SOLVABILITY CONDITIONS OF THE NEUMANN BOUNDARY VALUE PROBLEM FOR THE BIHARMONIC EQUATION IN THE UNIT BALL

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Abstract: The necessary and sufficient solvability condition of the Neumann boundary value problem for the inhomogeneous biharmonic equation in the unit ball is obtained. Solution with polynomial boundary data is constructed.

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1. Introduction

Consider the Neumann boundary value problem (see, for example, [1]) for the nonhomogeneous biharmonic equation in the unit ball

\[
\Delta^2 u(x) = f(x), \ x \in \Omega; \quad (1)
\]

\[
\frac{\partial u}{\partial \nu}_{|\partial \Omega} = \varphi_1(s), \quad \frac{\partial^2 u}{\partial \nu^2_{|\partial \Omega}} = \varphi_2(s), \ s \in \partial \Omega, \quad (2)
\]

where \(\Omega = \{x \in \mathbb{R}^n : |x| < 1\}\), \(\nu\) is the unit outer normal to \(\partial \Omega\), boundary functions in (2) and right hand side of (1) have the following smoothness \(f \in \)
$C(\Omega)$, $\varphi_1(s) \in C^1(\partial \Omega)$ and $\varphi_2(s) \in C^2(\partial \Omega)$. In the paper [2] by B.E. Kanguzhin and B.D. Koshanov, in particular, a solvability condition for the Neumann boundary value problem (1)-(2) has been obtained. This condition

$$\int_{|x|=1} (\varphi_1(s) - \varphi_2(s) + \int_{|y|<1} d_{4,n}(4-n) \times [(2-n)|x-y|^{-n}(1-(x,y)) + |x-y|^{2-n}(x,y)] f(y) dy) ds_x = 0,$$

where $d_{4,n} = (n/2 - 2)/(216\pi n^2)$ is hard to verify even for very simple functions $f(x)$, $\varphi_1(s)$ and $\varphi_2(s)$. In present paper we give the easily to verify necessary and sufficient solvability condition (3) for the problem (1)-(2). In the paper [3] solvability conditions of the boundary value problem for the Poisson equation $\Delta u(x) = f(x)$ with higher order polynomials on normal derivatives in the boundary condition $P_n(\partial/\partial \nu)u|_{\partial \Omega} = \varphi(s)$ are investigated. Similar problems for the Poisson equation with boundary operators of fraction order were investigated in [4]. Application of the Almansi formula for constructing polynomial solutions to the Dirichlet problem for a second-order equation is considered in [5]. We use the methods and results of the papers [3] and [6].

2. Main Results

Let $V[f](x)$ be the volume potential with the density $f(x)$ i.e.,

$$V[f](x) = -\frac{1}{\omega_n} \int_{\Omega} E(x, \xi) f(\xi) d\xi,$$

where $E(x, \xi) = (n-2)^{-1}|\xi - x|^{2-n}$ $(n > 2)$ is an elementary solution of the Laplace’s equation [7], and $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

**Theorem 1.** Solution of the Neumann boundary value problem (1)-(2) with $f \in C^1(\Omega)$, $\varphi_1 \in C^1(\partial \Omega)$ and $\varphi_2 \in C^2(\partial \Omega)$ exists if and only if the following equality holds true

$$\int_{\partial \Omega} (\varphi_1(s) - \varphi_2(s)) ds = \int_{\Omega} \frac{|x|^2 - 1}{2} f(x) dx. \quad (3)$$

**Proof.** Represent a solution of the Neumann boundary value problem (1)-(2) in the form $u(x) = V[V[f]] + w(x)$. It is known that $\Delta^2 V[V[f]] = f$ [7]. Then

$$\Delta^2 u(x) - f = \Delta^2 w(x)$$
and therefore for \( w(x) \) we obtain the following boundary value problem

\[
\Delta^2 w(x) = 0, \ x \in \Omega; \tag{4}
\]

\[
\frac{\partial w}{\partial \nu} \big|_{\partial \Omega} = \varphi_1(s), \quad \frac{\partial^2 u}{\partial \nu^2} \big|_{\partial \Omega} = \varphi_2(s), \ s \in \partial \Omega, \tag{5}
\]

where

\[
\varphi_1(s) = \varphi_1(s) - \frac{\partial}{\partial \nu} V[V[f]], \quad \varphi_2(s) = \varphi_2(s) - \frac{\partial^2}{\partial \nu^2} V[V[f]].
\]

From the results of the paper [6] it follows that the necessary and sufficient condition of solvability of the problem (4)-(5), and therefore of the problem (1)-(2) has the form

\[
\int_{\partial \Omega} (\varphi_1(s) - \varphi_2(s)) \, ds = 0.
\]

