A SCHUR-TYPE THEOREM FOR $\mathcal{I}$-CONVERGENCE AND MAXIMAL IDEALS

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Abstract: A Schur-type theorem in the setting of ideal convergence is proved. Some properties of ideal convergence in relation with usual convergence of suitable subsequences are investigated, and a characterization of maximal ideals is given. Furthermore some open problem is posed.

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1. Introduction

Ideal convergence was introduced in [20] and independently in [22] with the name of ”cofilter convergence”. It was recently studied in the literature by several authors, among which we quote for example [13, 14, 15, 16, 20, 21]. In
particular it has been deeply investigated in problems concerning convergence of functions (see for instance [4, 5, 19, 20]) and convergence of measures and integrals (see e.g.[4, 6, 7, 10, 11]).

In this paper we prove a Schur-type theorem, which connects weak and norm convergence in $\ell_1$ in the ideal context, investigate some properties of ideal convergence and study them in relation with classical convergence of suitable subsequences. We present a characterization of maximal ideals of $\mathbb{N}$ and pose some open problems. In this framework, some Schur-type theorems for ideal convergence are given in [11].

Recently, some slightly different versions of Schur-type theorems are proved with respect to the filter convergence. In particular, in [3] there are some necessary and/or sufficient conditions on filters of $\mathbb{N}$ to satisfy the Schur property. With similar techniques, these theorems have been extended to the context of absolutely summable ($\ell$)-group-valued sequences in [11] and, as applications and consequences, some versions of limit theorems ([7, 11]), Dieudonné-type theorems ([7, 11]) and uniform boundedness principle ([9]) are given with respect to filters. Furthermore, some related basic matrix theorems for ideal convergence are proved in [10], while in [5] also some other versions of limit theorems and some main properties of (weak) compactness in the ideal convergence setting were investigated.

2. Preliminaries

Throughout the paper we denote by $\ell_1$ the space of all real sequences whose associate series is absolutely convergent, $\ell_\infty$ the space of all bounded real sequences and $c_00$ the subspace of $\ell_\infty$ of the eventually null sequences.

We recall the fundamental properties of ideals (see also [13, 20, 21]).

**Definitions 2.1.** (a) A family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an ideal of $\mathbb{N}$ iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$.

(b) An ideal $\mathcal{I}$ of $\mathbb{N}$ is called non-trivial iff $\mathcal{I} \neq \emptyset$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible iff it contains all singletons.

(c) An ideal $\mathcal{I}$ of $\mathbb{N}$ is said to be maximal iff for every $A \subset \mathbb{N}$ we get either $A \in \mathcal{I}$ or $\mathbb{N} \setminus A \in \mathcal{I}$.

(d) An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is said to be a $P$-ideal iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there is a sequence $(B_j)_j$ of subsets of $\mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

We now give some notion of convergence in the ideal setting.
Definitions 2.2. (a) Let \((X, d)\) be a metric space, \(\mathcal{I}\) be an admissible ideal of \(\mathbb{N}\) and \(\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{N \setminus A : A \in \mathcal{I}\}\) be its dual filter. A sequence \((x_n)_n\) in \(X\) is called \(\mathcal{I}\)-convergent to \(x \in X\) iff for all \(\varepsilon > 0\), \(\{n \in \mathbb{N} : d(x_n, x) > \varepsilon\} \in \mathcal{I}\). We then write \(\mathcal{I} - \lim n x_n = x\).

(b) A sequence \((x_n)_n\) in \((X, d)\) is called \(\mathcal{I}\)-Cauchy iff for each \(\varepsilon > 0\) there exists \(q \in \mathbb{N}\) such that \(\{n \in \mathbb{N} : d(x_n, x_q) > \varepsilon\} \in \mathcal{I}\).

If \(X\) is a Banach space, then the concepts of norm \(\mathcal{I}\)-convergence and norm \(\mathcal{I}\)-Cauchy can be formulated analogously as above.

