International Journal of Pure and Applied Mathematics
Volume 81 No. 3 2012, 425-432
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

Superficial defective subschemes of $\mathbb{P}^3$

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Abstract: We prove that the zero-dimensional superficial schemes $Z \subset \mathbb{P}^3$ such that $\deg(Z) \le 3m$ and $h^1(\mathcal{I}_Z(m)) = 0$ are only the expected ones (" superficial " means that the Zariski tangent spaces have dimension ≤ 2).

AMS Subject Classification: 14N05
Key Words: postulation, zero-dimensional scheme, Hilbert function

1. Introduction

Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme. We say that Z is *superficial* if it has embedding dimension 2, i.e. if for each $P \in Z_{red}$ the Zariski tangent space of Z at P has dimension ≤ 2. Any zero-dimensional subscheme of $\mathbb{P}^2$ is superficial. If $r \ge 3$, then Bertini's theorem implies that Z is superficial if and only if it is contained in a complete intersection surface of very large degree. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional superficial subscheme. For any $P \in \mathbb{P}^3$ let $\ell_P : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$ denote the linear projection from P. Fix a zero-dimensional superficial subscheme $Z \subset \mathbb{P}^3$. Let G_Z be the set of all $P \in \mathbb{P}^3 \setminus Z_{red}$ such that $\ell_P|Z$ induces an embedding of Z into $\mathbb{P}^2$. Since Z is superficial, G_Z is a non-empty open subset of $\mathbb{P}^3$. We use it and [7] in $\mathbb{P}^2$ to prove the following result.

Received: July 18, 2012

Theorem 1. *Assume characteristic zero. Fix an integer $m \geq 4$. Let $Z \subset \mathbb{P}^3$ be a zero-dimensional superficial scheme such that $\deg(Z) \leq 3m$. If $\deg(Z) = 3m$, the assume $m \geq 19$. We have $h^1(\mathcal{I}_Z(m)) > 0$ if and only if there is $W \subseteq Z$ as in one of the following cases:*

(a) $\deg(W) \geq m+2$ *and W is contained in a line:*

(b) $\deg(W) \geq 2m+2$ *and W is contained in a plane conic;*

(c) $W = Z$, $\deg(W) = 3m$ *and Z is the complete intersection of a plane cubic and a degree m plane curve.*

Without the restriction on the characteristic and for any $m \geq 2$, Theorem 1 is true if Z is reduced (see [4]). Theorem 1 is true even for higher $\deg(Z)$ if Z is reduced. There are many technical problems when Z is not reduced. We overcame some of them for curvilinear schemes in [2]. Theorem 1 extends the case $r = 3$ of [2], Theorem 1, to the case $\deg(Z) = 3m$, but we require two additional assumptions: large m and characteristic zero. The main technical details in this paper are different from the ones in [1] and [2], although the general framework is similar.

We work over an algebraically closed field $\mathbb{K}$ with characteristic zero.

2. The Proof

Let $F \subset \mathbb{P}^r$ be a degree t hypersurface. For each zero-dimensional scheme $Z \subset \mathbb{P}^r$ let $\mathrm{Res}_F(Z)$ denote the residual scheme of Z with respect to F, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_F : \mathcal{I}_Z$ as its ideal sheaf. For every $y \in \mathbb{Z}$ we have the following exact sequence of coherent sheaves:

$$0 \to \mathcal{I}_{\mathrm{Res}_F(Z)}(y-t) \to \mathcal{I}_Z(y) \to \mathcal{I}_{Z\cap F,F}(y) \to 0 \qquad (1)$$

We have $\deg(Z) = \deg(Z \cap F) + \deg(\mathrm{Res}_F(Z))$.

Lemma 1. *Fix integers $m > t > 0$, a degree $F \subset \mathbb{P}^r$ hypersurface an a zero-dimensional scheme $Z \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_Z(m)) > 0$. Assume one of the following conditions:*

(a) $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(m-t)) = 0$;

(b) $\deg(\mathrm{Res}_F(Z)) \leq m-t+1$.

Then $h^1(\mathcal{I}_{Z\cap F}(m)) = h^1(F, \mathcal{I}_{Z\cap F}(m)) > 0$.

