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INDEPENDENT AND VERTEX COVERING NUMBER ON TENSOR PRODUCT OF FAN GRAPH

Siriluk Intaja[1], Thanin Sitthiwirattham[2 §]

[1,2]Department of Mathematics
Faculty of Applied Science
King Mongkut's University of Technology North Bangkok
Bangkok, 10800, THAILAND
[2]Centre of Excellence in Mathematics, CHE
Sri Ayutthaya Road, Bangkok, 10400, THAILAND

Abstract: Let $\alpha(G)$ and $\beta(G)$ be the independent number and vertex covering number of G, respectively. The Tensor product $G_1 \otimes G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, let G is a simple graph with order p, we prove that, $\alpha(F_{m,n} \otimes G) = \max\{(m+n)\alpha(G), p\max\{m, \lceil \frac{n}{2} \rceil\}\}$ and $\beta(F_{m,n} \otimes (G) = \min\{(m+n)\beta(G), p\min\{m + \lfloor \frac{n}{2} \rfloor, n\}\}$.

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let G_1 and G_2 be graphs. The Tensor product of graph G_1 and G_2, denote by $G_1 \otimes G_2$, is the graph with $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

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§Correspondence author

Next, we give the definitions about some graph parameters.

A subset U of the vertex set $V(G)$ of G is said to be an independent set of G if the induced subgraph $G[U]$ is a empty graph. An independent set of G with maximum number of vertices is called a maximum independent set of G. The number of vertices of maximum independent set of G is called the independent number of G, denoted by $\alpha(G)$.

A vertex of graph G is said to cover the edges incident with it, and a vertex cover of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G is called the vertex covering number of G, denoted by $\beta(G)$

By [4], we get $\alpha(F_{m,n}) = \max\{m, \lceil \frac{n}{2} \rceil\}$ and $\beta(F_{m,n}) = \min\{m + \lfloor \frac{n}{2} \rfloor, n\}$.

Proposition 1.1. *Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then:*

(i) $|V(H)| = |V(G_1)||V(G_2)|$

(ii) $|E(H)| = 2|E(G_1)||E(G_2)|$

(iii) *for every* $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Theorem 1.2. *Let G_1 and G_2 be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

Theorem 1.3. *Let G_1 and G_2 be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.*

Next we get that general from of graph of Tensor product of $F_{m,n}$ and a simple graph.

Proposition 1.4. *Let G be a connected graph of order p, the graph of $F_{m,n} \otimes G$ is*

$$\bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i \quad ; \quad H_i = \bigcup_{j=m+1}^{m+n} H_{ij} \quad \textit{and} \quad M_i = H_{i(i+1)}$$

where

$$V(H_{ij}) = S_i \cup S_j, \quad S_i = \{(i, 1), (i, 2), ..., (i, p)\}$$

and $E(H_{ij}) = \{(i, u)(j, v)/uv \in E(G)\}$.

Moreover, if G has no odd cycle then each H_{ij} has exactly two connected components isomorphic to G.

