SOME NEW NON-ABELIAN 2-GROUPS $G$ WITH EVERY AUTOMORPHISM FIXING $G/\Phi(G)$ ELEMENTWISE

Rasoul Soleimani
Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, IRAN

Abstract: In this paper, we exhibit an infinite family of non-abelian 2-groups $G$ in which each automorphism fix $G/\Phi(G)$ elementwise.

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1. Introduction

Let $G$ be a finite group and $N$ a characteristic subgroup of $G$. We let $\text{Aut}^N(G)$ denote the centralizer in $\text{Aut}(G)$ of $G/N$. Clearly $\text{Aut}^N(G) \trianglelefteq \text{Aut}(G)$, the automorphism group of $G$, and $\sigma \in \text{Aut}^N(G)$ if and only if $x^{\sigma}x^{-1} \in N$ for all $x \in G$. The groups $\text{Aut}^Z(G)$, $\text{Aut}^\Phi(G)$ and $\text{Aut}^{G'}(G)$ have been studied by several authors, where $Z$, $\Phi$ and $G'$ stand for the centre of $G$, the Frattini subgroup of $G$ and the derived subgroup of $G$, respectively; see for example (see [1], [3], [10], [12]). By a result of Adney and Yen [1], if a finite group $G$ has no proper abelian direct factor, then there is a bijection from $\text{Hom}(G, Z(G))$ onto $\text{Aut}^Z(G)$. This provides an efficient method to compute the order of $\text{Aut}^Z(G)$. Particularly, if all automorphisms of a finite group having no proper abelian direct factor are central, then one may determine $|\text{Aut}(G)|$. Various authors (see [4], [5], [8], [11]) have constructed some non-abelian finite $p$-groups $G$ in

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which every automorphism lies in Aut$^Z(G)$. More recently, Jamali [9] construct
an infinite family of non-abelian 2-groups $G$ for which all automorphisms fix
$G/\Phi(G)$ elementwise. Soleimani [14] exhibits an infinite family of 2-groups
$G$ for which Aut($G$) = Aut$^G(G)$, with $G' \neq Z(G)$ and $G' \neq \Phi(G)$. In this paper,
we take $N = \Phi(G)$, the Frattini subgroup of $G$, and construct a new infinite
family of 2-groups $G = G_m$ for which Aut($G$) = Aut$^\Phi(G)$ with $\Phi(G) \neq Z(G)$
and $\Phi(G) \neq G'$. Note that by a well-known theorem (Burnside-Hall) if $G$ is a
finite $p$-group then so is Aut$^\Phi(G)$. Our notation is standard and can be found
in [6,7], for example.

2. The Group $G_m$

For any positive integer $m$, we define the group $G_m$ by

$$G_m = \langle a, b | a^4 = b^2 = [a^2, b] = (aba^{-1}b)^{2m+1} = 1 \rangle,$$

and prove the following theorem.

**Theorem 2.1.** The group $G_m$ having order $2^{m+4}$ is of nilpotency class
$m + 2$ and has only automorphisms fixing $G_m/\Phi(G_m)$ elementwise. The au-
tomorphism group of $G_m$ has order $2^{2m+4}$. Furthermore Aut$^Z(G_m) \cong Z_4^2$,
Aut$^G(G_m) \cong \text{Inn}(G) \rtimes Z_2$, and Aut$^\Phi(G_m) \cong \text{Aut}^G(G_m) \rtimes (Z_2 \times Z_2)$.

3. Preliminary Lemmas

We begin with some lemmas which will be used in the proof of Theorem 2.1.
For simplicity, we set $G = G_m$, $c = aba^{-1}b$ and let $H = \langle a^2, c \rangle$.

**Lemma 3.1.** With the above notation, $H$ is an abelian normal subgroup
of $G$ with $|G : H| = 4$, $H \cong Z_2 \times Z_{2m+1}$ and $H = \Phi(G)$.

**Proof.** From the relation $[a^2, b] = 1$, we have $a^2 \in Z(G)$ and therefore $H$ is
abelian. To prove $H < G$, we observe that $a^{-1}ca = [b, a] = [a, b]^{-1} = c^{-1} \in H$,
b$^{-1}cb = baba^{-1} = c^{-1} \in H$.

We now determine a presentation for the group $H$ using the Modified Todd-
Coxeter algorithm in the form given in [13]. The algorithm gives a presentation
for $H$ on the generators $h = a^2$ and $k = c$ showing that $|G : H| = 4$. Elimin-
ating the redundant generators from the presentation obtained, we arrive at the
following presentation for $H$:

$$H = \langle h, k | h^2 = k^{2m+1} = 1, [h, k] = 1 \rangle.$$
Therefore, $|G| = 4|H| = 4 \cdot 2^{m+2} = 2^{m+4}$. Now since $G$ is a 2-generator group and $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we deduce that $H = \Phi(G)$, as required. 

Lemma 3.2. Let $x \in \Phi(G)$. Then:

(i) $xbx = b$ and $xax = a$,

(ii) $(xa)^4 = 1$ and $(xb)^2 = 1$.

Proof. (i) From the relations of $G$, we deduce that $c^ib^i = b$ and $c^iac^i = a$, for any integer $i$. Now for complete the proof it is sufficient to see that if $x \in \Phi(G)$ then $x$ can be written as $x = c^i$ or $x = a^2c^i$, for some integer $i$.

(ii) Let $x \in \Phi(G)$. We have $(xa)^4 = xaxaxxa = a^4 = 1$, by using (i). By a similar argument, $(xb)^2 = 1$.