Hence we can write this condition in terms of functions \( f(x), \varphi_1(s) \) and \( \varphi_2(s) \):

\[
\int_{\partial \Omega} (\varphi_1(s) - \varphi_2(s)) \, ds = \int_{\partial \Omega} \left( \frac{\partial}{\partial \nu} V[V[f]] - \frac{\partial^2}{\partial \nu^2} V[V[f]] \right) \, ds.
\]

Denote for simplicity \( f_1(x) = V[f](x) \). Normal derivative on \( \partial \Omega \) we can write in terms of homogeneous operator \( \Lambda = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \) in the form \( \partial^k / \partial \nu^k (f) = \Lambda^{[k]}(f) \) [8], where \( t^{[k]} = t(t-1)\ldots(t-k+1) \) is a factorial power of \( t \). Thus the solvability condition of the problem (1)-(2) can be written in the form

\[
\int_{\partial \Omega} (\varphi_1(s) - \varphi_2(s)) \, ds = \int_{\partial \Omega} \left( \Lambda V[f_1] - \Lambda^{[2]} V[f_1] \right) \, ds = \int_{\partial \Omega} \Lambda (2 - \Lambda) V[f_1] \, ds. \tag{6}
\]

In the paper [3] the following statement was proved. Let \( Q(t) = \sum_{i=0}^m a_i t^i \) be a polynomial, then denote

\[
Q^{(\lambda)}(t) = \sum_{i=0}^{m-1} a_{i+1} (t - \lambda)^i - \frac{a_0}{2\lambda + n - 2}.
\]

**Theorem 2.** If \( f(x) \) is a polynomial or \( f \in C_0^{k-1}(\Omega), \lambda \in N_0 = N \cup \{0\} \) and \( H_\lambda(x) \) is a homogeneous harmonic polynomial of degree \( \lambda \), then the following equality holds

\[
\int_{\partial \Omega} H_\lambda(s) Q(\Lambda) V[f](s) \, ds = \int_{\Omega} H_\lambda(x) Q^{(\lambda)}(\Lambda + 2) f(x) \, dx. \tag{7}
\]
Let \( P_n(x) \) be a consequence of polynomials uniformly converges on \( \bar{\Omega} \) together with all derivatives up to \((m-1)\)th degree to function \( f(x) \) i.e., \( \forall k \leq m-1, P_n^{(k)}(x) \Rightarrow f^{(k)}(x), \ n \to \infty. \) Then by virtue of properties of potentials \( \forall k \leq m, V^{(k)}[P_n](x) \Rightarrow V^{(k)}[f](x). \) Passing to limit in the equality (7) (it holds true for polynomials \( P_n(x) \)) when \( n \to \infty \) we obtain it correctness for \( f \in C^{k-1}(\bar{\Omega}). \) We apply this equality to the integral from the right hand side of (6), with \( f_1 \in C^1(\bar{\Omega}) \) since \( f \in C^1(\bar{\Omega}). \) Besides \( Q(t) = 2t-t^2, H_\lambda(x) = 1 \) and \( \lambda = 0 \) is a root of the polynomial \( Q(t). \) Therefore \( Q^{(0)}(t) = (2t-t^2)/t = 2 - t. \) Then

\[
\int_{\partial \Omega} \Lambda(2 - \Lambda) V[f_1] \, ds = - \int_{\Omega} \Lambda f_1(x) \, dx
\]

\[
= \frac{1}{\omega_n} \int_{|x|<1} \left( \Lambda_x \int_{|\xi|<1} E(x, \xi) f(\xi) \, d\xi \right) \, dx
\]

\[
= \frac{1}{\omega_n} \int_{|\xi|<1} f(\xi) \left( \int_{|x|<1} \Lambda_x E(x, \xi) \, dx \right) \, d\xi.
\]

Making use of the following representation from [3]

\[
E(x, \xi) = \sum_{m=0}^{\infty} \frac{|\xi|-2m+n-2}{2m+n-2} \sum_{i=1}^{h_m} H_m^{(i)}(x) H_m^{(i)}(\xi), \ |x| < |\xi|,
\]

where \( \{H_m^{(i)}(x) : i = 1, 2, \ldots, h_m\} \) is the full system of orthogonal harmonic polynomials of degree \( m \) normalized so that \( \int_{\partial \Omega} (H_m^{(i)}(\xi))^2 \, d\xi = \omega_n \) we can write