Example

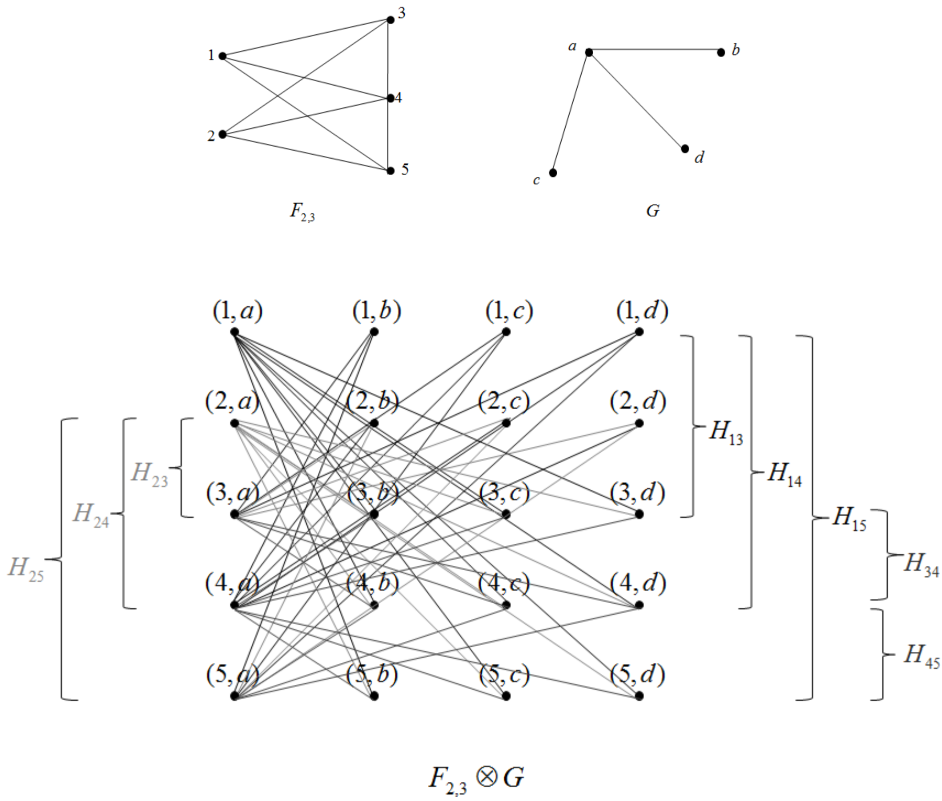


Figure 1: The graph of $Fm, n \otimes G$

2. Independent Number of the Graph of $F_{m,n} \otimes G$

We now state proposition and prove lemma before stating our main results. We begin this section by giving the proposition 2.1 show character of independent set and the lemma 2.2 show character of independent set for each H_{ij}.

Proposition 2.1. *Let $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph G if:*

(i) *v_i is not adjacent with v_j for all $i \neq j$ and $i, j = 1, 2, ..., k$.*

and

(ii) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

Lemma 2.2. *Let* $F_{m,n} \otimes G = \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i; H_i = \bigcup_{j=m+1}^{m+n} H_{ij}$ *and* $M_i = H_{i(i+1)}$, *then* $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) = 2\alpha(G)$.

Proof. Suppose G has no odd cycle, by proposition 1.4, we get $H_{ij} = H_{i(i+1)} = 2G$. So $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) = 2\alpha(G)$.

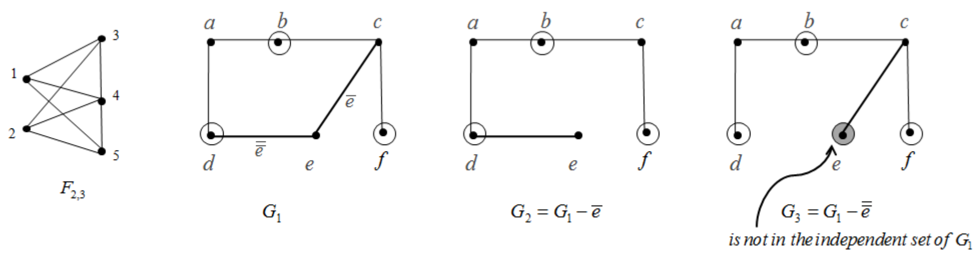
If G has odd cycle, for each H_{ij}, vertex $(u_i, v) \in S_i$ and $(u_j, v) \in S_j$ have $d_{H_{ij}}(u_i, v) = d_{H_{ij}}(u_j, v) = d_G(v)$.

Let $\bigcup_{i=1}^{m} \overline{H_i} \cup \bigcup_{i=m+1}^{m+n-1} \overline{M_i} = F_{m,n} \otimes (G - \overline{e}); i = 1, 2, ..., m$ when $\overline{e}$ is an edge in odd cycle, I be the maximum independent set of G. We get $\overline{H_{ij}} = \overline{H_{i(i+1)}} = 2(G - \overline{e})$ then

$$\alpha(\overline{H_{ij}}) = \alpha(\overline{H_{i(i+1)}}) = 2(G - \overline{e})$$
$$= \begin{cases} 2[\alpha(G) + 1], & \text{if } \overline{e} = xy \text{ then } x \in I,\ y \notin I \\ & \text{and is not adjacent with } z \in I \\ 2\alpha(G), & \text{otherwise.} \end{cases}$$

When we add $\overline{e}$ comeback, in the case $\alpha(G - \overline{e}) = \alpha(G) + 1$ be not impossible because the end vertices of edge $\overline{e}$ are in independent set of $G - \overline{e}$, so $\alpha(H_{ij}) = \alpha(\overline{H_{ij}}) - 1$. In the same, we get $\alpha(H_{i(i+1)}) = \alpha(\overline{H_{i(i+1)}}) - 1$.

