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E-EIGENVALUES OF TENSORS IN POSITIVE CHARACTERISTIC

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Abstract: We consider E-eigenvalues of tensors when the base field has positive characteristic. We extend some parts of the classical theory and we pose a question for symmetric tensors.

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1. Introduction

Fix integers $n \geq 1$ and $m \geq 2$ (we are only interested in the case $m \geq 3$, because the case $m = 2$ corresponds to the case of square matrices over a field). Fix an algebraically closed field $\mathbb{K}$. We consider m-tensors on the vector space $\mathbb{K}^n$ with the prescribed coordinates $x_1, \dots, x_n$, i.e. with the standard basis $e_i := (\delta_{ji})$, $j = 1, \dots, n$). Our definitions are not invariants by linear changes of coordinates. An m-tensor on $\mathbb{K}^n$ is just a set $A = (a_{i_1,\dots,i_m})_{1 \leq i_h \leq n}$ of n^m elements of $\mathbb{K}$. Eigenvalues of tensors are an important topic of applied linear algebra with applications given mainly in Engineering (see [3], [4], [5], [6], [7]). Two very different notions are used. Fix $\lambda \in \mathbb{K}$. We see the elements

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$x = (x_1, \dots, x_n) \in \mathbb{K}^n$ as column vectors. Set $Ax^{m-1} = (y_1, \dots, y_n)$ where

$$y_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{i,i_2,\dots,i_m} x_{i_2} \cdots x_{i_m}.$$

Fix $\lambda \in \mathbb{K}$. We say that λ is an E-eigenvalues of A and that x is an E-eigenvector of A if

$$Ax = \lambda x \ , \ x \neq 0. \tag{1}$$

In this case pair (λ, x) is called an E-eigenpair of A. In this note we extend some of the elementary parts of [6], [1] to the case $p := \mathrm{char}(\mathbb{K}) > 0$. Hence from now on we assume $\mathrm{char}(\mathbb{K} = p > 0$. For any $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ set $x \cdot x = x_1^2 + \cdots + x_n^2$. Notice that if $t \in \mathbb{K} \setminus \{0\}$, (λ, x) is an E-eigenpair of A if and only if $(t^{m-2}\lambda, tx)$ is an E-eigenpair of A. Since $\mathbb{K}$ is assumed to be algebraically closed, to know all E-eigenpairs of A it is sufficient to know all E-eigenpairs (λ, x) with $x \cdot x\{0, 1\}$. We say that λ is a *normalized* E-eigenvalue of A if there A has an E-eigenpair (λ, x) with $x \cdot x = 1$. We say that A is *regular* if it has no E-eigenpair $(0, x)$ with $x \cdot x = 0$. An E-eigenpair (λ, x) is said to be it normalized if $x \cdot x = 1$.

We may follow [1], §3, and [6] to prove the following result.

Theorem 1. *Assume that A is regular. Then either A has only finitely many normalized E-eigenvalues or for each $\lambda \in \mathbb{K}$ there is $x \in \mathbb{K}^n$ such that $Ax = \lambda x$ and $x \cdot x = 1$.*

Theorem 2. *Assume $n = 2$ (resp. $n = 3$). Assume that A is symmetric and assume $p > 2m$ (resp. $p \geq 2m^2 - 4m + 4$). Then A has only finitely many normalized E-eigenvalues.*

Question 1. Fix positive integer n, m. We ask if there is an integer $p_{m,n}$ with the following property. Assume $\mathrm{char}(\mathbb{K}) > p_{m,n}$. Let A be a symmetric tensor on $\mathbb{K}^n$. Then A has only finitely many E-eigenvalues.

There is a quite different concept (eigenvalues and eigenvectors) with a very different flavor (see [3], [7] and references therein). The extension of the elementary parts of the quoted papers to the case $\mathrm{char}(\mathbb{K}) > m$ is straightforward and hence it is omitted. We only ask the experts to get optimal bounds in positive characteristic (for symmetric tensor if $p|m$ the Euler's formula gives a way to distinguish several funny cases).