\[
\int_{|x|<1} \Lambda_x E(x, \xi) \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \left( \int_{|x|<|\xi|-\varepsilon} + \int_{|\xi|-\varepsilon<|x|<|\xi|+\varepsilon} + \int_{|\xi|+\varepsilon<|x|<1} \right) \Lambda_x E(x, \xi) \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{|x|<|\xi|} \Lambda_x \sum_{m=0}^{\infty} \frac{|\xi|-2m+n-2}{2m+n-2} \sum_{i=1}^{h_m} H_m^{(i)}(x) H_m^{(i)}(\xi) \, dx
\]

\[
+ \lim_{\varepsilon \to 0^+} \int_{|\xi|+\varepsilon<|x|<1} \Lambda_x \sum_{m=0}^{\infty} \frac{|x|-2m+n-2}{2m+n-2} \sum_{i=1}^{h_m} H_m^{(i)}(x) H_m^{(i)}(\xi) \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \sum_{m=0}^{\infty} \frac{m}{2m+n-2} \sum_{i=1}^{h_m} \int_{|x|<|\xi|-\varepsilon} H_m^{(i)}(x) \, dx H_m^{(i)}(\xi) |\xi|-2m+n-2
\]
\[ + \lim_{\varepsilon \to +0} \sum_{m=0}^{\infty} \frac{-(2m + n - 2) + m}{2m + n - 2} \]
\[ \times \sum_{i=1}^{h_m} \int_{|\xi|+\varepsilon<|x|<1} H_{m}^{(i)}(x)|x|^{-(2m+n-2)} \, dx H_{m}^{(i)}(\xi). \]

We were able to interchange the summation and integration because the functional series in the integration domain are uniformly converged. Because of orthogonality of the harmonic polynomials \( \{H_{m}^{(i)}(x) : i = 1, 2, \ldots, h_m\} \) on the sphere \( \partial \Omega \) in each sum it remains only one term with \( m = 0 \). The first sum at \( m = 0 \) gives zero but the second sum at \( m = 0 \) gives the item
\[- \int_{|\xi|<|x|<1} |x|^{-n+2} \, dx. \]

Thus we have
\[ \int_{|x|<1} \Lambda E(x, \xi) \, dx = - \int_{|\xi|<|x|<1} |x|^{-n+2} \, dx \]
\[ = - \int_{|\xi|}^{1} r^{n-1} \, dr \int_{|x|=1} r^{-n+2} \, ds = - \int_{|\xi|}^{1} r \omega_n \, dr = \frac{\omega_n}{2} (|\xi|^2 - 1). \]

Hence the solvability conditions of the boundary value problem (1)-(2) has the form
\[ \int_{\partial \Omega} (\varphi_1(s) - \varphi_2(s)) \, ds \]
\[ = - \int_{\Omega} \Lambda f_1(x) \, dx = \frac{1}{\omega_n} \int_{|\xi|<1} f(\xi) \left( \int_{|x|<1} \Lambda E(x, \xi) \, dx \right) \, d\xi \]
\[ = \frac{1}{\omega_n} \frac{\omega_n}{2} \int_{|\xi|<1} (|\xi|^2 - 1) f(\xi) \, d\xi \]
\[ = \frac{1}{2} \int_{\Omega} (|\xi|^2 - 1) f(\xi) \, d\xi. \]

Proof is complete. \( \square \)

### 3. Examples

The necessary and sufficient solvability condition (3) of the Neumann boundary value problem (1)-(2) obtained in Theorem 1 is easy to verify. For example, for \( f(x) = 1 \) we have
\[ \frac{1}{2} \int_{|x|<1} (|x|^2 - 1) \, d\xi = \frac{\omega_n}{2} \left( \frac{1}{n+2} - \frac{1}{n} \right) = - \frac{\omega_n}{n(n+2)} \]
and therefore the solvability condition (3) of the Neumann boundary value problem with \( f(x) = 1 \) has the form

\[
\frac{1}{\omega_n} \int_{\partial \Omega} (\varphi_1(s) - \varphi_2(s)) \, ds = -\frac{1}{n(n+2)}.
\] (8)

For example, the following Neumann boundary value problem

\[
\Delta^2 u(x) = 1, \ x \in \Omega;
\] (9)

\[
\frac{\partial u}{\partial \nu|_{\partial \Omega}} = \frac{1}{2n+4}, \quad \frac{\partial^2 u}{\partial \nu^2|_{\partial \Omega}} = \frac{1}{2n},
\] (10)

is solvable because the condition (8) is fulfilled. A solution, as it is not difficult to verify, has the form

\[
u(x) = \frac{|x|^4}{8n(n+2)} + \frac{(n-1)|x|^2}{4n(n+2)} + C.
\] (11)

