INDEPENDENT AND VERTEX COVERING NUMBER ON TENSOR PRODUCT OF FAN GRAPH

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Abstract: Let α(G) and β(G) be the independent number and vertex covering number of G, respectively. The Tensor product G₁ ⊗ G₂ of graph of G₁ and G₂ has vertex set V(G₁ ⊗ G₂) = V(G₁) × V(G₂) and edge set E(G₁ ⊗ G₂) = {(u₁v₁)(u₂v₂) | u₁u₂ ∈ E(G₁) and v₁v₂ ∈ E(G₂)}. In this paper, let G is a simple graph with order p, we prove that, α(Fₘ,ₙ ⊗ G) = max{(m + n)α(G), p max{m, ⌈n²/2⌉}} and β(Fₘ,ₙ ⊗ G) = min{(m + n)β(G), p min{m + ⌊n²/2⌋, n}}.

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let G₁ and G₂ be graphs. The Tensor product of graph G₁ and G₂, denote by G₁ ⊗ G₂, is the graph with V(G₁ ⊗ G₂) = V(G₁) × V(G₂) and E(G₁ ⊗ G₂) = {(u₁v₁)(u₂v₂) | u₁u₂ ∈ E(G₁) and v₁v₂ ∈ E(G₂)}.
Next, we give the definitions about some graph parameters. A subset $U$ of the vertex set $V(G)$ of $G$ is said to be an independent set of $G$ if the induced subgraph $G[U]$ is an empty graph. An independent set of $G$ with maximum number of vertices is called a maximum independent set of $G$. The number of vertices of maximum independent set of $G$ is called the independent number of $G$, denoted by $\alpha(G)$.

A vertex of graph $G$ is said to cover the edges incident with it, and a vertex cover of a graph $G$ is a set of vertices covering all the edges of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the vertex covering number of $G$, denoted by $\beta(G)$.

By [4], we get $\alpha(F_{m,n}) = \max\{m, \lceil \frac{n}{2} \rceil \}$ and $\beta(F_{m,n}) = \min\{m + \lfloor \frac{n}{2} \rfloor, n\}$.

**Proposition 1.1.** Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then:

(i) $|V(H)| = |V(G_1)||V(G_2)|$

(ii) $|E(H)| = 2|E(G_1)||E(G_2)|$

(iii) for every $(u, v) \in V(H)$, $d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

**Theorem 1.2.** Let $G_1$ and $G_2$ be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if $G_1$ or $G_2$ contains an odd cycle.

**Theorem 1.3.** Let $G_1$ and $G_2$ be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.

Next we get that general form of graph of Tensor product of $F_{m,n}$ and a simple graph.

**Proposition 1.4.** Let $G$ be a connected graph of order $p$, the graph of $F_{m,n} \otimes G$ is

$$
\bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i; \ H_i = \bigcup_{j=m+1}^{m+n} H_{ij} \text{ and } M_i = H_{i(i+1)}
$$

where

$$
V(H_{ij}) = S_i \cup S_j, \text{ } S_i = \{(i,1), (i,2), \ldots, (i,p)\}
$$

and $E(H_{ij}) = \{(i,u)(j,v) | uv \in E(G)\}$.

Moreover, if $G$ has no odd cycle then each $H_{ij}$ has exactly two connected components isomorphic to $G$.

**Example**
We now state proposition and prove lemma before stating our main results. We begin this section by giving the proposition 2.1 show character of independent set and the lemma 2.2 show character of independent set for each $H_{ij}$.

**Proposition 2.1.** Let $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph $G$ if:

(i) $v_i$ is not adjacent with $v_j$ for all $i \neq j$ and $i, j = 1, 2, ..., k$.

and

(ii) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.
Lemma 2.2. Let $F_{m,n} \otimes G = \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i$; $H_i = \bigcup_{j=m+1}^{m+n} H_{ij}$ and $M_i = H_{i(i+1)}$, then $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) = 2\alpha(G)$.

Proof. Suppose $G$ has no odd cycle, by proposition 1.4, we get $H_{ij} = H_{i(i+1)} = 2G$. So $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) = 2\alpha(G)$.

If $G$ has odd cycle, for each $H_{ij}$, vertex $(u_i, v) \in S_i$ and $(u_j, v) \in S_j$ have $d_{H_{ij}}(u_i, v) = d_{H_{ij}}(u_j, v) = d_G(v)$.

Let $\bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i = F_{m,n} \otimes (G - \overline{\tau})$; $i = 1, 2, ..., m$ when $\overline{\tau}$ is an edge in odd cycle, $I$ be the maximum independent set of $G$. We get $\overline{H_{ij}} = \overline{H_{i(i+1)}} = 2(G - \tau)$. Then

$$\alpha(\overline{H_{ij}}) = \alpha(\overline{H_{i(i+1)}}) = 2(G - \tau)$$

$$= \begin{cases} 
2[\alpha(G) + 1], & \text{if } \tau = xy \text{ then } x \in I, y \notin I \\
\text{and is not adjacent with } z \in I & \text{otherwise.}
\end{cases}$$

When we add $\tau$ comeback, in the case $\alpha(G - \tau) = \alpha(G) + 1$ be not impossible because the end vertices of edge $\tau$ are in independent set of $G - \tau$, so $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) - 1$. In the same, we get $\alpha(H_{i(i+1)}) = \alpha(H_{i(i+1)}) - 1$.

