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SOME NEW NON-ABELIAN 2-GROUPS G WITH EVERY AUTOMORPHISM FIXING $G/\Phi(G)$ ELEMENTWISE

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Abstract: In this paper, we exhibit an infinite family of non-abelian 2-groups G in which each automorphism fix $G/\Phi(G)$ elementwise.

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1. Introduction

Let G be a finite group and N a characteristic subgroup of G. We let $\mathrm{Aut}^N(G)$ denote the centralizer in $\mathrm{Aut}(G)$ of G/N. Clearly $\mathrm{Aut}^N(G) \lhd \mathrm{Aut}(G)$, the automorphism group of G, and $\sigma \in \mathrm{Aut}^N(G)$ if and only if $x^\sigma x^{-1} \in N$ for all $x \in G$. The groups $\mathrm{Aut}^Z(G)$, $\mathrm{Aut}^\Phi(G)$ and $\mathrm{Aut}^{G'}(G)$ have been studied by several authors, where Z, Φ and G' stand for the centre of G, the Frattini subgroup of G and the derived subgroup of G, respectively, see for example (see [1], [3], [10], [12]). By a result of Adney and Yen [1], if a finite group G has no proper abelian direct factor, then there is a bijection from $\mathrm{Hom}(G, Z(G))$ onto $\mathrm{Aut}^Z(G)$. This provides an efficient method to compute the order of $\mathrm{Aut}^Z(G)$. Particularly, if all automorphisms of a finite group having no proper abelian direct factor are central, then one may determine $|\mathrm{Aut}(G)|$. Various authors (see [4], [5], [8], [11]) have constructed some non-abelian finite p-groups G in

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which every automorphism lies in $\mathrm{Aut}^Z(G)$. More recently, Jamali [9] construct an infinite family of non-abelian 2-groups G for which all automorphisms fix $G/\Phi(G)$ elementwise. Soleimani [14] exhibits an infinite family of 2-groups G for which $\mathrm{Aut}(G) = \mathrm{Aut}^{G'}(G)$, with $G' \neq Z(G)$ and $G' \neq \Phi(G)$. In this paper, we take $N = \Phi(G)$, the Frattini subgroup of G, and construct a new infinite family of 2-groups $G = G_m$ for which $\mathrm{Aut}(G) = \mathrm{Aut}^{\Phi}(G)$ with $\Phi(G) \neq Z(G)$ and $\Phi(G) \neq G'$. Note that by a well-known theorem (Burnside-Hall) if G is a finite p-group then so is $\mathrm{Aut}^{\Phi}(G)$. Our notation is standard and can be found in [6,7], for example.

2. The Group G_m

For any positive integer m, we define the group G_m by

$$G_m = \langle a, b | a^4 = b^2 = [a^2, b] = (aba^{-1}b)^{2^{m+1}} = 1\rangle,$$

and prove the following theorem.

Theorem 2.1. *The group G_m having order 2^{m+4} is of nilpotency class $m+2$ and has only automorphisms fixing $G_m/\Phi(G_m)$ elementwise. The automorphism group of G_m has order 2^{2m+4}. Furthermore $\mathrm{Aut}^Z(G_m) \cong \mathbb{Z}_2^4$, $\mathrm{Aut}^{G'}(G_m) \cong \mathrm{Inn}(G) \rtimes \mathbb{Z}_{2^m}$, and $\mathrm{Aut}^{\Phi}(G_m) \cong \mathrm{Aut}^{G'}(G_m) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.*

3. Preliminary Lemmas

We begin with some lemmas which will be used in the proof of Theorem 2.1. For simplicity, we set $G = G_m$, $c = aba^{-1}b$ and let $H = \langle a^2, c\rangle$.

Lemma 3.1. *With the above notation, H is an abelian normal subgroup of G with $|G : H| = 4$, $H \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{m+1}}$ and $H = \Phi(G)$.*

Proof. From the relation $[a^2, b] = 1$, we have $a^2 \in Z(G)$ and therefore H is abelian. To prove $H \lhd G$, we observe that $a^{-1}ca = [b, a] = [a, b]^{-1} = c^{-1} \in H$, $b^{-1}cb = baba^{-1} = c^{-1} \in H$.