Proof. Since F is arithmetically Cohen-Macaulay, for each zero-dimensional scheme $A \subset F$ we have $h^1(\mathcal{I}_A(m)) = h^1(F, \mathcal{I}_A(m))$. First assume

$$h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(m-t)) = 0.$$

The exact sequence (1) gives $h^1(F, \mathcal{I}_{Z \cap F}(m)) > 0$. If $\deg(\mathrm{Res}_F(Z)) \le m - t + 1$, then $h^1(\mathcal{I}_{\mathrm{Res}_F(Z)}(m-t)) = 0$ (see [3], Lemma 34). □

Lemma 2. *For any superficial $Z \subset \mathbb{P}^3$, any integer t and any $P \in G_Z$ we have $h^1(\mathcal{I}_Z(t)) \le h^1(\mathbb{P}^2, \mathcal{I}_{\ell_P(Z)}(t))$.*

Proof. Since $P \in G_Z$, we have $\deg(\ell_P(Z)) = \deg(Z)$. We have $h^1(\mathcal{I}_Z(t)) = h^0(\mathcal{I}_Z(t)) - \binom{m+3}{3} + \deg(Z)$ and $h^1(\mathbb{P}^2, \mathcal{I}_{\ell_P(Z)}(t)) = h^0(\mathbb{P}^2, \mathcal{I}_{\ell_P(Z)}(t)) - \binom{m+2}{2} + \deg(\ell_P(Z))$. Identify $H^0(\mathbb{P}^2, \mathcal{I}_{\ell_P(Z)}(t))$ with the linear subspace of $H^0(\mathcal{I}_Z(t))$ which parametrizes the cones with vertex containing P. □

Proof of Theorem 1. The " if " part is easy and well-known (see [1]). Hence we only prove the " only if " part. Assume $h^1(\mathcal{I}_Z(m)) > 0$. If $\deg(Z) \le 2m+1$, then use [3], Lemma 34. Hence we may assume $2m+2 \le \deg(Z) \le 3m$. Taking a smaller subscheme if necessary, we may assume $h^1(\mathcal{I}_A(m)) = 0$ for all $A \subsetneq Z$. Fix any $P \in G_Z$ and set $E := \ell_P(Z)$ and $z := \deg(E)$. Let τ be the maximal integer t such that $h^1(\mathcal{I}_E(t)) > 0$. Lemma 2 gives $h^1(\mathbb{P}^2, \mathcal{I}_E(m)) > 0$, i.e. $\tau \ge m$. Hence $\tau \ge 3 - 3 + z/3$. Hence we may apply the case $d := z$ and $s := 3$, of [7], Corollaire 2. We get that E is as in one of the following cases:

(a1) there is $W' \subseteq E$ with $\deg(W') \ge m+2$ and W' contained in a line;

(a2) there is $W'' \subseteq E$ with $\deg(W'') \ge 2m+2$ and W'' is contained in a conic;

(a3) $z = 3m$ and E is the complete intersection of a plane cubic and a degree m plane curve.

(i) Assume the existence of a line $D \subset \mathbb{P}^2$ such that $\deg(D \cap E) \ge m+2$. Let $H \subset \mathbb{P}^3$ be the only plane such that $P \in H$ and $\ell_P(H \setminus \{P\}) = D$. Hence $\deg(\mathrm{Res}_H(Z)) \le 2m-2$. First assume $h^1(H, \mathcal{I}_{Z \cap H}(m)) > 0$. We may apply [7], Corollaire 2, with respect to the scheme $Z \cap H$, the integer $s = 3$ and $d := \deg(H \cap Z)$ and see that $H \cap Z$ is as in (a), (b) or (c) (of course, in case (c) we must have $\deg(H \cap Z) = 3m$ and hence $Z \subset H$). Now assume $h^1(H, \mathcal{I}_{Z \cap H}(m)) = 0$. The exact sequence (1) gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) > 0$. Since $\deg(\mathrm{Res}_H(Z)) \le 3m - m - 2 \le 2(m-1)+1$, there is a line $D' \subset \mathbb{P}^3$ such that $\deg(D' \cap \mathrm{Res}_H(Z)) \ge m+1$. We may assume $\deg(Z \cap D') = m+1$ (otherwise we are in case (a1)). Let H' be any plane containing D'. Let $H + H'$