Lemma 3.3. $G' = \langle c \rangle \cong \mathbb{Z}_{2^{m+1}}$, $Z(G) \leq \Phi(G)$ and $Z(G) = \langle a^2, c^{2^m} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Let $N = \langle c \rangle$. As $c^a = c^b = c^{-1} \in N$, $N$ is a normal subgroup of $G$. Now since $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $N = G' = \langle c \rangle \cong \mathbb{Z}_{2^{m+1}}$, as required.

For the second part of the Lemma, according to Lemma 3.1, we have $G = \Phi \bigcup \Phi a \bigcup \Phi b \bigcup \Phi ab$, where $\Phi$ stands for $\Phi(G)$. It is seen that $Z(G)$ intersects neither of $\Phi a, \Phi b$, and $\Phi ab$. It follows that $Z(G) \leq \Phi(G)$. Now it is straightforward to see that $Z(G) = \langle a^2, c^{2^m} \rangle$.

Lemma 3.4. The cosets $\Phi a, \Phi b$ and $\Phi ab$ have all their elements of orders 4, 2 and $2^{m+2}$, respectively.

Proof. By using Lemma 3.2, we establish only for the elements of $\Phi ab$. If $x \in \Phi$ then we have

$$(xab)^2 = xabxab = xax^{-1}bx^{-1}x^2ab = abx^2ab = abx^2ax^2x^{-2}bx^{-2}x^2 = (ab)^2x^2.$$ 

So $(xab)^4 = c^2x^4$. Now since $(xab)^{2^{m+1}} = c^{2^m}$, $|xab| = 2^{m+2}$.
4. Proof of Theorem

In this section first we give a result which will be used in the rest of the paper.

**Theorem 4.1.** (see [2], Theorem 3.2) Let $G = \langle a, b \rangle$ be a two generated metabelian group. Then the following are equivalent:

(i) For all $u, v \in G'$ there is an automorphism of $G$ that maps $a$ to $au$ and $b$ to $bv$;

(ii) $G$ is nilpotent.

Now we proceed to prove the main theorem. To see this we consider the four cosets of $\Phi(G)$ in $G$, namely $\Phi, \Phi a, \Phi b$ and $\Phi ab$. By Lemma 3.4, the cosets $\Phi a, \Phi b$, and $\Phi ab$ have all their elements of orders 4, 2, and $2^{m+2}$, respectively. Thus each coset stays fixed under all automorphisms of $G$. This means that all automorphisms of $G$ fix $G/\Phi$ elementwise, that is, $\text{Aut}(G) = \text{Aut}^{\Phi}(G)$. To compute the order of $\text{Aut}(G)$, we define $\tau : \{a, b\} \rightarrow G$ by setting $a^{\tau} = xa$ and $b^{\tau} = yb$, where $x, y$ are some fixed elements of $\Phi(G)$. Obviously $G = \langle xa, yb \rangle$.

We below show that for all relations $r$ of $G$, the result of substituting $xa$ for $a$ and $yb$ for $b$ in $r$ yields the identity of $G$. So $\tau$ extends to an automorphism of $G$ and hence $|\text{Aut}(G)| = |\Phi(G)|^2 = 2^{2m+4}$. We observe that by Lemma 3.2, $(xa)^4 = 1$, $(yb)^2 = 1$, $[(xa)^2, yb] = [a^2, yb] = 1$ and

$$(xayba^{-1}x^{-1}yb)^{2m+1} = (xaxx^{-1}ybx^{-1}y(x^{-1}y)^{-1}a^{-1}(x^{-1}y)^{-1}(x^{-1}y)^2b)^{2m+1} = (aba^{-1}(x^{-1}y)^2b(x^{-1}y)^2(x^{-1}y)^{-2})^{2m+1} = (c_y^{-2}x^2)^{2m+1} = c^{2m+1} = 1.$$ 

We now proceed to determine the nilpotency class of $G$. Taking $G_1 = G/Z(G)$ gives

$$G_1 = \langle a, b | a^2 = b^2 = 1, (ab)^{2m+1} = 1 \rangle.$$ 

So $G_1 \cong D_{2^{m+2}}$, the dihedral group of order $2^{m+2}$. Since $D_{2^{m+2}}$ is of maximal class, we conclude that $G$ is of class $m + 2$.

By Lemma 3.3, $G$ has no non-trivial central direct factor, and hence

$$|\text{Aut}^Z(G)| = |\text{Hom}(G/G', Z(G))| = 16.$$ 

Now it is obvious that the automorphisms $\theta$ sending $a$ to $z_1a$ and $b$ to $z_2b$, where $z_1, z_2 \in Z(G)$, has order 2.

Now, there are automorphisms $\alpha, \beta$ and $\gamma$ defined by $a^{\alpha} = ac, b^{\alpha} = bc, a^{\beta} = ac, b^{\beta} = b$ and $a^{\gamma} = a, b^{\gamma} = bc^2$. It is then easy to check that $\text{Inn}(G) =$
\langle \alpha, \beta \rangle \text{ and } |\gamma| = 2^m. \text{ Next by Theorem 4.1, } |\text{Aut}^G(G)| = |G'|^2 = 2^{2m+2}. \text{ Comparing the orders of Inn}(G) \text{ and } \gamma, \text{ we see that } \text{Aut}^G(G) = \text{Inn}(G) \rtimes \langle \gamma \rangle \cong \text{Inn}(G) \rtimes \mathbb{Z}_{2^m}. \text{ Finally, there are automorphisms, } \delta \text{ and } \psi \text{ defined by } a^\delta = a^{-1}, b^\delta = b, \text{ and } a^\psi = a, \ b^\psi = ba^2. \text{ It is then easy to check that } \text{Aut}^G(G_m) = \text{Aut}^G(G_m) \rtimes (\langle \delta, \psi \rangle) \cong \text{Aut}^G(G_m) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2). \text{ This completes the proof. }

References