(c) Let \(X\) be a Banach space. A sequence \((x_n)_n\) is called weakly-\(\mathcal{I}\)-convergent iff the sequence \((x^*(x_n))_n\) is \(\mathcal{I}\)-convergent for every \(x^* \in X^*\) (the dual space of \(X\)). A sequence \((x_n)_n\) is said to be weakly-\(\mathcal{I}\)-Cauchy iff the sequence \((x^*(x_n))_n\) is \(\mathcal{I}\)-Cauchy for every \(x^* \in X^*\).

(d) A sequence \((x_n)_n\) in \((X, d)\) \(\mathcal{I}^*\)-converges to \(x \in X\) iff there exists a set \(A_0 \in \mathcal{F}(\mathcal{I})\) with \(\lim_{n \in A_0} x_n = x\).

Examples 2.3. (see [20]) (i) If \(\mathcal{I}_{\text{fin}} = \{A \subset \mathbb{N} : A \text{ finite}\}\), then \(\mathcal{I}_{\text{fin}}\) is a \(P\)-ideal of \(\mathbb{N}\) and \(\mathcal{I}_{\text{fin}}\)-convergence coincides with the ordinary convergence.

(ii) Let \(A \subset \mathbb{N}\). The asymptotic density of \(A\) is defined as

\[
\delta(A) := \lim_{n} \frac{|A \cap \{1, \ldots, n\}|}{n},
\]

where \(|\cdot|\) denotes the cardinality of the set in brackets. If \(\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}\), then \(\mathcal{I}_\delta\) is a \(P\)-ideal of \(\mathbb{N}\) (see [20]), but it is not maximal, because, if \(\mathbb{E}\) denotes the set of all even subsets of \(\mathbb{N}\), then neither \(\mathbb{E}\) nor \(\mathbb{N} \setminus \mathbb{E}\) belongs to \(\mathcal{I}_\delta\).

(iii) Let \(\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j\) be a partition of \(\mathbb{N}\) such that \(\Delta_j\) is an infinite set for every \(j \in \mathbb{N}\) and \(\mathcal{I}_g = \{A \subset \mathbb{N} : A \text{ intersects only a finite number of } \Delta_j\text{'s }\}\). Then \(\mathcal{I}_g\) is not a \(P\)-ideal.

We now give the following results.

Proposition 2.4. Let \(\mathcal{I}\) be an admissible ideal, \(\mathcal{F}(\mathcal{I})\) be its dual filter, \((x_n)_n\) be a sequence in \((X, d)\), such that \(\mathcal{I} - \lim n x_n = x \in X\). Then there exists a strictly increasing subsequence \((x_{n_q})_q\) of \((x_n)_n\), such that \(\lim q x_{n_q} = x\).

Moreover, if \(\mathcal{I}\) is a \(P\)-ideal, then the subsequence \((x_{n_q})_q\) can be chosen in such a way that \(\{n_1 < n_2 < \ldots < n_q < \ldots\} \in \mathcal{F}(\mathcal{I})\).

Proof. Choose arbitrarily \(\varepsilon > 0\). By definition of ideal convergence, in correspondence with \(\varepsilon\) there exists a positive integer \(n_1\) with \(d(x_{n_1}, x) \leq \varepsilon\). At
the second step, in correspondence with
\[ \varepsilon_2 := \frac{\min\{\varepsilon, d(x_1, x), d(x_2, x), \ldots, d(x_{n_1}, x)\}}{2}, \] (1)
there is \( n_2 \in \mathbb{N} \) such that \( d(x_{n_2}, x) \leq \varepsilon_2 \). By virtue of (1), we get \( n_2 > n_1 \). Proceeding by induction, supposed that \( \varepsilon_{q-1} \) and \( n_{q-1} \) have been determined, let
\[ \varepsilon_q := \frac{\min\{\varepsilon_{q-1}, d(x_1, x), d(x_2, x), \ldots, d(x_{n_{q-1}}, x)\}}{2}. \] (2)
By \( \mathcal{I} \)-convergence, there exists an integer \( n_q \in \mathbb{N} \) with \( d(x_{n_q}, x) \leq \varepsilon_q \). By (2), we have \( n_q > n_{q-1} \). Since \( \lim_{q} \varepsilon_q = 0 \), then we get \( \lim_{q} d(x_{n_q}, x) = 0 \), and hence \( \lim_{q} x_{n_q} = x \). This concludes the proof of the first part.