denote the reducible quadric $H \cup H'$ if $H' \neq H$ and the double quadric $2H$ if $H = H'$. Set $Z'' := Z \cap (H + H')$. Since $\deg(Z \cap (H + H')) \geq \deg(Z \cap H) + \deg(D' \cap \mathrm{Res}_H(Z)) \geq 2m + 3$, we have $\deg(\mathrm{Res}_{H+H'}(Z)) \leq m - 3 \leq m - 1$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_{H+H'}(Z)}(m-2)) = 0$. The exact sequence (1) gives $h^1(\mathcal{I}_{Z''}(m)) = 0$. Hence $Z'' = Z$. Taking another general plane $H'' \subset D'$, we get $Z \subset (H + H') \cap (H + H'')$. This implies that Z_{red} is contained in H and that $Z \cap (\mathbb{P}^3 \setminus D') \subset H$. Set $L := D'$. Fix a plane $M \supset L$. We may assume $h^1(\mathcal{I}_{Z \cap M}(m)) = 0$. Hence (1) gives $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(m-1)) > 0$. As above we get the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(R \cap \mathrm{Res}_M(Z)) \geq m + 1$. Let H_1 be a plane containing R. Since $\deg(\mathrm{Res}_{M+R}(Z)) \leq 3m - 2m - 2 \leq m - 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{M+R}(Z)}(m-2)) = 0$. Lemma 1 gives $Z \subset M \cup H_1$.

(i1) In this step we assume $R \neq L$. First assume $R \cap L \neq \emptyset$. We are in case (b) with respect to the reducible conic $R \cup L$. Now assume $R \cap L \neq \emptyset$. There is a smooth quadric surface Q containing $R \cup L$. Since $\deg(\mathrm{Res}_Q(Z)) \leq 3m - 2m - 2$, Lemma 1 gives $Z \subset Q$. Call $(1, 0)$ the ruling of Q containing the lines R and L. Since $R \cup L$ is an effective Cartier divisor of Q, one may define the residual scheme $\mathrm{Res}_{R \cup L}(Z)$. Since $\deg(L \cap Z) = \deg(R \cap Z) = m + 1$, we have $h^1(L \cup R, \mathcal{I}_{Z \cap (L \cup R), L \cup R}(m)) = 0$. Since $\deg(\mathrm{Res}_{R \cup L}(Z)) \leq 3m - 2m - 2 \leq m - 1$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_{R \cup L}(Z), Q}(m-2, m)) = 0$. We have an exact sequence similar to (**??**). We get $h^1(Q, \mathcal{I}_{Z,Q}(m)) = 0$. Hence $h^1(\mathcal{I}_Z(m)) = 0$, a contradiction.

(i2) Assume $R = L$. Let $L^{(1)}$ denote the first infinitesimal neighborhood of L in $\mathbb{P}^3$, i.e. the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_L)^2$ as its reduction. The scheme $L^{(1)}$ is a degree 3 locally Cohen-Macaulay scheme of pure dimension 1 with L as its reduction. In this case we proved that for any two planes $H_1, H_2 \supset L$ Z is contained in the reducible quadric $H_1 \cup H_2$. We get $Z \subset L^{(1)}$. Fix a connected component A of Z and set $\{O\} := A_{red}$. We have $O \in L$. Set $\beta := \deg(L \cap A)$. For every plane $M \supset L$ we have $\mathrm{Res}_M(A) \subseteq A$ and hence $\deg(\mathrm{Res}_M(A) \cap L) \leq \beta$. Since $\mathrm{Res}_M(Z)$ is contained in any plane containing L, we have $\mathrm{Res}_M(Z) \subset L$. Hence $\deg(\mathrm{Res}_M(A) \cap L) = \deg(\mathrm{Res}_M(A))$ and $\deg(A) = \beta + \deg(\mathrm{Res}_M(A))$. Since $\deg(\mathrm{Res}_M(Z) = m + 1 = \deg(Z \cap L)$ and $\deg(\mathrm{Res}_M(A) \cap L) \leq \deg(A \cap L)$ for each connected component A of Z, we get $\deg(Z) = 2m + 2$ and $\deg(\mathrm{Res}_M(A)) = \beta$. Since $\beta > 0$, we have $A \cap L \neq A$. Hence there is a plane $N \subset L$ such that $\deg(N \cap A) > \deg(A \cap L)$. Taking N instead of M we get $\deg(\mathrm{Res}_N(Z)) < m + 1$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_N(Z)}(m-1)) = 0$. Hence the exact sequence (1) gives $h^1(\mathcal{I}_{N \cap Z}(m)) > 0$. Hence $Z \subset N$. Hence we are in case (a2) with respect to the double line of N with L as its support.

