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MULTISCALE SOLUTIONS OF THE ELECTROMAGNETIC CONTINUITY DIFFERENTIAL EQUATION USING PACKETS OF HARMONIC WAVELETS

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Abstract: This paper proposes an approach to the analysis of the electromagnetic continuity differential equation of the Maxwell equations. Solutions through dyadic harmonic wavelets at different levels of resolution are presented. Wavelets approach, through their different space-time levels of resolution, can favorably describe the segmented space structure of some kinds of technical applications.

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1. Introduction and Motivation

Wavelets and wavelet packets are well known to provide an orthonormal base with a good localization both in time and spaces for instance. Since some years wavelets have been used as a tool to deal with more problems, such as processing of signal [1], detection of singularity, numerical solution of differential

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equations [2]. In this paper, we suggest a possible solution to the electromagnetic continuity differential equation by using wavelet packets. In particular, harmonic wavelet packets are used to solve this equation. In electromagnetic field, harmonic analysis of the phenomena is often demanded. This paper suggests therefore the use of harmonic wavelets to calculate the solution. The use of packets of wavelets allows to segment the domain of the definition of the solution. The solution is also calculated through this segmentation. Through an adaptive choice of this segmentation space or time, resolution of the solution can be emphasized. The paper is organized as follows. Section 2 deals with the problem formulation. Section 3 presents the structure of the harmonic wavelets and Section 4 shows a possible solution of the proposed problem. Conclusions and an outlook close the paper.

1.1. Notation

Through the paper the following notation is used:

$\vec{E}$ represents an electrical field.

$\vec{H}$ represents the density flux of a magnetic field.

$\rho(x, \vec{J})$ represents the density of charge as a function of distance x with respect to an initial point and of current density $\vec{J}$.

ϵ is the dielectric constant.

With the notation *div* and *rot* the operators divergence and rotor of a vector field are respectively indicated.

i is the unit complex value.

2. Problem Formulation

Assuming a model with just one dimension, assuming the charge density $\rho(x, \vec{J})$ as a function of the distance with respect to an initial point and as a function of the current density, as described above, the well known following Maxwell equation

$$div\vec{E} = \vec{\nabla}\vec{E} = \frac{\rho(x, \vec{J})}{\epsilon} \tag{1}$$

becomes

$$\frac{\partial E_x}{\partial x} = \frac{\rho(x, \vec{J})}{\epsilon}, \tag{2}$$

If an one-dimensional model is considered, the Maxwell equation

$$rot\vec{H} = \vec{\nabla} \times \vec{H} = \Big(\vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t}\Big) \tag{3}$$

applying the *div* operator becomes

$$div(rot\vec{H}) = div\Big(J + \epsilon \frac{\partial E_x}{\partial t}\Big) = 0. \tag{4}$$

Relation (4) represents the known continuity differential equation in the electromagnetic phenomena.

Problem 1. If $\rho = \rho_0 - a\vec{J}$ is an affine function of the current density, with "a" a positive constant is indicated, $\rho_0 > 0$ is the initial condition of the charge density. Assuming the $\rho > 0$ and combining eq. (4) with (3), considering that $\frac{\partial \rho_0}{\partial x} = \rho_0$ the following equation is obtained:

$$\frac{\partial^2 E_x}{\partial x^2} = a\epsilon \frac{\partial^2 E_x}{\partial x \partial t} + \rho_0. \tag{5}$$

Find a solution structure in wavelet domain depending on the scale level of approximation.

In the following part we will sketch a multiscale solution based on series of harmonic wavelets (see e.g. [3, 4] and references therein). This choice is due to the localization property of harmonic wavelets in frequency domain, moreover they allow us to represent the solution in a complex domain.

3. Periodic Harmonic Wavelets

Periodic harmonic wavelets are the complex valued functions [4, 5, 6]

$$\Psi_k^n(x) \equiv 2^{-n/2}\Psi(2^n x - k) = 2^{-n/2} \sum_{s=2^n}^{2^{n+1}-1} e^{-2\pi\ i\ s(x-k/2^n)}, \tag{6}$$

with $n, k \in \mathbb{N} \cup 0$. They are defined in the interval $(-\infty, +\infty)$ with slow decay, but their Fourier transforms $\widehat{\Psi}_k^n(\omega)$ are disjoint rectangle functions:

$$\widehat{\Psi}_k^n(\omega) = \begin{cases} 1/(2^{n+1}\pi) & 2^{n+1}\pi < \omega < 2^{n+2}\pi \\ 0 & \text{elsewhere}\ , \end{cases}$$

with compact support at each frequency, thus having a good localization in frequency. More in general, we can consider harmonic wavelets based on the interval $[0, 2^{-m})$, with fixed $m \in \mathbb{N} \cup \{0\}$,

$$\Psi_k^n(x) \equiv 2^{-n/2} \sum_{s=2^n}^{2^{n+1}-1} e^{-2^{m+1}\pi \ i \ s(x-k/2^n)} \ ,$$

with period 2^{-m} and dyadic intervals $(k2^{-m}, (k+1)2^{-m}] \ , \ k \in \mathbb{Z}$.