For the last part, see [20, Theorem 3.2].

**Proposition 2.5.** (see also [20, Theorem 3.2]) The \( \mathcal{I}^* \)-convergence of sequences implies always the \( \mathcal{I} \)-convergence.

Moreover, if \( (x_n)_n \) is a sequence in \( (X, d) \), \( \mathcal{I} \)-convergent to \( x \in X \), and \( \mathcal{I} \) is a \( P \)-ideal, then \( (x_n)_n \) \( \mathcal{I}^* \)-converges to \( x \).

**Proposition 2.6.** Let \( (x_{i,j})_{i,j} \) be a double sequence in \( (X, d) \), \( \mathcal{I} \) be any \( P \)-ideal, \( \mathcal{F} = \mathcal{F}(\mathcal{I}) \) be its dual filter, and let us suppose that \( \mathcal{I} - \lim_{i} x_{i,j} = x_j \) for every \( j \in \mathbb{N} \). Then there exists \( B_0 \in \mathcal{F} \) such that \( \lim_{h \in B_0} x_{h,j} = x_j \) for all \( j \in \mathbb{N} \).

**Proof.** Since \( \mathcal{I} \) is a \( P \)-ideal, by virtue of Proposition 2.5 we get
\[ \mathcal{I}^* - \lim_{i} x_{i,j} = x_j \]
for every \( j \in \mathbb{N} \). Hence there is a sequence \( (A_j)_j \) in \( \mathcal{F} \) such that \( \lim_{i \in A_j} x_{i,j} = x_j \) for all \( j \in \mathbb{N} \). As \( \mathcal{I} \) is a \( P \)-ideal, there is a sequence of sets \( (B_j)_j \) in \( \mathcal{F} \) such that \( A_j \Delta B_j \) is finite for all \( j \in \mathbb{N} \) and \( B_0 := \bigcap_{j=1}^{\infty} B_j \in \mathcal{F} \). Since \( \lim_{i \in A_j} x_{i,j} = x_j \) for all \( j \), then we get also \( \lim_{i \in B_j} x_{i,j} = x_j \) for all \( j \). Let \( B_0 = \{p_1 < \ldots < p_h < \ldots\} \) and choose arbitrarily \( j \in \mathbb{N} \): then, since \( B_0 \subset B_j \), in correspondence with \( \varepsilon \) an integer \( h = \overline{h}(\varepsilon,j) \) can be found, with the property that \( |x_{p_h,j} - x_j| \leq \varepsilon \) whenever \( h > \overline{h} \). This concludes the proof.

The following lemma will be useful to prove our Schur-type theorem.
Lemma 2.7. Let \((x_k)_k \in \ell_1\). For every \(p, q \in \mathbb{N}\) with \(p < q\), let \(S_{p,q}\) be the class of all subsets of \(\{p, p+1, \ldots, q\}\). Then we get
\[
\sum_{k=p}^{q} |x_k| \leq 2 \max_{S \in S_{p,q}} \left| \sum_{k \in S} x_k \right|.
\]

Proof. Let us define \(\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{R}\) as follows:
\[
\mu(A) := \sum_{k \in A} x_k, \quad A \subset \mathbb{N}.
\]

By [8, Lemma 3.6], we get
\[
\sum_{k=p}^{q} |x_k| = \sum_{k=p}^{q} |\mu(\{k\})| \leq 2 \max_{S \in S_{p,q}} |\mu(S)| = 2 \max_{S \in S_{p,q}} \left| \sum_{k \in S} x_k \right|.
\]

This ends the proof. \(\Box\)

3. The Main Results

We begin with a Schur-type theorem in the context of ideal convergence, which uses a sliding hump technique (for a related literature, see also [2, 3, 10, 12]).