(ii) Assume the existence of a conic $C \subset \mathbb{P}^2$ such that $\deg(C \cap E) \geq 2m+2$. Let $T \subset \mathbb{P}^3$ be the quadric cone with vertex containing P and with $\ell_P(T \backslash \{P\}) = C$. Set $Z' := Z \cap T$. We have $\deg(Z') = \deg(E \cap C) \geq 2m+2$. Hence $\deg(\mathrm{Res}_T(Z)) \leq 3m - 2m - 2 \leq m-1$. Lemma 1 gives $Z' = Z$, i.e. $Z \subset T$. First assume that C is a double line, say $C = 2L$. In this case T is a double plane. Set $H := T_{red}$. If $h^1(\mathcal{I}_{Z \cap H}(m)) > 0$, then $Z = Z \cap H$ and we may again apply [7], Corollaire 2, with $s = 3$. Now assume $h^1(\mathcal{I}_{Z \cap H}(m)) = 0$. The exact sequence (1) gives $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m)) > 0$. Since $\deg(\mathrm{Res}_H(Z)) \leq 2m-1$, we get the existence of a line $L \subset \mathbb{P}^3$ such that $\deg(R \cap \mathrm{Res}_H(Z)) \geq m+1$. Obviously $\ell_P(L) = D$. We have $\deg(L \cap Z) \geq m+1$. Excluding case (a1) we may assume $\deg(Z \cap L) = m+1$. We are in the case killed in step (i2).

The same proof works if C is a reduced, but reducible conic (see the case $R \cap L \neq \emptyset$ of step (i1)).

Now assume that C is a smooth conic. Hence Z is contained in an irreducible quadric cone U_P with vertex $P \in G_Z$. Assume for the moment that this is true for a general $(Q, O) \in G_Z \times G_Z$. In particular we assume $O \neq Q$, $O \notin U_Q$ and $Q \notin U_O$. Hence $U_Q \cap U_O$ is a degree 4 curve (not necessarily irreducible) containing no line. Hence either $U_Q \cap U_O$ is an irreducible complete intersection or $U_Q \cap U_O$ is the union of two smooth plane conics or it is a double structure on a smooth plane conic. We have $Z \subset U_Q \cap U_O$.

First assume that $U_Q \cap U_O$ is an integral degree 4 curve. Since $h^1(\mathcal{I}_{U_Q \cap U_O}(m)) = 0$, we have $h^1(U_Q \cap U_O, \mathcal{I}_{Z, U_Q \cap U_O}(m)) = h^1(\mathcal{I}_Z(m)) > 0$. Since $p_a(U_Q \cap U_O) = 1$ by the adjunction formula and $\deg(\mathcal{O}_{U_Q \cup U_O}(m)) - \deg(Z) = 4m - \deg(Z) \geq 2p_a(U_Q \cap U_O) - 1$, we have $h^1(U_Q \cap U_O, \mathcal{I}_{Z, U_Q \cap U_O}(m)) = 0$.

