.15}$$

If $\varepsilon=-1$ then the ξ-sectional curvature is positive.

From (3.12), we have

$$\begin{aligned} R(X,Y,Z,W) =&(2a-\frac{r}{2})(g(Y,Z)g(X,W)-g(X,Z)g(Y,W)) \\ &+b(\eta(Y)g(X,W)-\eta(X)g(Y,W))\eta(Z). \end{aligned} \tag{3.16}$$

Replacing Y by ϕX, Z by X and W by ϕX in the above equation, we get

$$R(X,\phi X,X,\phi X)=\frac{r}{2}-2\left(\frac{r}{2}-\left(\alpha^2-\beta^2\right)\right).$$

The above equation can be written as

$$K(X,\phi X)=2\left(\alpha^2-\beta^2\right)-\frac{r}{2}. \tag{3.17}$$

Thus we can state the following:

Theorem 2. *The ξ-sectional curvature and the ϕ-sectional curvature of a three dimensional(ε, δ)-trans-Sasakian manifold M are given by* (3.15) *and* (4.4) *respectively. Further the ξ-sectional curvature of M is positive if $\varepsilon = -1$ and $\alpha \geq \beta$.*

4. Curvature Tensor of a (ε, δ)-Trans-Sasakian Manifold

Let M be an (ε, δ)-trans-Sasakian manifold. From (2.6) and (3.1), a direct computation of the Riemann curvature tensor of M gives

$$\begin{aligned} R(X,Y)\phi Z =& \phi R(X,Y)Z \\ &+ (\varepsilon\alpha^2 - \delta\beta^2)(g(X,Z)\phi Y - g(Y,Z)\phi X \\ &+ g(\phi X, Z)Y - g(\phi Y, Z)X) + (\varepsilon + \delta)\alpha\beta(g(\phi X, Z)\phi Y \\ &- g(\phi Y, Z)\phi X) - 2\delta\alpha\beta(g(Y,Z)X - g(X,Z)Y) \\ &+ (X\alpha)(g(Y,Z)\xi - \varepsilon\eta(Z)Y) \\ &+ (X\beta)(g(\phi Y, Z)\xi - \delta\eta(Z)\phi Y) \\ &+ (Y\alpha)(g(X,Z)\xi - \varepsilon\eta(Z)X) \\ &+ (Y\beta)(g(\phi X, Z)\xi - \delta\eta(Z)\phi X). \end{aligned} \tag{4.1}$$

For constants α and β, we have

$$\begin{aligned} R(X,Y)\phi Z =& \phi R(X,Y)Z + (\varepsilon\alpha^2 - \delta\beta^2)(g(X,Z)\phi Y - g(Y,Z)\phi X \\ &+ g(\phi X, Z)Y - g(\phi Y, Z)X) + (\varepsilon + \delta)\alpha\beta(g(\phi X, Z)\phi Y \\ &- g(\phi Y, Z)\phi X) - 2\delta\alpha\beta(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{4.2}$$

From (2.1) and (4.2), we obtain

$$\begin{aligned} R(X,Y)Z =& - \phi R(X,Y)\phi Z + (\varepsilon\alpha^2 - \delta\beta^2)(g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X) + (\varepsilon + \delta)\alpha\beta(g(\phi Y, Z)X \\ &- g(\phi X, Z)Y) - 2\delta\alpha\beta(g(Y,Z)\phi X - g(X,Z)\phi Y). \end{aligned} \tag{4.3}$$