Theorem 3.1. Let \((y_{n,k})_{n,k}\) be a double sequence in \(\mathbb{R}\) such that \((y_{n,k})_k \in \ell_1\) for each \(n \in \mathbb{N}\), \(\mathcal{I}\) be a fixed \(P\)-ideal of \(\mathbb{N}\), and \(\mathcal{F} = \mathcal{F}(\mathcal{I})\) be its dual filter. Suppose that there exist a sequence \((y_k)_k \in \ell_1\), with

a) \(\mathcal{I} - \lim_n y_{n,k} = y_k\) for every \(k \in \mathbb{N}\)

and an element \(F \in \mathcal{F}\), such that

b) \(\mathcal{I} - \lim_{r} \sum_{k \in E} y_{i_r,k} = \sum_{k \in E} y_k\) for each strictly increasing sequence \((i_r)_r\) in \(F\)

and for every \(E \subset \mathbb{N}\) such that both \(E\) and \(\mathbb{N} \setminus E\) are infinite.

Then \(\mathcal{I} - \lim_n \sum_{k=1}^{\infty} |y_{n,k} - y_k| = 0\).

Proof. For every \(p, q \in \mathbb{N}\) with \(p < q\), let \(S_{p,q}\) be as in Lemma 2.7. For each \(n, k \in \mathbb{N}\) put \(b_{n,k} := y_{n,k} - y_k\).

First of all note that, by a) and Proposition 2.6, a set \(K \in \mathcal{F}\) can be found, with

\[
\lim_{n \in K} b_{n,k} = 0 \quad (3)
\]
for any $k \in \mathbb{N}$. So, in order to prove the result, it will be enough to show that

$$
\lim_{n \in F \cap K} \sum_{k=1}^{\infty} |b_{n,k}| = 0,
$$

(4)

since $F \cap K \in \mathcal{F}$ and by the first part of Proposition 2.5. If (4) is not true, then there is a positive real number $C$ with the property that for every $q \in K$ there exists $l_q \in F \cap K$, $l_q > q$, with

$$
\sum_{k=1}^{\infty} |b_{l_q,k}| > C.
$$

(5)

From (5) and (3) it follows that there is a positive integer $i_1 \in F \cap K$, with

$$
\sum_{k=1}^{\infty} |b_{i_1,k}| > C \quad \text{and} \quad |b_{i_1,1}| \leq \frac{C}{8}.
$$

(6)

At the first step, let $k_1 := 1$. Since $(y_k)_k$ and $(y_{i_1,k})_k$ belong to $\ell_1$, then

$$
\sum_{k=1}^{\infty} |b_{i_1,k}| < +\infty,
$$

and so we can choose a natural number $k_2 > k_1$ with

$$
\sum_{k=k_2}^{\infty} |b_{i_1,k}| \leq \frac{C}{8}.
$$

From this and (6) we obtain

$$
\sum_{k=2}^{k_2-1} |b_{i_1,k}| > \frac{3C}{4}.
$$

(7)

From (7) and Lemma 2.7 we have

$$
\max_{A \in \mathcal{S}_{2,k_2-1}} \left| \sum_{k \in A} b_{i_1,k} \right| > \frac{3C}{8}.
$$

At the second step, taking $q = i_1$ in (5), we get the existence of an element $i_2 \in F \cap K$, $i_2 > i_1$, with

$$
\sum_{k=1}^{\infty} |b_{i_2,k}| > C \quad \text{and} \quad \sum_{k=1}^{k_2} |b_{i_2,k}| \leq \frac{C}{8}:
$$

(8)
such a choice is possible, by virtue of (5) and (3) respectively. By proceeding analogously as above, we can find an integer $k_3 > k_2$ with

$$\sum_{k=k_3}^{\infty} |b_{i_{k},k}| \leq \frac{C}{8}.$$  

From this and (8) we get

$$\sum_{k=k_2+1}^{k_3-1} |b_{i_{k},k}| > \frac{3C}{4}. \quad (9)$$

From (9) and Lemma 2.7 it follows that

$$\max_{A \in S_{k_{r}+1,k_{r+1}-1}} \left| \sum_{k \in A} b_{i_{r},k} \right| > \frac{3C}{8}.$$  