Hence $\alpha(H_{ij}) = \alpha(H_{i(i+1)}) = 2\alpha(G)$. \qed

Next, we establish theorem 2.3 for a maximum independent number of $F_{m,n} \otimes G$.

Lemma 2.3. Let $G$ be connected graph order $p$, then $\alpha(F_{m,n} \otimes G) = \max\{(m + n)\alpha(G), p \max\{m, \lceil \frac{n}{2} \rceil\}\}$. 
Proof. Let $V(F_{m,n}) = \{u_i/i = 1, 2, ..., m+n\}$, $V(G) = \{v_j/j = 1, 2, ..., p\}$, 
$S_i = \{(u_i, v_j) \in V(F_{m,n})/j = 1, 2, ..., p\}, i = 1, 2, ..., m + n$ and since $\alpha(F_{m,n}) = \max\{m,\left\lceil \frac{n}{2} \right\rceil\}$. Assume that the maximum independent set of $F_{m,n}, G$ be

$$I_1 = \begin{cases} \{u_1, u_2, ..., u_m\}, & m > \left\lceil \frac{n}{2} \right\rceil \\ \{u_{m+1}, u_{m+3}, ..., u_{m+2\left\lceil \frac{n}{2} \right\rceil-1}\}, & m < \left\lceil \frac{n}{2} \right\rceil \end{cases} \quad I_2, \text{respectively.}$$

For $H_1$, by lemma 2.2 we have $\alpha(H_{1j}) = 2\alpha(G)$, $j = m+1, m+2, ..., m+n$. Since every $H_{1j}$ have $\alpha(G)$ common vertices in their independent set which is $S_1$. So the independent set of $H_1$ be in $S_1 \cup \bigcup_{j=m+1}^{m+n} S_j$.

Similarly, for the independent set of $H_2, H_3, ..., H_m$ have $\alpha(G)$ common vertices in their independent set which is in $S_2, S_3, ..., S_m$, respectively. So the
independent set of \( H_i \) be in \( S_i \cup \bigcup_{j=m+1}^{m+n} S_j \); \( i = 2, 3, \ldots, m \).

For \( M_i ; i = m + 1, m + 2, \ldots, m + n - 1 \); clearly that the independent set of \( M_i \) are subset of \( \bigcup_{j=m+1}^{m+n} S_j \). Hence \( \alpha(F_{m,n} \otimes G) \geq \max\{(m+n)\alpha(G)\} \).

In the other hand, we get a independent set of \( F_{m,n} \otimes G \) be \( \bigcup_{i=1}^{m} S_i \) or \( \bigcup_{j=m+1}^{m+n} S_j \), then \( \alpha(F_{m,n} \otimes G) \geq \max\{p \max\{m, \lceil n/2 \rceil\}\} \).

Hence \( \alpha(F_{m,n} \otimes G) \geq \max\{(m+n)\alpha(G), p \max\{m, \lceil n/2 \rceil\}\} \).

Figure 4: The region of \( W, S \) when \( n \) is odd

Suppose that \( \alpha(F_{m,n} \otimes G) > \max\{(m+n)\alpha(G), p \max\{m, \lceil n/2 \rceil\}\} \), then there exists \( uv_j \in V(F_{m,n} \otimes G) - W \); \( j = k + 1, k + 2, \ldots, m \); \( u \not\in I_1 \) which is not adjacent with another vertices in \( W, W = \{u_i v/ u_i \in I_1\} \). It not true.

Hence \( \alpha(F_{m,n} \otimes G) = \max\{(m+n)\alpha(G), p \max\{m, \lceil n/2 \rceil\}\} \).
3. Vertex Covering Number of the Graph of $F_{m,n} \otimes G$

We begin this section by giving Lemma 3.1 that shows a relation of independent number and vertex covering number.

**Lemma 3.1.** (see [2]) Let $G$ be a simple graph with order $n$. Then $\alpha(G) + \beta(G) = n$.

Next, we establish theorem 3.2 for a minimum vertex covering number of $F_{m,n} \otimes G$.

**Theorem 3.2.** Let $G$ be connected graph order $p$, then $\beta(F_{m,n} \otimes G) = \{\min(m + n)\beta(G), p \min \{m + \lceil \frac{n}{2} \rceil, n\}\}$.

**Proof.** By theorem 2.3 and lemma 3.1, we can also show that

$$\begin{align*}
\alpha(F_{m,n} \otimes G) + \beta(F_{m,n} \otimes G) & = (m + n)p \\
\beta(F_{m,n} \otimes G) & = (m + n)p - \max\{(m + n)\alpha(G), p \max \{m, \lceil \frac{n}{2} \rceil\}\} \\
& = (m + n)p \\
& \quad + \min\{-(m + n)\alpha(G), -p \max \{m, \lceil \frac{n}{2} \rceil\}\} \\
& = \min\{(m + n)(p - \alpha(G)), p \min \{m + \lceil \frac{n}{2} \rceil, n\}\} \\
& = \min\{(m + n)\beta(G), p \min \{m + \lceil \frac{n}{2} \rceil, n\}\}. \quad \square
\end{align*}$$

**References**