Harmonic wavelets have an exact analytical expression and are infinitely differentiable functions, so that from (6), the first and second derivatives are:

$$\begin{cases} \dfrac{\mathrm{d}\Psi_k^n(x)}{\mathrm{d}x} &= -2^{-n/2+1} \displaystyle\sum_{s=2^n}^{2^{n+1}-1} i \ \pi s \ e^{-2\pi \ i \ s(x-k/2^n)} \\ \dfrac{\mathrm{d}^2\Psi_k^n(x)}{\mathrm{d}x^2} &= 2^{-n/2+2} \displaystyle\sum_{s=2^n}^{2^{n+1}-1} \pi s^2 e^{-2\pi \ i \ s(x-k/2^n)} \ . \end{cases} \tag{7}$$

Harmonic wavelets form an orthogonal set of independent periodic functions locally concentrated at the values $k/2^n, (k, n \in Z)$, with unit period, i.e. based on the unit interval $[0, 1]$.

3.1. Connection Coefficients

Harmonic wavelets are periodic functions, thus we restrict to the unit interval $[0, 1]$ and there we assume as scalar product

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x \ ,$$

where the bar stands for the complex conjugate.

From the definition (6) and the equations (7), it easily follows for the linear connection coefficients (see also [6]), applying the Plancherel-Fubini theorem:

$$\begin{aligned} \gamma_{kh}^{nm} &\equiv \langle \frac{\mathrm{d}}{\mathrm{d}x}\Psi_k^n(x), \overline{\Psi}_h^m(x)\rangle \\ &= -2^{-(n+m)/2+1} \sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} i \ \pi s \int_{-\infty}^{\infty} e^{2\pi \ i \ [(r-s)x-(h/2^m-k/2^n)]}\mathrm{d}x, \end{aligned}$$

so that the unvanishing components of the connection coefficients are those for which $n = m$.

There follows:

$$\gamma_{kh}^{nm} = -2^{-(n+m)/2+1} \left[\sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} i\ \pi s\ e^{-2\pi\ i\ (h/2^m - k/2^n)} \delta_{rs} \right] \delta^{nm} \ ,$$

and explicitly,

$$\gamma_{kh}^{nm} = \begin{cases} -2^{1-n}\ \pi\ i \displaystyle\sum_{s,r=2^n}^{2^{n+1}-1} s\ e^{-2^{1-n}\pi\ i\ (h-k)} \delta_{rs} & for \quad n \ = \ m \ , \\ 0 & for \quad n \ \neq \ m \ . \end{cases} \tag{8}$$

$$\begin{aligned} \eta_{kh}^{nm} &\equiv \langle \frac{\mathrm{d}^2}{\mathrm{d}x^2} \Psi_k^n(x), \overline{\Psi}_h^m(x) \rangle \\ &= 2^{-(n+m)/2+2} \sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} \pi s^2 \int_{-\infty}^{\infty} e^{2\pi\ i\ [(r-s)x - (h/2^m - k/2^n)]} \mathrm{d}x, \end{aligned}$$

so that the unvanishing components of the connection coefficients are those for which $n = m$.

There follows:

$$\eta_{kh}^{nm} = 2^{-(n+m)/2+2} \left[\sum_{s=2^n}^{2^{n+1}-1} \sum_{r=2^m}^{2^{m+1}-1} \pi s^2\ e^{-2\pi\ i\ (h/2^m - k/2^n)} \delta_{rs} \right] \delta^{nm} \ ,$$

and explicitly,

$$\eta_{kh}^{nm} = \begin{cases} 2^{2-n}\ \pi \displaystyle\sum_{s,r=2^n}^{2^{n+1}-1} s^2\ e^{-2^{1-n}\pi\ i\ (h-k)} \delta_{rs} & for \quad n \ = \ m \ , \\ 0 & for \quad n \ \neq \ m \ . \end{cases} \tag{9}$$