By induction, we can construct two strictly increasing sequences $(i_r)_r$ and $(k_r)_r$ in $F \cap K$ and $\mathbb{N}$ respectively, such that $k_1 = 1$ and

1) $\sum_{k=1}^{k_r} |b_{i_{r},k}| \leq \frac{C}{8};$

2) $\sum_{k=k_{r}+1}^{\infty} |b_{i_{r},k}| \leq \frac{C}{8};$

3) $\max_{A \in S_{k_{r}+1,k_{r+1}-1}} \left| \sum_{k \in A} b_{i_{r},k} \right| > \frac{3C}{8}$

for every $r \geq 2$. From 3) it follows that for such $r$'s there exists a set $E_r \in S_{k_{r}+1,k_{r+1}-1}$ with

4) $\left| \sum_{k \in E_r} b_{i_{r},k} \right| > \frac{3C}{8}.$

Let now $E := \bigcup_{r=1}^{\infty} E_r$. Note that, by construction, $E$ is infinite and $E \cap \{k_r : r \geq 2\} = \emptyset$, and hence $\mathbb{N} \setminus E$ is infinite too. By b) and Proposition 2.4, we can associate to the sequence $(i_r)_r$ and the set $E$ a subsequence $(i_{r_s})_s$ of $(i_r)_r$, with

$$\lim_{s} \left( \sum_{k \in E} b_{i_{r_s},k} \right) = 0,$$
where the limit is intended in the usual sense. Thus, in correspondence with \( \frac{C}{8} \), there is a positive integer \( s_0 \in \mathbb{N} \) such that for every \( s \geq s_0 \) we get:

\[
\left| \sum_{k \in E_{rs}} b_{ir_s,k} \right| = \left| \sum_{k \in E} b_{ir_s,k} - \sum_{k \in E, k = 1}^{k_{rs}} b_{ir_s,k} - \sum_{k \in E, k = k_{rs} + 1}^{\infty} b_{ir_s,k} \right| \\
\leq \left| \sum_{k \in E} b_{ir_s,k} \right| + \sum_{k = 1}^{k_{rs}} |b_{ir_s,k}| + \sum_{k = k_{rs} + 1}^{\infty} |b_{ir_s,k}| \\
\leq \frac{C}{8} + \frac{C}{8} + \frac{C}{8} = \frac{3C}{8}.
\]

This contradicts 4) and proves the theorem.

In [12] the following Schur-type theorem was proved (see also [2]).

**Theorem 3.2.** Let \((y_{n,k})_{n,k}\) be a double sequence in \( \mathbb{R} \), and assume that:

(i) \( \lim_{n} \sum_{k \in A} y_{n,k} \) exists in \( \mathbb{R} \) for every subset \( A \subset \mathbb{N} \);

(ii) \( \lim_{n} y_{n,k} = y_k \) exists for all \( k \in \mathbb{N} \).

Then we get:

(a) \( (y_k)_{k} \in \ell_1 \);

(b) \( \lim_{n} \sum_{k=1}^{\infty} |y_{n,k} - y_k| = 0 \).

**Remarks 3.3.** (a) Theorem 3.2 has also the following interpretation.

Let \( x_n = (y_{n,k})_{k} \in \ell_1 \), for each \( n \in \mathbb{N} \). Theorem 3.2 says that \( (x_n)_{n} \) is a Cauchy sequence in the topology \( \sigma(\ell_1, c_{00}) \). The conclusion is that \( (x_n)_{n} \) is norm convergent in \( \ell_1 \). In particular, this implies that any weakly convergent sequence in \( \ell_1 \) is norm convergent (that is the classical Schur theorem).

(b) Observe that Theorem 3.1 is an extension of Theorem 3.2 in the context of \( P \)-ideals.