We now determine a presentation for the group H using the Modified Todd-Coxeter algorithm in the form given in [13]. The algorithm gives a presentation for H on the generators $h = a^2$ and $k = c$ showing that $|G : H| = 4$. Eliminating the redundant generators from the presentation obtained, we arrive at the following presentation for H:

$$H = \langle h, k | h^2 = k^{2^{m+1}} = 1, [h, k] = 1\rangle.$$

Therefore, $|G| = 4|H| = 4.2^{m+2} = 2^{m+4}$. Now since G is a 2-generator group and $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we deduce that $H = \Phi(G)$, as required. □

Lemma 3.2. *Let $x \in \Phi(G)$. Then:*

(i) $xbx = b$ and $xax = a$,

(ii) $(xa)^4 = 1$ and $(xb)^2 = 1$.

Proof. (i) From the relations of G, we deduce that $c^ibc^i = b$ and $c^iac^i = a$, for any integer i. Now for complete the proof it is sufficient to see that if $x \in \Phi(G)$ then x can be written as $x = c^i$ or $x = a^2c^i$, for some integer i.

(ii) Let $x \in \Phi(G)$. We have $(xa)^4 = xaxaxaxa = a^4 = 1$, by using (i). By a similar argument, $(xb)^2 = 1$. □

Lemma 3.3. $G' = \langle c \rangle \cong \mathbb{Z}_{2^{m+1}}$*, $Z(G) \leq \Phi(G)$ and $Z(G) = \langle a^2, c^{2^m} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $N = \langle c \rangle$. As $c^a = c^b = c^{-1} \in N$, N is a normal subgroup of G. Now since $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $N = G' = \langle c \rangle \cong \mathbb{Z}_{2^{m+1}}$, as required.

For the second part of the Lemma, according to Lemma 3.1, we have $G = \Phi \dot{\bigcup} \Phi a \dot{\bigcup} \Phi b \dot{\bigcup} \Phi ab$, where Φ stands for $\Phi(G)$. It is seen that $Z(G)$ intersects neither of $\Phi a, \Phi b$, and Φab. It follows that $Z(G) \leq \Phi(G)$. Now it is straightforward to see that $Z(G) = \langle a^2, c^{2^m} \rangle$. □

Lemma 3.4. *The cosets $\Phi a, \Phi b$ and Φab have all their elements of orders $4, 2$ and 2^{m+2}, respectively.*

Proof. By using Lemma 3.2, we establish only for the elements of Φab. If $x \in \Phi$ then we have

$$(xab)^2 = xabxab = xaxx^{-1}bx^{-1}x^2ab = abx^2ab = abx^2ax^2x^{-2}bx^{-2}x^2 = (ab)^2x^2.$$

So $(xab)^4 = c^2x^4$. Now since $(xab)^{2^{m+1}} = c^{2^m}$, $|xab| = 2^{m+2}$. □

4. Proof of Theorem

In this section first we give a result which will be used in the rest of the paper.

Theorem 4.1. (see [2], Theorem 3.2) *Let $G = \langle a, b\rangle$ be a two generated metabelian group. Then the following are equivalent:*

(i) For all $u, v \in G'$ there is an automorphism of G that maps a to au and b to bv;

(ii) G is nilpotent.

Now we proceed to prove the main theorem. To see this we consider the four cosets of $\Phi(G)$ in G, namely $\Phi, \Phi a, \Phi b$ and Φab. By Lemma 3.4, the cosets $\Phi a, \Phi b$, and Φab have all their elements of orders $4, 2$, and 2^{m+2}, respectively. Thus each coset stays fixed under all automorphisms of G. This means that all automorphisms of G fix G/Φ elementwise, that is, $\mathrm{Aut}(G) = \mathrm{Aut}^{\Phi}(G)$. To compute the order of $\mathrm{Aut}(G)$, we define $\tau : \{a, b\} \longrightarrow G$ by setting $a^{\tau} = xa$ and $b^{\tau} = yb$, where x, y are some fixed elements of $\Phi(G)$. Obviously $G = \langle xa, yb\rangle$. We below show that for all relations r of G, the result of substituting xa for a and yb for b in r yields the identity of G. So τ extends to an automorphism of G and hence $|\mathrm{Aut}(G)| = |\Phi(G)|^2 = 2^{2m+4}$. We observe that by Lemma 3.2, $(xa)^4 = 1$, $(yb)^2 = 1$, $[(xa)^2, yb] = [a^2, yb] = 1$ and

$$\begin{aligned}(xayba^{-1}x^{-1}yb)^{2^{m+1}} &= (xaxx^{-1}ybx^{-1}y(x^{-1}y)^{-1}a^{-1}(x^{-1}y)^{-1}(x^{-1}y)^2b)^{2^{m+1}}\\ &= (aba^{-1}(x^{-1}y)^2b(x^{-1}y)^2(x^{-1}y)^{-2})^{2^{m+1}} = (cy^{-2}x^2)^{2^{m+1}} = c^{2^{m+1}} = 1.\end{aligned}$$

