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INDEPENDENT AND VERTEX COVERING NUMBER ON STRONG PRODUCT OF COMPLETE GRAPHS

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Abstract: Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number, respectively. The strong Product $G_1 \boxtimes G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \boxtimes G_2) = \{(u_1v_1)(u_2v_2)|[u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)] \cup [u_1u_2 \in E(G_1) \text{ and } v_1 = v_2]\}$. In this paper, let G is a simple graph with order m, we prove that, $\alpha(K_n \boxtimes G) = \alpha(G)$ and $\beta(K_n \boxtimes G) = mn - \alpha(G)$.

AMS Subject Classification: 05C69, 05C70, 05C76
Key Words: strong product, independent number, vertex covering number

1. Introduction

In this paper, graphs must be simple graphs which can be the trivial graph. Let G_1 and G_2 be graphs. The strong product of graph G_1 and G_2, denote by $G_1 \boxtimes G_2$, is the graph with $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \boxtimes G_2) =$

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$\{(u_1v_1)(u_2v_2)|[u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)] \cup [u_1 = u_2$ and $v_1v_2 \in E(G_2)] \cup [u_1u_2 \in E(G_1)$ and $v_1 = v_2]\}$. There are some properties about strong product of graph. We recall these here.

Proposition 1. *Let $H = G_1 \boxtimes G_2 = (V(H), E(H))$ then*
(i) $|V(H)| = |V(G_1)||V(G_2)|$
(ii) $|E(H)| = 2|E(G_1)||E(G_2)| + |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|$
(iii) *for every* $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u) + d_{G_2}(v)$.

Theorem 2. *Let G_1 and G_2 be connected g graphs, The graph $H = G_1 \boxtimes G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

Next we get that general form of graph of strong Product of K_n and a simple graph.

Proposition 3. *Let G be a simple graph of order m, the graph*

$$K_n \boxtimes G \quad is \quad \bigcup_{i=1}^{n-1} H_i \cup \bigcup_{i=1}^{n} R_i \cup \bigcup_{j=1}^{m} S_j \ \ ; \quad H_i = \bigcup_{j=i+1}^{n} H_{ij}$$

where $V(H_{ij}) = W_i \cup W_j$, $W_i = \{(i,1),(i,2),...,(i,m)\}$, $W_j = \{(j,1),(j,2),...,(j,m)\}$; $i < j$ and $E(H_{ij}) = \{(i,u)(j,v)/uv \in E(G)\}$. $V(R_i) = W_i$ and $E(R_i) = \{(i,u)(i,v)/uv \in E(G)\}$. $V(S_j) = \{(1,j),(2,j),...,(n,j)\}$ and $E(S_j) = \{(u,j)(v,j)/uv \in E(K_n)\}$. Moreover, if G has no odd cycle then each H_{ij} has exactly two connected components isomorphic to G. And $R_i \cong G, S_j \cong K_n$.

Example

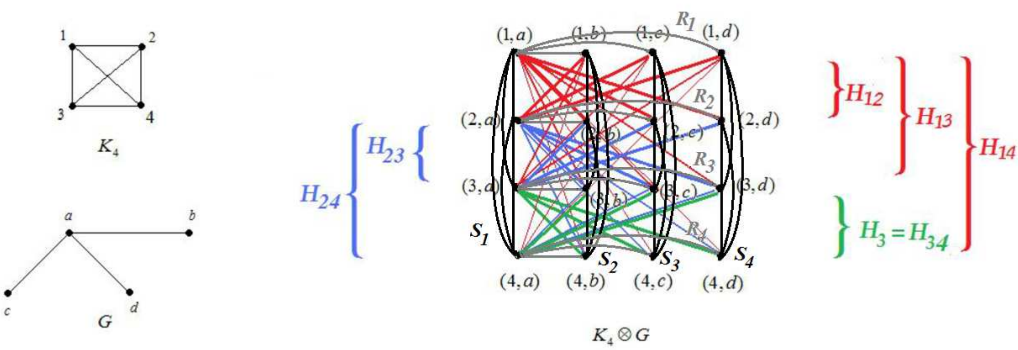


Figure 1: The graph of $K_4 \boxtimes G$

Next, we give the definitions about some graph parameters. A subset U of the vertex set $V(G)$ of G is said to be an independent set of G if the induced subgraph $G[U]$ is a trivial graph. An independent set of G with maximum number of vertices is called a maximum independent set of G . The number of vertices of a maximum independent set of G is called the independent number of G , denoted by $\alpha(G)$.

A vertex of graph G is said to cover the edges incident with it, and a vertex cover of a graph G is a set of vertices covering all the edges of G . The minimum cardinality of a vertex cover of a graph G is called the vertex covering number of G , denoted by $\beta(G)$.

By definitions, clearly that $\alpha(K_n) = 1$ and $\beta(K_n) = n - 1$.

2. Independent Number of the Graph of $K_n \boxtimes G$

We now state proposition and prove lemma before stating our main results. We begin this section by giving the proposition 4 show character of independent set and the lemma 5 show character of independent set.