We now give a characterization of maximal ideals, related with the \( I \)-limit of the subsequences of real bounded sequences. Note that, if we assume the continuum hypothesis, then there exists a large class of maximal \( P \)-ideals (see [18]).
Proposition 3.4. An admissible ideal $I$ of $\mathbb{N}$ is not maximal if and only if, for any bounded sequence $(a_n)_n$ in $\mathbb{R}$ with the property that $I - \lim_{h} a_{i_h}$ exists in $\mathbb{R}$ for each strictly increasing sequence $(i_h)_h$ in $\mathbb{N}$, we get that the limit $l := \lim_{n} a_n$ exists in $\mathbb{R}$ in the classical sense. In this case, $I - \lim_{h} a_{i_h} = l$ is the same for every strictly increasing sequence $(i_h)_h$.

Proof. We begin with the ”if” part. Suppose by contradiction that $I$ is maximal. Then there exists a bounded real sequence $(a_n)_n$, which does not admit limit in the classical sense. For every strictly increasing sequence $(i_h)_h$ in $\mathbb{N}$, the sequence $(a_{i_h})_h$ is obviously bounded too, and hence, since $I$ is maximal, the limit $I - \lim_{h} a_{i_h}$ exists in $\mathbb{R}$ (see also [4]). This leads to a contradiction, and concludes the proof of the ”if” part.

We now turn to the ”only if” part. First of all we claim that, if $I$ is any not maximal ideal of $\mathbb{N}$, then there exist two disjoint elements $B_1 \not\subseteq I$, $B_2 \not\subseteq I$ whose union is $\mathbb{N}$. Otherwise, for each partition of $\mathbb{N}$ formed by two elements $(B_1, B_2)$ of $I$, then either $B_1$ or $B_2$ belongs to $I$. If $B_1 \in I$, then $B_2 \not\subseteq I$, otherwise $\mathbb{N} \in I$, and hence $I$ should be trivial. Similarly, if $B_2 \in I$, then $B_1 \not\subseteq I$. Thus the ideal $I$ should be maximal. This leads to a contradiction and proves the claim. Moreover note that, since $I$ is admissible, then every finite subset of $\mathbb{N}$ belongs to $I$, and hence the two involved sets $B_1$, $B_2$ turn out to be infinite. Thus we can represent them in the form

$$B_1 := \{t_1 < t_2 < \ldots < t_j < \ldots\}, \quad B_2 := \{r_1 < r_2 < \ldots < r_j < \ldots\}. \quad (10)$$

Now suppose by contradiction that $\lim_{n} a_n$ does not exist in $\mathbb{R}$. So, since $(a_n)_n$ is bounded, there are two sequences in $\mathbb{N}$, $(p'_h)_h$, $(q'_h)_h$ such that $\lim_{h} a_{p'_h} = l_1$, $\lim_{h} a_{q'_h} = l_2$, where

$$\liminf_{n} a_n := l_1 < l_2 := \limsup_{n} a_n.$$

Set $P := \{p'_j : j \in \mathbb{N}\}$, $Q := \{q'_l : l \in \mathbb{N}\}$. Let now $p_1 := p'_1$, and choose $q_1 > p_1$, $q_1 \in Q$: such an element does exist, since $Q$ is infinite. Pick now $p_2 \in P$ such that $p_2 > q_1$: such a choice is possible, because $P$ is infinite. Keeping on by induction, it is possible to construct two sequences $(p_h)_h$, $(q_h)_h$, with the properties that: $\lim_{j} a_{p_j} = l_1$, $\lim_{s} a_{q_s} = l_2$, and

$$p_1 < q_1 < p_2 < \ldots < q_{h-1} < p_h < q_h < p_{h+1} < \ldots$$

For example, if we have just defined $p_1 < q_1 < \ldots < p_{h-1} < q_{h-1}$, let us choose $p_h \in P$ such that $p_h > q_{h-1}$ and $q_h \in Q$ with $q_h > p_h$: this is possible, since $P$ and $Q$ are infinite.
Let now $B_1, B_2$ be as in (10). For every $n \in \mathbb{N}$ there exists one natural number $j$ that $n = t_j$, or there is one positive integer $s$ such that $n = r_s$. In the first case put $b_n := a_{p_j}$, and in the second case set $b_n := a_{q_s}$.