Hence $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) = 2\alpha(G)$. □



Next, we establish theorem 2.3 for a maximum independent number of $F_{m,n} \otimes G$.

Lemma 2.3. Let G be connected graph order p, then $\alpha(F_{m,n} \otimes G) = \max\{(m+n)\alpha(G), p\max\{m, \lceil \frac{n}{2} \rceil\}\}$.

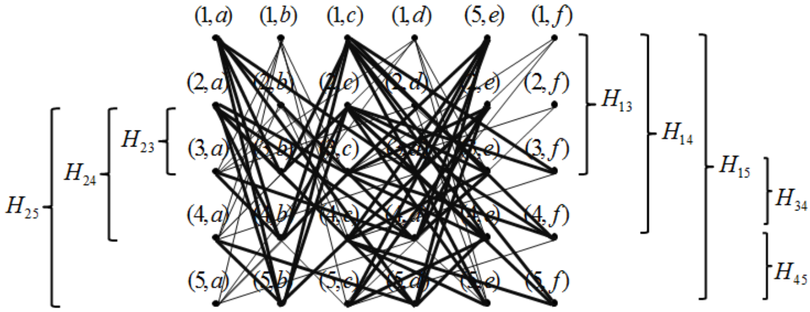


Figure 2: The graph of $F_{2,3} \otimes G_1$

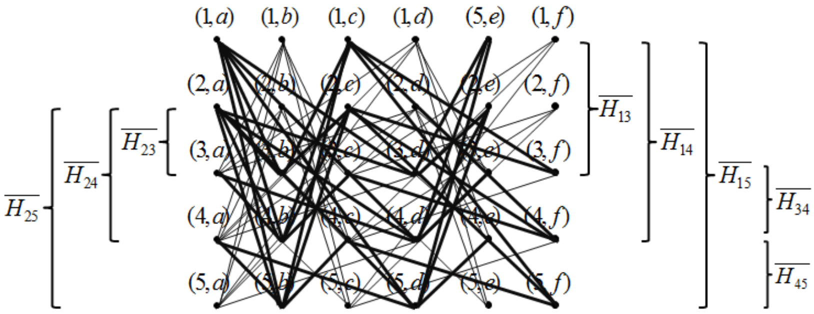


Figure 3: The graph of $F_{2,3} \otimes G_2$

Proof. Let $V(F_{m,n}) = \{u_i/i = 1, 2, ..., m + n\}$, $V(G) = \{v_j/j = 1, 2, ..., p\}$, $S_i = \{(u_i, v_j) \in V(F_{m,n})/j = 1, 2, ..., p\}, i = 1, 2, ..., m + n$ and since $\alpha(F_{m,n}) = max\{m, \lceil \frac{n}{2} \rceil\}$. Assume that the maximum independent set of $F_{m,n}, G$ be

$$I_1 = \begin{cases} \{u_1, u_2, ..., u_m\}, & m > \lceil \frac{n}{2} \rceil \\ \{u_{m+1}, u_{m+3}, ..., u_{m+2\lceil \frac{n}{2} \rceil - 1}\}, & m < \lceil \frac{n}{2} \rceil; \quad I_2, respectively. \end{cases}$$

For H_1, by lemma 2.2 we have $\alpha(H_{1j}) = 2\alpha(G)$, $j = m+1, m+2, ..., m+n$. Since every H_{1j} have $\alpha(G)$ common vertices in their independent set which is S_1. So the independent set of H_1 be in $S_1 \cup \bigcup_{j=m+1}^{m+n} S_j$.

Similarly, for the independent set of $H_2, H_3, ..., H_m$ have $\alpha(G)$ common vertices in their independent set which is in $S_2, S_3, ..., S_m$, respectively.So the independent set of H_i be in $S_i \cup \bigcup_{j=m+1}^{m+n} S_j$; $i = 2, 3, ..., m$.

For M_i ; $i = m+1, m+2, ..., m+n-1$; clearly that the independent set of M_i are subset of $\bigcup_{j=m+1}^{m+n} S_j$. Hence $\alpha(F_{m,n} \otimes G) \geq max\{(m+n)\alpha(G)\}$.