Now assume that $U_Q \cap U_O$ in the union of two smooth plane conics or it is a double structure on a smooth plane conic. In this case there is a plane M such that $\deg(Z \cap M) \geq \deg(Z)/2$. If $h^1(\mathcal{I}_{Z \cap M}(m)) > 0$, then $Z = Z \cap M$ and we conclude by [7], Corollaire 2, with $s = 3$. If $h^1(\mathcal{I}_{Z \cap M}(m)) = 0$, then we first get $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(m-1)) > 0$ and then the existence of a line D' such that $\deg(D' \cap \mathrm{Res}_M(Z)) \geq m+1$. Since $\ell_P(D')$ is a line and C is a smooth conic, we have $\deg(\ell_P(D') \cap C) \leq 2$. Since $\ell_P(Z \cap D') \subset C$ and $m+1 > 2$, we get a contradiction. If for a general $Q \in G_Z$ we are not in case (ii) with C irreducible, then we conclude by step (i) or by the proof of the Claim in the next step.

(iii) Assume that $z = 3m$ and that E is the complete intersection of a cubic plane curve J and a degree m plane curve. We get $\deg(Z) = 3m$. To prove that we are in case (a3) it is sufficient to find a plane $H \supset Z$ and then apply [7], Corollaire 2, with $s = 3$. Let $H \subset \mathbb{P}^3$ be any plane such that $\alpha := \deg(H \cap Z)$

is maximal. If $\alpha = 3m$, then we are done. Now assume $\alpha < 3m$.

Claim. *If $\alpha \geq m$, then we are in one of the cases (a1), (a2), (a3) and hence the lemma is true in this case.*

Proof of the Claim. If $h^1(H, \mathcal{I}_{Z\cap H}(m)) > 0$, then $Z \subset H$ and hence we may apply [7], Corollaire 2, with $s = 3$. Now assume $h^1(H, \mathcal{I}_{Z\cap H}(m)) = 0$. By (1) we have $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(m-1)) > 0$. Since $\deg(\mathrm{Res}_H(Z)) \leq 3m - m < 3(m-1)$. Hence either there is a line $D \subset \mathbb{P}^3$ such that $\deg(D \cap \mathrm{Res}_H(Z)) \geq m+1$ or there is a conic D' such that $\deg(D' \cap \mathrm{Res}_H(Z)) \geq 2m$. The latter case gives $\alpha \geq 2m$ and hence $z \geq \alpha + 2m > 3m$, a contradiction. Since $\deg(D\cap Z) \geq m+1$ and $P \in G_Z$, we have $P \notin D$. Hence $\ell_P(D \cap Z)$ is a degree $m+1$ collinear subscheme of J. Hence $\ell_P(Z)$ is not a complete intersection of J and a degree m plane curve, a contradiction.

By the Claim we may assume $\alpha < m$. If $J' \subset J$ is a line, then $\deg(\ell_P(Z) \cap J') = m$, because $\ell_P(Z)$ is a complete intersection. Taking the plane though P with J' as its image we get $\alpha \geq m$, a contradiction. Hence J is an integral curve.

Let $Y_P \subset \mathbb{P}^3$ be the cubic cone with vertex P such that $\ell_P(Y_P \setminus \{P\}) = J$. Since J is reduced and irreducible, Y_P is reduced and irreducible. Since $P \in G_Z$ and $\ell_P(Z) \subset E$, we have $Z \subset Y \setminus \{P\}$. Since E is a complete intersection of J and a degree m surface, we get $\deg(H \cap Z) \leq m$ for each plane $H \subset \mathbb{P}^3$ and $\deg(T \cap Z) \leq 2m$ for each quadric cone T. Hence Z is in case (a3) for all $P \in G_Z$. Fix $Q \in G_Z$ such that $P \neq Q$. We get a cubic cone $Y_Q \supset Z$. As above we may assume that Y_Q is irreducible. Hence the only lines of Y_P or Y_Q are the one through its vertex. For a general $Q \in G_Z$ we may assume $Q \notin Y_P$. Hence $Y_P \cap Y_Q$ is a degree 9 complete intersection and no irreducible component of $(Y_P \cap Y_Q)_{red}$ is a line. Since J_Q is irreducible, we get that each irreducible component of $(Y_P \cap Y_Q)_{red}$ has degree $\{3, 6, 9\}$.