Again from (4.2), we get the following set of equations

$$\begin{aligned} R(X,Y,\phi Z,\phi W) =& R(X,Y,Z,W) + (\varepsilon\alpha^2 - \delta\beta^2)[g(X,Z)g(Y,W) \\ &- g(Y,Z)g(X,W) + g(\phi X, Z)g(Y,\phi W)g(\phi Y, Z)g(X,\phi W)] \\ &+ (\varepsilon + \delta)\alpha\beta[g(\phi X, Z)g(Y,W) - g(\phi Y, Z)g(X,\phi W)] \\ &+ 2\delta\alpha\beta[g(Y,Z)g(X,\phi W) - g(X,Z)g(Y,\phi W)], \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
R(\phi X,\phi Y,\phi Z,\phi W) =& R(X,Y,Z,W)+(\varepsilon\alpha^2-\delta\beta^2)[g(X,Z)\eta(Y)\eta(W)\\
&-g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z)\\
&-g(Y,Z)\eta(X)\eta(W)](\varepsilon+\delta)\alpha\beta[g(\phi Z,X)g(Y,W)\\
&-2g(\phi W,X)g(Y,Z)-g(\phi Y,W)g(X,Z)\\
&+\varepsilon g(\phi Y,W)\eta(X)\eta(Z)-\varepsilon g(\phi X,W)\eta(Y)\eta(Z)]\\
&-2\delta\alpha\beta[2g(\phi Y,Z)g(X,W)-g(\phi Y,W)g(X,Z)\\
&-\varepsilon g(\phi Y,Z)\eta(X)\eta(W)-g(\phi X,Z)g(Y,W)\\
&+\varepsilon g(\phi X,Z)\eta(Y)\eta(W)].
\end{aligned}
\tag{4.5}
$$

Now we can write (4.2) as

$$
\begin{aligned}
R(X,Y,\phi Z,W) =& g(\phi R(X,Y)Z,W)+(\varepsilon\alpha^2-\delta\beta^2)[g(X,Z)g(\phi Y,W)\\
&-g(Y,Z)g(\phi X,W)+g(\phi X,Z)g(Y,W)\\
&-g(\phi Y,Z)g(X,W)+(\varepsilon+\delta)\alpha\beta g(\phi X,Z)g(\phi Y,W)\\
&-g(\phi Y,Z)g(\phi X,W)]-2\delta\alpha\beta[g(Y,Z)g(X,W)\\
&-g(X,Z)g(Y,W)].
\end{aligned}
\tag{4.6}
$$

If we set $g(\phi X,Y)=\cos\theta$, $0\leq\theta\leq\pi$, for any orthonormal pair of vectors $\{X,Y\}$ orthogonal to ξ, we have

$$R(X,Y,X,\phi Y)=R(X,Y,X,Y)-(\varepsilon+\delta)\,\alpha\beta\cos^2\theta+2\delta\alpha\beta. \tag{4.7}$$

From (4.5), we deduce

$$R(\phi X,\phi Y,\phi X,\phi Y)=R(X,Y,X,Y), \tag{4.8}$$

and

$$R(\phi Y,X,\phi Y,X)=R(\phi X,Y,\phi X,Y)+(\varepsilon+\delta)\,\alpha\beta\cos 2\theta. \tag{4.9}$$

From (4.4) and (4.6), we can deduce the following set of equations

$$R(X,Y,\phi X,\phi Y)=R(X,Y,X,Y)+\left(\varepsilon\alpha^2-\delta\beta^2\right)\sin^2\theta, \tag{4.10}$$

$$R(Y,\phi X,X,\phi Y)=R(X,\phi Y,X,\phi Y)+\left(\varepsilon\alpha^2-\delta\beta^2\right)\sin^2\theta. \tag{4.11}$$

By the use of Bianchi identity

$$R(X,Y,\phi X,\phi Y)+R(Y,\phi X,X,\phi Y)+R(\phi X,X,Y,\phi Y)=0,$$

and from (4.10) and (4.11), we have

$$R(X,\phi X,Y,\phi Y) \\ = R(X,Y,X,Y) + 2\left(\varepsilon\alpha^2 - \delta\beta^2\right)\sin^2\theta + R(X,\phi Y,X,\phi Y). \quad (4.12)$$