Indeed, \( \Delta^2 u = 1 \) and

\[
\frac{\partial u}{\partial \nu|_{|x|=1}} = \frac{1}{2n(n+2)} + \frac{n-1}{2n(n+2)} = \frac{1}{2(n+2)},
\]

\[
\frac{\partial^2 u}{\partial \nu^2|_{|x|=1}} = \frac{3 + n - 1}{2n(n+2)} = \frac{1}{2n}.
\]

Another Neumann boundary value problem

\[
\Delta^2 u(x) = 1, \ x \in \Omega;
\]

\[
\frac{\partial u}{\partial \nu|_{\partial \Omega}} = 0, \quad \frac{\partial^2 u}{\partial \nu^2|_{\partial \Omega}} = 0
\]

is not solvable because it do not satisfy to condition (8). Indeed, let this problem is solvable and \( u(x) \) is its solution. Then according to the equality \( \Delta \Lambda u = (\Lambda + 2)\Delta u \) (formula (50) from [8]) the function \( v = \Lambda u \) is a solution of the following Dirichlet boundary value problem

\[
\Delta^2 v(x) = 4, \ x \in \Omega;
\]

\[
v|_{\partial \Omega} = 0, \quad \Lambda v|_{\partial \Omega} = 0.
\]

Solution of this problem (see [9], formula (17)) is a polynomial

\[
v(x) = \frac{(|x|^2 - 1)^2}{2n(n+2)}.
\]
However, it is impossible to find a biharmonic function \( u(x) \) from the equation

\[
\Lambda u = \frac{1}{2n(n+2)}(|x|^4 - 2|x|^2 + 1), \quad x \in \Omega
\]

because analytic in \( \Omega \) function \( \Lambda u \) do not have a term of the form \( cx^0 \), but the polynomial on the right hand side of this equation has such a term: \( 1/(2n(n+2)) \). Therefore, the original Neumann boundary value problem is not solvable.

4. Polynomial Solutions

Let the functions \( f(x), \varphi_1(s) \) and \( \varphi_2(s) \) from the Neumann boundary value problem (1)-(2) be polynomials. Acting by the operator \( \Lambda+4 \) on the biharmonic operator we obtain:

\[(\Lambda + 4)\Delta^2 u = \Delta^2 \Lambda u = \Delta^2 v, \quad v = \Lambda u.\]

Therefore the problem (1)-(2) can be reduced to the Dirichlet problem

\[
\Delta^2 v(x) = (\Lambda + 4)f(x), \quad x \in \Omega; \\
v|_{\partial \Omega} = \varphi_1(s), \quad \frac{\partial v}{\partial n}|_{\partial \Omega} = \varphi_1(s) + \varphi_2(s), \quad s \in \partial \Omega,
\]

with polynomial data. Using Theorem 6 from [9] we can write a polynomial solution \( v(x) \) of this problem in the form

\[
v(x) = \varphi_1(x) + \frac{|x|^2 - 1}{2} (\varphi_1(x) + \varphi_2(x) - \Lambda \varphi_1(x)) \\
\quad + \frac{(|x|^2 - 1)^2}{4} \int_0^1 \sum_{s=0}^\infty \frac{(1 - \alpha|x|^2)^s(1 - \alpha)^s}{(2s)!!(2s + 2)!!} \Delta^s \left( \Delta(\Lambda \varphi_1 - \varphi_1 - \varphi_2) + \frac{1 - \alpha}{2s + 4} ((\Lambda + 4)f - \Delta^2 \varphi_1) \right) (\alpha x) \alpha^{n/2 - 1} d\alpha.
\]

Then we need to solve the equation \( v = \Lambda u \) in \( \bar{\Omega} \). Obviously, the condition \( v(0) = 0 \) must be fulfilled. Our polynomial solution has the form

\[
u(x) = \varphi_1(x) - \frac{1}{2n+4} + \frac{|x|^2 - 1}{2} \left( \frac{1}{2n+4} + \frac{1}{2n} \right) + \frac{(|x|^2 - 1)^2}{8}
\]
\[ \times \int_0^1 (1 - \alpha)\alpha^{n/2-1} \, d\alpha = \frac{1}{2(n+2)} + (|x|^2 - 1)\frac{n+1}{2n(n+2)} \]
\[ + \frac{(|x|^2 - 1)^2}{2n(n+2)} = \frac{1}{2n(n+2)}(n-1)|x|^2 + |x|^4). \]

Using (15) we obtain
\[ u(x) = \frac{1}{2n(n+2)}\left(\frac{n-1}{2}|x|^2 + \frac{1}{4}|x|^4\right) + C. \]

This coincides with the solution (11).

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**References**