In particular, taking into account that $k = 0, \dots, 2^n - 1,\ h = 0, \dots, 2^m - 1$, we have:

$$\gamma_{00}^{00} = -2\pi\ i \ , \quad and \quad \eta_{00}^{00} = 4\pi. \tag{10}$$

Considering, for instance, $n = m = 1$, the (first order) connection coefficients are the matrices

$$\gamma_{kh}^{11} = (5\pi) \begin{pmatrix} -\ i & i \\ i & -\ i \end{pmatrix}, \quad \eta_{kh}^{11} = (17\pi) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{11}$$

4. Periodic Harmonic Wavelet Solutions

We assume as a wavelet solution of equation (5) the following function depending on the (scale) level of approximation $N \leq \infty$

$$E_x(x,t) = \sum_{n=0}^{2^N-1} \sum_{k=0}^{2^n-1} \beta_k^n(t)\Psi_k^n(x), \tag{12}$$

with $\Psi_k^n(x)$ given by (6). Taking into account the orthogonality as property of the harmonic wavelets with the corresponding conjugate functions, equation (5) becomes:

$$\sum_{n,k} \beta_k^n(t)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_k^n(x) = a\epsilon \sum_{n,k}\left[\frac{\mathrm{d}}{\mathrm{d}t}\beta_k^n(t)\right]\frac{\mathrm{d}}{\mathrm{d}x}\Psi_k^n(x) + \rho_0,$$

in the unknown functions $\beta_k^n(t)$ with $n = 0,\dots,2^N-1,\ k = 0,\dots,2^n-1$ and, $\sum_{n,k} = \sum_{n=0}^{2^N-1}\sum_{k=0}^{2^n-1}$. So that according to the definition of the connection coefficients (9) and the orthonormality of wavelets it is

$$\sum_{n,k} \beta_k^n(t)\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_k^n(x)\right]\Psi_h^m(x)$$
$$= a\epsilon \sum_{n,k}\left[\frac{\mathrm{d}}{\mathrm{d}t}\beta_k^n(t)\right]\frac{\mathrm{d}}{\mathrm{d}x}\Psi_k^n(x)\Psi_h^m(x) + \rho_0\Psi_k^n(x)\Psi_h^m(x),$$

and

$$\sum_{n,k} \beta_k^n(t)\int_{-\infty}^{\infty}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_k^n(x)\right]\Psi_h^m(x)\mathrm{d}x$$
$$= a\epsilon \sum_{n,k}\frac{\mathrm{d}}{\mathrm{d}t}\beta_k^n(t)\left[\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\Psi_k^n(x)\Psi_h^m(x)\mathrm{d}x\right] + \int_{-\infty}^{\infty}\rho_0\ \Psi_k^n(x)\Psi_h^m(x)\mathrm{d}x$$

from where

$$\sum_{n,k} \beta_k^n(t)\eta_{kh}^{nm} = a\epsilon\left[\frac{\mathrm{d}}{\mathrm{d}t}\beta_h^m(t)\right]\gamma_{kh}^{nm} + \rho_0\ \delta_h^m. \tag{13}$$

4.1. Harmonic Wavelet Solution at the Level $N = 0$

At the lowest resolution level ($N = 0 \Rightarrow n = k = 0$), it is $\gamma^{00}_{00} = -2\pi i$ so that, we have, from (13), the following equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\beta^0_0(t) &= \frac{1}{a\epsilon}\left[\beta^0_0(t)\eta^{00}_{00} - \rho_0\right](\gamma^{00}_{00})^{-1}, \\ &= \frac{1}{a\epsilon}\left[-2\pi i\beta^0_0(t) - \rho_0\right], \end{aligned} \tag{14}$$

with a solution given by

$$\beta^0_0(t) = c_1 e^{-2\pi i t/(a\epsilon)} + \frac{\rho_0}{2\pi i} \tag{15}$$

and since $\Psi^0_0(x) = e^{-2\pi i x}$, *the periodic harmonic wavelet solution at the level* $N = 0$ is

$$E_x(x,t) = c_1 e^{-2\pi i[x + t/(a\epsilon)]} + \frac{\rho_0}{2\pi i} e^{-2\pi i x}\ . \tag{16}$$

5. Conclusion and Outlook

In this paper we have shown a harmonic wavelet packet based solution structure of the known electromagnetic continuity differential equation. This approach gives the possibility to segment the domain and to calculate the solution at each segment.

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