Proposition 4. *Let $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph G if*

(i) v_i is not adjacent with v_j for all $i \neq j$ and $i, j = 1, 2, ..., k$

and (ii) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

Lemma 5. *Let $K_n \boxtimes G$ is* $\bigcup_{i=1}^{n-1} H_i \cup \bigcup_{i=1}^{n} R_i \cup \bigcup_{j=1}^{m} S_j; H_i = \bigcup_{j=i+1}^{n} H_{ij}; H_i = \bigcup_{j=i+1}^{n} H_{ij}, i < j$. *Then* $\alpha(H_{ij}) = 2\alpha(R_i) = 2\alpha(G)$ *and* $\alpha(S_j) = 1$.

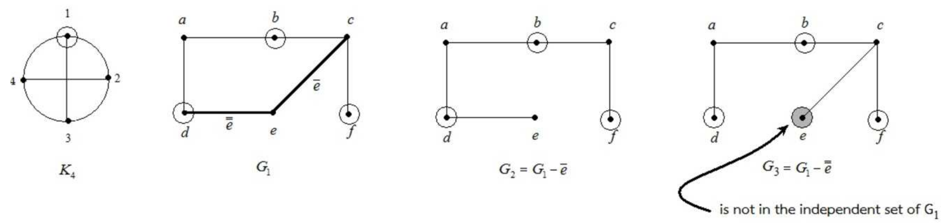
Proof. Suppose G has no odd cycle, by proposition 3, we get H_{ij}=$2G$. So $\alpha(H_{ij}) = 2\alpha(G)$.

If G has odd cycle, for each H_{ij} , vertex $(u_i, v) \in W_i$ and $(u_{i+1}, v) \in W_{i+1}; i < j$ have $d_{H_{ij}}((u_i, v)) = d_{H_{ij}}(u_{i+1}, v)) = d_G(v)$. Let $\bigcup_{j=i+1}^{n} \overline{H_{ij}} = K_n \boxtimes (G - \overline{e}); i = 1, 2, ..., n-1$ when $\overline{e}$ is an edge in odd cycle, I be the maximum independent set of G . We get $\overline{H_{ij}} = 2(G - \overline{e})$ then

$$\alpha(\overline{H_{ij}}) = 2\alpha(G - \overline{e}) = \begin{cases} 2[\alpha(G) + 1], & \text{if } \overline{e} = xy \text{ then } x \in I,\ y \notin I \\ & \text{and is not adjecent with } z \in I \\ 2\alpha(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\alpha(G-\overline{e}) = \alpha(G)+1$ be not impossible because the end vertices of edge $\overline{e}$ are in independent set of $G-\overline{e}$, so $\alpha(H_{ij}) = \alpha(\overline{H_{ij}}) - 1$.

Hence $\alpha(H_{ij}) = 2\alpha(G) = 2\alpha(R_i)$(since $R_i \cong G$). From $S_j \cong K_n$, we get $\alpha(S_j) = \alpha(K_n) = 1$. □



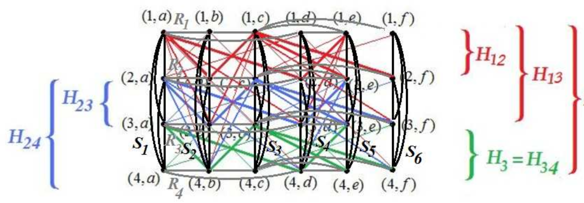


Figure 2: The graph of $K_4 \boxtimes G_1$

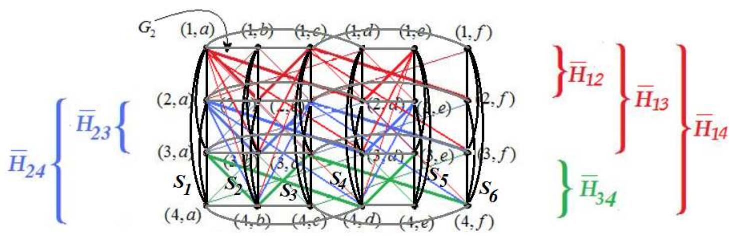


Figure 3: The graph of $K_4 \boxtimes G_2$

Next, we establish theorem 6 for a maximum independent number of $K_n \boxtimes G$.

Theorem 6. *Let G be connected graph order m, then $\alpha(K_n \boxtimes G) = \alpha(G)$.*

Proof. Let $V(K_n) = \{u_i, i = 1,2,...,n\}$, $V(G) = \{v_i, i = 1,2,...,m\}$, $S_i = \{(v_i, u_j) \in V(K_n \boxtimes G)/j = 1,2,...,m\}, i = 1,2,...,n$ and since $\alpha(K_n) = 1$.

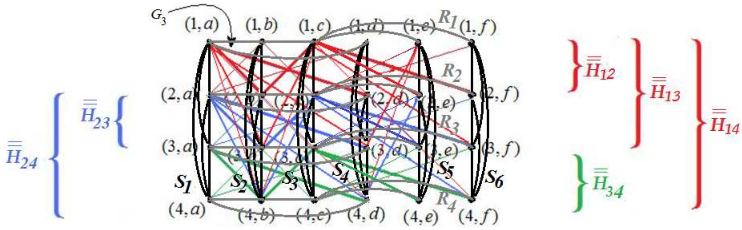


Figure 4: The graph of $K_4 \boxtimes G_3$

Assume that the maximum independent set of G be I.