2. The Proof

Proof of Theorem 1. We may assume that A has infinitely many normalized and regular E-eigenvalues. Let $\pi : \mathbb{K}^n \times \mathbb{K} \to \mathbb{K}$ and $\eta : \mathbb{P}^n \times \mathbb{K} \to \mathbb{K}$ denote the projections onto the last factor. We take homogeneous coordinates $x_1, \dots, x_n, t$ on $\mathbb{P}^n$ with $x_1, \dots, x_n$ coordinates on $\mathbb{K}^n$ and we use them to identify $\mathbb{K}^n$ (resp. $\mathbb{K}^n \times \mathbb{K}$) as the open subset $t \neq 0$ of $\mathbb{P}^n$ (resp. $\mathbb{P}^n \times \mathbb{K}$). The closure $\overline{Q}$ in $\mathbb{P}^n$ of the smooth affine quadric $\{x \cdot x = 1\}$ is the smooth quadric hypersurface $\{x \cdot x = t^2\}$. Set $T := \{(x, c) \in \mathbb{K}^n \times \mathbb{K} : x \cdots x = 1, Ax = cx\}$. Since A is regular, T is closed in $\mathbb{P}^n \times \mathbb{K}$. Since η is a proper map, we get that $\eta(T)$ is closed in $\mathbb{K}$, i.e. either $\eta(T) = \mathbb{K}$ or $\eta(T)$ is finite. Since $\eta(T) = \pi(T)$, either every $\lambda \in \mathbb{K}$ is a regular E-eigenvalue of A or A has only finitely many normalized E-eigenvalues. □

Proof of Theorem 2. Set $Q := \{x \cdot x = 1\} \subset \mathbb{K}^n$ and $\overline{Q} = \{(x, t) \in \mathbb{P}^n : x^2 = t^2\}$. Since $p > m$, A is uniquely determined by a degree m polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$. Since the case $A = 0$ is obvious, we may assume $f \neq 0$. Notice that f induces a morphism $f|Q : Q \to \mathbb{K}$. Set $\psi := f|Q$. Since $p > m$ the normalized regular E-eigenvalues of A are the critical values of the regular function The only problem to extend [1], Proof of Proposition 3.3, to the positive characteristic case is that Sard's lemma is often false in positive characteristic.

First assume $n = 2$. In this case Q is a smooth, affine and connected curve. In this case a general $\lambda \in \mathbb{K}$ is a regular value of ψ if and only if either ψ is constant or ψ is separable. Assume that ψ is neither constant nor separable. In this case a general fiber of ψ is a disjoint union of finitely many zero-dimensional scheme, each of them with the same length β and $\beta \equiv 0 \pmod{p}$. Since any fiber of ψ has degree $\leq 2m$, we get $p \leq 2m$, a contradiction.

Now assume $n = 3$. Fix a general fiber U of ψ, say $U = \psi^{-1}(c)$, and let V be its closure in $\overline{Q}$. V is an effective divisor of type (m, m) of $\overline{Q}$ whose equation is the restriction to $\overline{Q}$ of the surface $\{f(x_1, x_2, x_3) = ct^m = 0\}$. Since $p > 2m$, as in the case $n = 2$ we see that ψ is separable and that each irreducible component of V appears with multiplicity one. Hence V is reduced. The adjunction formula gives $p_a(V) = m^2 - 2m + 1$. First assume that V is irreducible. Let g be the geometric genus of V. By [8], Corollary 2, we have $p_a(V) - g \geq (p-1)/2$. Hence $p \leq 2m^2 - 4m + 3$, a contradiction. Now assume that V is reducible and call V_i, $1 \leq i \leq s$, the irreducible components of V. Let (a_i, b_i) be the bidegree of V_i as a divisor of $\overline{Q}$. Since $(p-1)/2 > p_a(V_i)$, we get that each $V_i \cap Q$ is smooth. Taking a modification $\widetilde{Q} \to \overline{Q}$ which is the identity on Q we may extend ψ to a morphism $\phi : \widetilde{Q} \to \mathbb{P}^1$. Since V is reducible, but each irreducible component of V has multiplicity $< p$, a form of Bertini's second theorem proved by Zariski says that ϕ is composed with an involution (see [2], page 27). Hence $a_i = a_j$ and $b_i = b_j$ for all $i \neq j$. Hence $a_i = b_i = m/s$ for all i. Since $U = V \cap Q$ is singular, while each $V_i \cap Q$ is singular and a general fiber of a degree $s \geq 2$ separable morphism between integral curves is not connected, we get a contradiction. □

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