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SOME DIVISIBILITY PROPERTIES OF GENERALIZED REPUNITS

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Abstract: We will concentrate on properties of generalized repunits $R_n(k)$, where k is any nonnegative integer and n is any positive integer greater than 1. In this paper a new result on divisibility of generalized repunits is stated.

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1. Introduction

A repunit R_n is any integer written in decimal form as a string of 1's. Thus repunits have the form $R_n = \frac{10^n-1}{9}$. The term *repunit* was coined by Beiler [1] in 1966. The great effort was devoted to testing of primality and finding all their prime factors. It easily can be seen that R_2 is prime. Hoppe [4] proved R_{19} to be prime in 1916 and Lehmer [7] and Kraitchik [6] independently found R_{23} to be prime in 1929. Williams proved that R_{317} is prime in 1978 and Williams and Dubner [10] proved that R_{1031} is prime in 1986. No other repunit primes are not known, but in recent time four probably prime repunits have known. In 1999 Dubner [3] found R_{49081}, Baxter discovered R_{86453} in 2000, Dubner found R_{109297} in 2007 and Voznyy and Budnyy found R_{270343} in 2007.

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Snyder [9] extended the notation repunit to one in which for some integer $b \geq 2$ by this way

$$R_n(b) = \frac{b^n - 1}{b - 1}. \tag{1}$$

They are called as *generalized repunits* or *repunits to base b and* consist of a string of 1's when written in base b. It is easy to see that if n is divisible by a, then $R_n(b)$ is divisible by $R_a(b)$. The other facts on the divisibility and primality of $R_n(b)$ can be found in Jaroma [5] and Dubner [2]. Generalized repunits are a generalization of the Mersenne numbers $M_n = 2^n - 1$ as they can be obtained by choice $b = 2$. It is well-known that the Mersenne numbers have various connections to many objects in the number theory. Probably, one of little known facts is the property, that $n \nmid M_n$ for any integer $n > 1$ (a proof can be found for example in [8]).

In this paper we will study whether generalized repunits $R_n(b)$ have the similar property $n \nmid R_n(b)$ for any $b > 2$. For the simplicity of notation we will be write $M_n(k)$ instead of $R_n(k+1)$ in the rest of this text. It is obvious that the numbers $M_n(k)$ are connected with the binomial theorem, concretely with the identity $(k+1)^n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i}$.

2. The Main Results

The main result established in this paper is expressed in the following theorem.

Theorem 1. *Let s be any positive integer, let $k = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$, where $p_1 < p_2 < \cdots < p_s$ be any primes and a_1, a_2,..., a_s be any positive integers. Let i be any nonnegative integer. Then*

$$p_j{}^i \mid M_{p_j{}^i}(k)$$

for $j = 1, 2, \ldots, s$.

Thus, the assertion of Theorem 1 says that there are infinitely many positive integers n which divide $M_n(k)$ for an arbitrary k.

3. Some Preliminary Results

Before we prove the main result of this article we derive some congruences and two relations on divisibility of binomial coefficients.

Definition 2. Let p be any prime and let n be any positive integer. The p-adic order (or valuation) of n, we use notation $\nu_p(n)$, is the exponent of the highest power of a prime p which divides n.

Now we derive some congruences and then we prove some facts on the divisibility of binomial coefficients.

Lemma 3. *Let p be any prime, let i, k and m be any positive integers. Then*

$$\frac{p^i - k}{k} \equiv \begin{cases} m^{p-2} - 1 \pmod p, & \text{iff } \nu_p(k) = i; \\ -1 \pmod p, & \text{iff } \nu_p(k) < i, \end{cases} \tag{2}$$

where $m = \frac{k}{p^{\nu_p(k)}}$.