For H_1 , by lemma 5 we have $\alpha(H_{1j}) = 2\alpha(G), j = 2, 3, ..., n$. Since every $H_{1j}, H_{1k}; j \neq k; k = 2, 3, ..., n$ have $\alpha(G)$ common vertices in their independent set which is in S_1. So the independent set of H_1 be in $\bigcup_{i=1}^{n} S_i$.

Similarly, for the independent set of $H_2, H_3, ..., H_{n-1}$ have $\alpha(G)$ common vertices in their independent set which is in $S_2, S_3, ..., S_{n-1}$, respectively.

But the independent set of $H_2, H_3, ..., H_{n-1}$ are subset of the independent set of H_1.

Suppose the independent set of $\bigcup_{i=1}^{n-1}(H_i)$ is $I = \{(u_,1v_1), ...$
$, (u_1, v_{\alpha(G)}, (u_2, v_1), ..., (u_2, v_{\alpha(G)}, ..., (u_3, v_1), ..., (u_3, v_{\alpha(G)})\}$.

Since for each $S_j \cong K_n$, we get the vertex (u_1, v_j) are adjacent with the vertices $(u_2, v_j), ..., (u_n, v_j), j = 1, 2, ..., \alpha(G)$.

Similarly, for each $R_i \cong G$, we get $\alpha(R_i) = \alpha(G)$, but all vertices of R_i are adjacent with all vertices of $R_p, i \neq p$.

So $I_i = \{(u_i, v_k)/v_k \in I\}$ are independent set of $K_n \boxtimes G; i = 1, 2, ...$ or n.

Hence $\alpha(K_n \boxtimes G) \geq \alpha(G)$.

Suppose that $\alpha(K_n \boxtimes G) > \alpha(G)$, then there exists $uv_j \in V(K_n \boxtimes G) - I_i; j = \alpha(G) + 1, \alpha(G) + 2, ..., m$ which is not adjacent with another vertices in $I_i = \{(u_i, v_k)/v_k \in I\}$. It is not true.

Hence $\alpha(K_n \boxtimes G) = \alpha(G)$. □

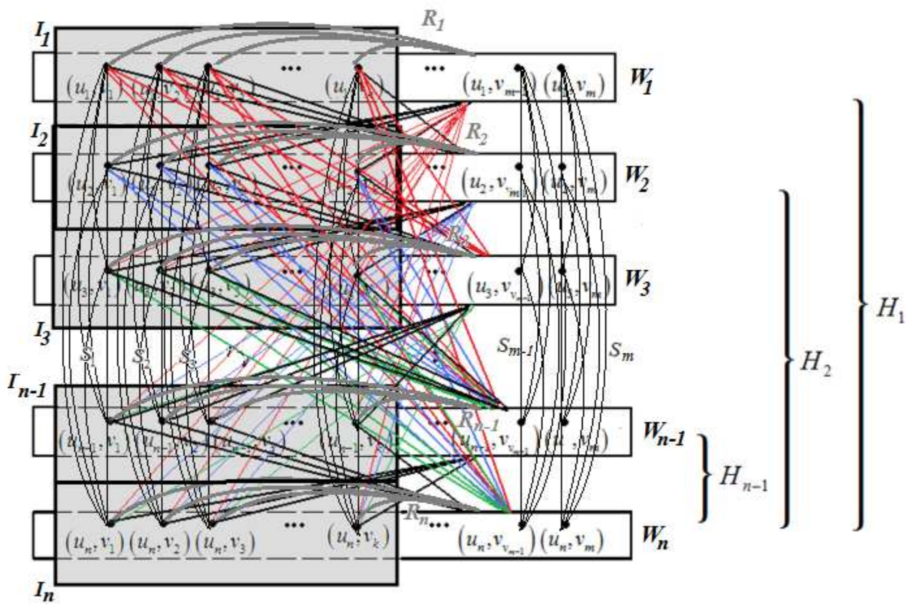


Figure 5: The independent set of $K_n \boxtimes G$

3. Vertex Covering Number of the Graph of $K_n \boxtimes G$

We begin this section by giving the lemma 7 that shows a relation of independent number and vertex covering number.

Lemma 7. (see [2]) *Let G be a simple graph with order n. Then $\alpha(G) + \beta(G) = n$*

Next we establish theorem 8 for a minimum vertex covering number of $K_n \boxtimes G$.

Theorem 8. *Let G be connected graph order m, then $\beta(K_n \boxtimes G) = mn - \alpha(G)$*

Proof. By theorem 6 and lemma 7, we can also show that

$$\begin{aligned} \alpha(K_n \boxtimes G) + \beta(K_n \boxtimes G) &= nm \\ \alpha(G) + \beta(K_n \boxtimes G) &= nm \\ \beta(K_n \boxtimes G) &= nm - \alpha(G) \end{aligned}$$

□

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