Replacing X by $X+Y$ and by $X-Y$ and adding the resulting equations, we get

$$\begin{aligned} R(X+Y,\phi X+\phi Y,\phi X+\phi Y,X+Y) + R(X+Y,\phi X+\phi Y,\phi X+\phi Y,X+Y) \\ =2\{R(X,\phi X,\phi X,X) + R(Y,\phi Y,\phi Y,Y) \\ + 2R(X,\phi X,Y,\phi Y) \\ + 2R(X,\phi Y,Y,\phi X) \\ + R(X,\phi Y,X,\phi Y) + R(Y,\phi X,Y,\phi X). \end{aligned} \quad (4.13)$$

By virtue of (4.9), (4.11), (4.12), (4.13), for a unit vector X, we have

$$\begin{aligned} H\left(\frac{X+Y}{|X+Y|}\right) + H\left(\frac{X+Y}{|X+Y|}\right) \\ = 2H(X) + H(X) + 5R(X,\phi X,Y,\phi Y) \\ - 3R(X,Y,X,Y) - 3\left(\varepsilon\alpha^2-\delta\beta^2\right)\sin^2\theta - (\varepsilon+\delta)\alpha\beta\cos 2\theta \end{aligned}$$

The above equation may be rewritten as

$$\begin{aligned} B(X,Y) = \frac{3}{5}K(X,Y) + \frac{1}{10}H\left(\frac{X+Y}{|X+Y|}\right) + H\left(\frac{X+Y}{|X+Y|}\right) - 2H(X) \\ - 2H(Y) + 6\left(\varepsilon\alpha^2-\delta\beta^2\right)\sin^2\theta + 2(\varepsilon+\delta)\alpha\beta\cos 2\theta. \end{aligned} \quad (4.14)$$

This establishes the relation among the sectional curvature, the totally real bisectional curvature and the $\phi-$sectional curvature.

Thus we have,

Theorem 4.1. *The sectional curvature, the totally real bisectional curvature and the ϕ-sectional curvature satisfy the relation 4.14.*

5. 3-Dimensional $(\varepsilon - \delta)$-Trans-Sasakian Manifolds

Let M be a three dimensional $(\varepsilon - \delta)$-trans-Sasakian Manifold.

Then from (3.12), we have

$$R(X,Y)\phi Z = \left(2a - \frac{r}{2}\right)[g(Y,\phi Z)X - g(X,\phi Z)Y] \\ + b[g(Y,\phi Z)\eta(X)\xi - g(X,\phi Z)\eta(Y)\xi] \quad (5.1)$$

and

$$\phi R(X,Y)Z = \left(2a - \frac{r}{2}\right)[g(Y,Z)\phi X - g(X,Z)\phi Y] \\ + b\eta(Z)[\eta(Y)\phi X - \eta(X)\phi Y]. \quad (5.2)$$

Now from (4.2), (5.1) and (5.2), we have

$$\begin{aligned} & a[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] \\ & + b[g(Y,\phi Z)\eta(X)\xi - g(X,\phi Z)\eta(Y)\xi + (\eta(Y)\phi X - \eta(X)\phi Y)\eta(Z)] \\ = & \left(\varepsilon\alpha^2 - \delta\beta^2\right)[g(X,Z)\phi Y - g(Y,Z)\phi X + g(\phi X,Z)Y - g(\phi Y,Z)X] \quad (5.3) \\ & + (\epsilon+\delta)\alpha\beta[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] \\ & + 2\delta\alpha\beta[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

where $a = \frac{r}{2} - 2(\alpha^2 - \beta^2)$ and $b = 3(\alpha^2 - \beta^2) - \varepsilon\frac{r}{2}$.

Let $\{\xi, e_1, \phi e_1\}$ be an orthonormal basis of the tangent space T_xM of M.

Setting $X = \xi, Y = e_1, Z = \phi e_1$, we see that $a = -(\varepsilon\alpha^2 - \delta\beta^2)$ or

$$r = (2-\varepsilon)\alpha^2 - (2-\delta)\beta^2.$$

From this it follows that r is negative if $\alpha \leq \beta$ and positive otherwise.

Thus we have

Theorem 3. *A 3-dimensional (ε, δ)-trans-Sasakian manifold is of negative curvature if $\beta \leq \alpha$.*

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