We now proceed to determine the nilpotency class of G. Taking $G_1 = G/Z(G)$ gives

$$G_1 = \langle a, b | a^2 = b^2 = 1, (ab)^{2^{m+1}} = 1\rangle.$$

So $G_1 \cong D_{2^{m+2}}$, the dihedral group of order 2^{m+2}. Since $D_{2^{m+2}}$ is of maximal class, we conclude that G is of class $m + 2$.

By Lemma 3.3, G has no non-trivial central direct factor, and hence

$$|\mathrm{Aut}^Z(G)| = |\mathrm{Hom}(G/G', Z(G))| = 16.$$

Now it is obvious that the automorphisms θ sending a to z_1a and b to z_2b, where $z_1, z_2 \in Z(G)$, has order 2.

Now, there are automorphisms α, β and γ defined by $a^{\alpha} = ac$, $b^{\alpha} = bc$, $a^{\beta} = ac$, $b^{\beta} = b$ and $a^{\gamma} = a$, $b^{\gamma} = bc^2$. It is then easy to check that $\mathrm{Inn}(G) =$

$\langle \alpha, \beta \rangle$ and $|\gamma| = 2^m$. Next by Theorem 4.1, $|\mathrm{Aut}^{G'}(G)| = |G'|^2 = 2^{2m+2}$. Comparing the orders of $\mathrm{Inn}(G)$ and γ, we see that $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G) \rtimes \langle \gamma \rangle \cong \mathrm{Inn}(G) \rtimes \mathbb{Z}_{2^m}$. Finally, there are automorphisms, δ and ψ defined by $a^\delta = a^{-1}$, $b^\delta = b$, and $a^\psi = a$, $b^\psi = ba^2$. It is then easy to check that $\mathrm{Aut}^\Phi(G_m) = \mathrm{Aut}^{G'}(G_m) \rtimes (\langle \delta, \psi \rangle) \cong \mathrm{Aut}^{G'}(G_m) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. This completes the proof.

References

[1] J.E. Adney, T. Yen, Automorphisms of a p-group, *Ill. J. Math.*, **9** (1965), 137-143.

[2] A. Caranti, C.M. Scoppola, Endomorphisms of two-generated metabelian groups that induce the identity modulo the derived subgroup, *Arch. Math.*, **56(3)**(1991), 218-227.

[3] Y. Cheng, On finite p-groups with cyclic commutator subgroup, *Arch. Math.*, **39** (1982), 295-298.

[4] M.J. Curran, A non-abelian automorphism group with all automorphisms central, *Bull. Austral. Math. Soc.*, **26** (1982), 393-397.

[5] S.P. Glasby, 2-groups with every automorphism central, *J. Austral. Math. Soc. (Series A)*, **41** (1986), 233-236.

[6] D.J. Gorenstein, *Finite Group*, Harper and Row, New York (1968).

[7] B. Huppert, *Endliche Gruppen I*, Springer - Verlag, New York (1967).

[8] A.R. Jamali, Some new non-abelian 2-groups with abelian automorphism groups, *J. Group Theory.*, **1(5)** (2002), 53-57.

[9] A.R. Jamali, 2-groups G with every automorphism fixing $G/\Phi(G)$ elementwise, *Southeast Asian Bull. Math.*, **31** (2007), 255-257.

[10] H. Liebeck, The automorphism group of finite p-groups, *J. Algebra.*, **4** (1966), 426-432.

[11] J.J. Malone, P-groups with non-abelian automorphism groups and all automorphisms central, *Bull. Austral. Math. Soc.*, **29** (1984), 35-37.

[12] O. Müller, On p-automorphisms of finite p-groups, *Arch. Math.*, **32** (1979), 533-538.

[13] J. Neubüser, An Elementary Introduction to Coset Table Methods in Computational Group Theory, In *Groups St. Andrews 1981,* C. M. Campbell and E. F. Robertson, Eds; Cambridge Univ. Prees: Cambridge, 137-154, 1982. London Math. Soc. Lecture Notes Vol. 71.

[14] R. Soleimani, 2-Groups G with every automorphism fixing G/G' elementwise, *Southeast Asian Bull. Math.*, **34(3)** (2010), 489-495.