The next step is to prove that for all $l \in \mathbb{R}$ there exists $\delta(l) > 0$, such that

$$\{n \in \mathbb{N} : |b_n - l| > \delta\} \not\in \mathcal{I}. \tag{11}$$

First of all, let us consider the case $l \neq l_1$. Take $\delta := \frac{|l - l_1|}{2}$ and set

$$\varepsilon := \frac{|l - l_1|}{4} = \frac{\delta}{2}.$$ 

By the definition of limit, we get: $|a_{p_j} - l_1| \leq \frac{|l - l_1|}{4}$ in the complement of a finite number of indexes $j$. So there exists a finite subset $N_1 \subset \mathbb{N}$ such that, if $n \in B_1 \setminus N_1$, then $|b_n - l_1| \leq \frac{|l - l_1|}{4}$. This implies that for all $n \in B_1 \setminus N_1$ we get: $|b_n - l| > \frac{|l - l_1|}{2}$. Otherwise we should have:

$$|l - l_1| \leq |l - b_n| + |b_n - l_1| \leq \frac{|l - l_1|}{2} + \frac{|l - l_1|}{4} = \frac{3}{4} |l - l_1|.$$ 

This is possible if and only if $l = l_1$, but this is absurd, because it contradicts our assumption.

Thus the set $\{n \in \mathbb{N} : |b_n - l| > \delta\}$ contains $B_1 \setminus N_1$, and so it does not belong to $\mathcal{I}$, since $B_1 \not\in \mathcal{I}$, $N_1$ is finite and $\mathcal{I}$ is admissible. Thus (11) is proved, at least when $l \neq l_1$.

We now turn to the case $l = l_1$. Take $\delta := \frac{l_2 - l_1}{2}$. Note that $\delta > 0$, since $l_1 < l_2$. Analogously as above, we get $|a_{r_s} - l_2| \leq \frac{l_2 - l_1}{4}$ in the complement of finitely many indexes $s$. Thus there is a finite subset $N_2 \subset \mathbb{N}$ such that $|b_n - l_2| \leq \frac{l_2 - l_1}{4}$ whenever $n \in B_2 \setminus N_2$. This implies that for all $n \in B_2 \setminus N_2$ we have: $|b_n - l| > \frac{l_2 - l_1}{2}$. Otherwise, we get:

$$0 < l_2 - l_1 \leq |l_1 - b_n| + |b_n - l_2| \leq \frac{l_2 - l_1}{2} + \frac{l_2 - l_1}{4} = \frac{3}{4} (l_2 - l_1) < l_2 - l_1,$$

a contradiction. Thus the set $\{n \in \mathbb{N} : |b_n - l| > \delta\}$ contains $B_1 \setminus N_1$, and so it does not belong to $\mathcal{I}$, since $B_2 \not\in \mathcal{I}$, $N_2$ is finite and $\mathcal{I}$ contains all the finite subsets of $\mathbb{N}$. This proves (11) in the case $l = l_1$. 
From (11) it follows that the sequence \((b_n)_n\) does not have \(\mathcal{I}\)-limit. By construction, it follows easily that the sequence
\[
(a_{p_1}, a_{q_1}, a_{p_2}, \ldots, a_{q_{h-1}}, a_{p_h}, a_{q_h}, a_{p_{h+1}}, \ldots)
\]
does not have \(\mathcal{I}\)-limit. Thus the assertion follows.

\[\square\]

**Remark 3.5.** Observe that Proposition 3.4 is a strengthening of [1, Theorem 2.1].

**Open Problems**

(a) Investigate the ideals \(\mathcal{I}\) for which weak \(\mathcal{I}\)-convergence in \(\ell_1\) implies norm \(\mathcal{I}\)-convergence. A similar investigation considering filters was done in [3] and, in the \((\ell)\)-group setting, in [11].

(b) Study the ideals \(\mathcal{I}\) for which weak* \(\mathcal{I}\)-convergence in \(\ell_*\) implies weak \(\mathcal{I}\)-convergence of sequences (see also [17, pp. 103-104]).

(c) Find results analogous to Proposition 3.4 when rearrangements of the initial sequence \((a_n)_n\) are considered.

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**References**