Proof. We will consider three cases. Firstly, we obtain using Fermat's little theorem the following for $p \nmid k$

$$\frac{p^i - k}{k} \equiv \frac{p^i - k}{k} k^{p-1} \equiv p^i k^{p-2} - k^{p-1} \equiv -1 \pmod p.$$

Secondly, we have for $0 < \nu_p(k) = a < i$

$$\frac{p^i - mp^a}{mp^a} m^{p-1} \equiv p^{i-a} m^{p-2} - m^{p-1} \equiv -1 \pmod p.$$

Finally, we get for $\nu_p(k) = i$

$$\frac{p^i - mp^i}{mp^i} \equiv \frac{1-m}{m} m^{p-1} \equiv m^{p-2} - m^{p-1} \equiv m^{p-2} - 1 \pmod p. \qquad \square$$

Lemma 4. *Let p be any prime, let a, k and l be any nonnegative integers, $k < p$, $l < p^a$, $k + l > 0$. Then*

$$\frac{p^a}{kp^a + l} \equiv \begin{cases} 0 \pmod p, & l \neq 0; \\ k^{p-2} \pmod p, & l = 0. \end{cases} \tag{3}$$

Proof. We use Fermat's little theorem. For $l = 0$ we have

$$\frac{p^a}{kp^a + l} = \frac{1}{k} \equiv \frac{1}{k} k^{p-1} \equiv k^{p-2} \pmod p.$$

If $l > 0$, then we can write l in the form $l = mp^{\nu_p(l)}$, where m is any positive integer, $0 \le \nu_p(l) < a$ and $p \nmid m$. We obtain

$$\begin{aligned}\frac{p^a}{kp^a+l} &= \frac{p^a}{kp^a+mp^{\nu_p(l)}} \equiv \frac{p^{a-\nu_p(l)}}{kp^{a-\nu_p(l)}+m}(kp^{a-\nu_p(l)}+m)^{p-1}\\ &\equiv p^{a-\nu_p(l)}(kp^{a-\nu_p(l)}+m)^{p-2} \pmod p\\ &\equiv 0 \pmod p. \quad \square\end{aligned}$$

Lemma 5. *Let p be any prime and let i, m be any positive integers, $m < p^i$. Then*

$$p^{i+1} \mid \binom{p^i}{m} p^m.$$

Proof. Clearly we can rewritte the assertion as

$$\binom{p^i}{m} p^{m-i} \equiv 0 \pmod p,$$

which we prove using Lemma 3 by the following way

$$\begin{aligned}p^{m-i}\binom{p^i}{m} &= p^{m-i}\left(p^i \frac{p^i-1}{1}\frac{p^i-2}{2}\cdots\frac{p^i-(m-1)}{(m-1)}\frac{1}{m}\right)\\ &\equiv p^m(-1)^{m-1}\frac{1}{m} \equiv 0 \pmod p. \quad \square\end{aligned}$$

The following lemma is a stronger version of Lemma 5.

Lemma 6. *Let p be any prime and let a, i, k, l be any nonnegative integers, $k < p$, $a < i$, $l < p^a$, $k + l > 0$. Then*

$$\frac{1}{p^{i-a}}\binom{p^i}{kp^a+l} \equiv \begin{cases} 0 \pmod p, & l \neq 0;\\ (-1)^{kp^a-1}k^{p-2} \pmod p, & l = 0.\end{cases} \tag{4}$$

Proof. The assertion clearly gives $kp^a + l < p^i$, thus using (2) we have

$$\begin{aligned}&\frac{1}{p^{i-a}}\binom{p^i}{kp^a+l} \qquad (5)\\ &= \frac{1}{p^{i-a}}\left(p^i\frac{p^i-1}{1}\frac{p^i-2}{2}\cdots\frac{p^i-(kp^a+l-1)}{(kp^a+l-1)}\frac{1}{kp^a+l}\right)\\ &\equiv (-1)^{kp^a+l-1}\frac{p^a}{kp^a+l} \pmod p.\end{aligned}$$

Hence using Lemma 4 we obtain the assertion. $\square$

4. The Proof of the Main Result

Proof of Theorem 1. As $k = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ we can write k in the form $k = lp$, where p is any of the primes p_j, $j = 1, 2, \ldots, s$. Thus without loss of generality we have

$$M_{p_j{}^i}(k) = M_{p^i}(lp) = \frac{(lp+1)^{p^i} - 1}{lp}$$

and the assertion can be rewritten as $M_{p^i}(lp) \equiv 0 \pmod{lp^{i+1}}$ or equivalently

$$(lp+1)^{p^i} \equiv 1 \pmod{lp^{i+1}}. \tag{6}$$

To prove the last congruence we use the binomial theorem

$$(lp+1)^{p^i} = \sum_{m=0}^{p^i} \binom{p^i}{m} (lp)^m = 1 + \sum_{m=1}^{p^i} \binom{p^i}{m} (lp)^m.$$

Now congruence (6) follows from Lemma 5. □

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