(iii1) In this step we assume that $Y_P \cap Y_Q$ is an integral curve. Since $Y_Q \cap Y_P$ is a complete intersection, we have $h^1(\mathcal{I}_{Y_Q\cap Y_P}(m)) = 0$. Hence $h^1(Y_Q \cap Y_O, \mathcal{I}_{Z,Y_Q\cap Y_P}(m)) = h^1(\mathcal{I}_Z(m)) > 0$. We have $\omega_{Y_Q\cap Y_P} \cong \mathcal{O}_{Y_Q\cap Y_P}(2))$ (adjunction formula). Since $\deg(\mathcal{O}_{Y_Q\cap Y_P}(m)) - \deg(\omega_{Y_Q\cap Y_O}) = 9(m-2) > \deg(Z)$, we have $h^1(Y_Q \cap Y_P, \mathcal{I}_{Z,Y_Q\cap Y_P}(m)) = 0$, a contradiction.

(iii2) By step (iii1) we may assume that $Y_O \cap Y_Q$ is not integral for a general $(O, Q) \in G_Z \times G_Z$. Since $\deg(Y_O \cap Y_Q) = 9$ and each irreducible component of $(Y_P \cap Y_Q)_{red}$ has degree $\{3, 6, 9\}$, there is an irreducible component F of $(Y_P \cap Y_Q)_{red}$ with $\deg(F) = 3$.

First assume that F is degenerate. Hence F is a plane section both of Y_P and of Y_Q. We have $P \notin F$, $Q \notin F$ and Y_P and Y_Q are projectively equivalent. The scheme $Y_P \cap Y_Q$ is a multiplicity 3 structure supported by F. Hence $\deg(F \cap Z) \geq m$. The Claim shows that the lemma is true in this case.

From now on we assume that F is non-degenerate. Hence F is a rational normal curve. Take $I \in G_Z$ such that $Y_O \cap Y_I$ and $Y_Q \cap Y_I$ are not integral. If $Y_O \cap Y_Q \cap Y_I$ is a zero-dimensional scheme, then it has degree 3^3. Since $Z \subseteq Y_O \cap Y_Q \cap Y_I$ we have $\deg(Z) \leq 3^3$. Indeed, by the postulation of complete intersection curves we may be more precise and get $3m = \deg(Z) < 3^3$. Hence $m \leq 8$. Now assume that $Y_O \cap Y_Q \cap Y_I$ contains a reduced curve G. First assume $\deg(G) = 6$ (for a general $(P, Q, O) \in G_Z^3$). We saw that G is linked by the surfaces Y_P and Y_O to a rational normal curve F_1. A rational normal curve is arithmetically Cohen-Macaulay. Since G is linked to a rational normal curve, F_1, by Y_O and Y_Q, G is arithmetically Cohen-Macaulay and $p_a(G) = 3$ (see [5], Exercise 21.23, [8]). The Hilbert function of G is uniquely determined by the one of F_1 and the degrees of Y_P and Y_O (see [8]). We get $h^0(\mathcal{I}_G(3)) = 4$. We found a non-empty open subset of a 3-dimensional projective space parametrizing cubic cones with as vertex a general point of $\mathbb{P}^3$. Since $\mathrm{char}(\mathbb{K}) = 0$, Bertini's theorem gives a contradiction. Now assume $\deg(G) = 3$. We saw that we may assume that G is a rational normal curve. Let $T \subset \mathbb{P}^3$ be a smooth quadric surface containing G. Let G_2 be the degree 6 curves linked to G by Y_P and Y_Q. Since $Z \subset Y_P \cap Y_Q \cap Y_O$, we have $\deg(Z \cap G_2) \leq 18$. If $m \geq 19$, we get $h^1(\mathcal{I}_{\mathrm{Res}_{T(G)}}(m-2)) = 0$. Hence $h^1(\mathcal{I}_{Z\cap T}(m)) > 0$ by (1). Hence $Z \subset T$. Since $\mathcal{I}_G(2)$ is spanned and $Z \subset T$ for every $T \in |\mathcal{I}_G(2)|$, we get $Z \subset G$. Since $G \cong \mathbb{P}^1$ and $\deg(\mathcal{I}_{Z,G}(m)) = 0$, we have $h^1(G, \mathcal{I}_{Z,G}(m)) = 0$. Since G is arithmetically Cohen-Macaulay, we get $h^1(\mathcal{I}_Z(m)) = 0$, a contradiction. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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