In the other hand, we get a independent set of $F_{m,n} \otimes G$ be $\bigcup_{i=1}^{m} S_i$ or $\bigcup_{j=m+1}^{m+n} S_j$, then $\alpha(F_{m,n} \otimes G) \geq \max\{p \max\{m, \lceil \frac{n}{2} \rceil\}\}$.

Hence $\alpha(F_{m,n} \otimes G) \geq \max\{(m+n)\alpha(G), p \max\{m, \lceil \frac{n}{2} \rceil\}\}$.

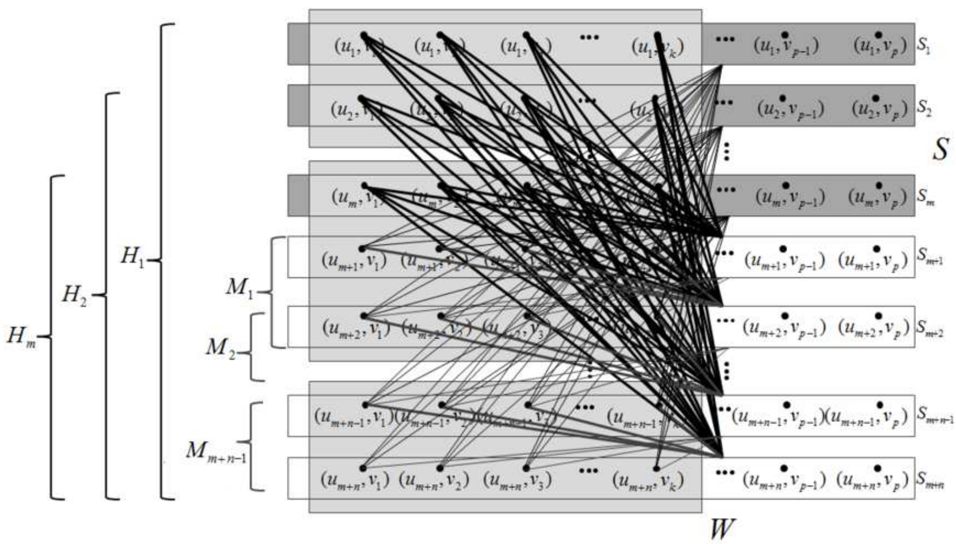


Figure 4: The region of W, S when n is odd

Suppose that $\alpha(F_{m,n} \otimes G) > \max\{(m+n)\alpha(G), p \max\{m, \lceil \frac{n}{2} \rceil\}\}$, then there exists $uv_j \in V(F_{m,n} \otimes G) - W$; $j = k+1, k+2, ..., m$; $u \notin I_1$ which is not adjacent with another vertices in $W, W = \{u_i v / u_i \in I_1\}$. It not true.

Hence $\alpha(F_{m,n} \otimes G) = \max\{(m+n)\alpha(G), p \max\{m, \lceil \frac{n}{2} \rceil\}\}$. □

3. Vertex Covering Number of the Graph of $F_{m,n} \otimes G$

We begin this section by giving Lemma 3.1 that shows a relation of independent number and vertex covering number.

Lemma 3.1. (see [2]) *Let G be a simple graph with order n. Then $\alpha(G) + \beta(G) = n$.*

Next, we establish theorem 3.2 for a minimum vertex covering number of $F_{m,n} \otimes G$.

Theorem 3.2. *Let G be connected graph order p, then $\beta(F_{m,n} \otimes G) = \{\min(m+n)\beta(G), p\min\{m + \lfloor \frac{n}{2} \rfloor, n\}\}$.*

Proof. By theorem 2.3 and lemma 3.1, we can also show that

$$\begin{aligned}\alpha(F_{m,n} \otimes G) + \beta(F_{m,n} \otimes G) &= (m+n)p \\ \beta(F_{m,n} \otimes G) &= (m+n)p - \max\{(m+n)\alpha(G), p\max\{m, \lceil \frac{n}{2} \rceil\}\} \\ &= (m+n)p \\ &\quad + \min\{-(m+n)\alpha(G), -p\max\{m, \lceil \frac{n}{2} \rceil\}\} \\ &= \min\{(m+n)(p-\alpha(G)), p\min\{m + \lfloor \frac{n}{2} \rfloor, n\}\} \\ &= \min\{(m+n)\beta(G)), p\min\{m + \lfloor \frac{n}{2} \rfloor, n\}\}. \qquad \square\end{aligned}$$

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