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SOLVABILITY CONDITIONS OF THE NEUMANN BOUNDARY VALUE PROBLEM FOR THE BIHARMONIC EQUATION IN THE UNIT BALL

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Abstract: The necessary and sufficient solvability condition of the Neumann boundary value problem for the inhomogeneous biharmonic equation in the unit ball is obtained. Solution with polynomial boundary data is constructed.

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1. Introduction

Consider the Neumann boundary value problem (see, for example, [1]) for the nonhomogeneous biharmonic equation in the unit ball

$$\Delta^2 u(x) = f(x),\ x \in \Omega; \tag{1}$$

$$\frac{\partial u}{\partial \nu}_{|\partial\Omega} = \varphi_1(s), \quad \frac{\partial^2 u}{\partial \nu^2}_{|\partial\Omega} = \varphi_2(s),\ s \in \partial\Omega, \tag{2}$$

where $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, ν is the unit outer normal to $\partial\Omega$, boundary functions in (2) and right hand side of (1) have the following smoothness $f \in$

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$C(\Omega)$, $\varphi_1(s) \in C^1(\partial\Omega)$ and $\varphi_2(s) \in C^2(\partial\Omega)$. In the paper [2] by B.E. Kanguzhin and B.D. Koshanov, in particular, a solvability condition for the Neumann boundary value problem (1)-(2) has been obtained. This condition

$$\int_{|x|=1} \Big(\varphi_1(s) - \varphi_2(s) + \int_{|y|<1} d_{4,n}(4-n)$$
$$\times[(2-n)|x-y|^{-n}(1-(x,y))^2 + |x-y|^{2-n}(x,y)]f(y)\,dy\Big)\,ds_x = 0,$$

where $d_{4,n} = (n/2-2)/((2)16\pi^{n/2})$ is hard to verify even for very simple functions $f(x)$, $\varphi_1(s)$ and $\varphi_2(s)$. In present paper we give the easily to verify necessary and sufficient solvability condition (3) for the problem (1)-(2). In the paper [3] solvability conditions of the boundary value problem for the Poisson equation $\Delta u(x) = f(x)$ with higher order polynomials on normal derivatives in the boundary condition $P_n(\partial/\partial\nu)u_{|\partial\Omega} = \varphi(s)$ are investigated. Similar problems for the Poisson equation with boundary operators of fraction order were investigated in [4]. Application of the Almansi formula for constructing polynomial solutions to the Dirichlet problem for a second-order equation is considered in [5]. We use the methods and results of the papers [3] and [6].

2. Main Results

Let $V[f](x)$ be the volume potential with the density $f(x)$ i.e.,

$$V[f](x) = -\frac{1}{\omega_n}\int_\Omega E(x,\xi)f(\xi)\,d\xi,$$

where $E(x,\xi) = (n-2)^{-1}|\xi - x|^{2-n}$ $(n>2)$ is an elementary solution of the Laplace's equation [7], and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Theorem 1. *Solution of the Neumann boundary value problem* (1)-(2) *with $f \in C^1(\bar\Omega)$, $\varphi_1 \in C^1(\partial\Omega)$ and $\varphi_2 \in C^2(\partial\Omega)$ exists if and only if the following equality holds true*

$$\int_{\partial\Omega} (\varphi_1(s) - \varphi_2(s))\,ds = \int_\Omega \frac{|x|^2-1}{2} f(x)\,dx. \tag{3}$$

Proof. Represent a solution of the Neumann boundary value problem (1)-(2) in the form $u(x) = V[V[f]] + w(x)$. It is known that $\Delta^2 V[V[f]] = f$ [7]. Then

$$\Delta^2 u(x) - f = \Delta^2 w(x)$$

and therefore for $w(x)$ we obtain the following boundary value problem

$$\Delta^2 w(x) = 0,\ x \in \Omega; \tag{4}$$

$$\frac{\partial w}{\partial \nu}_{|\partial\Omega} = \tilde{\varphi}_1(s), \quad \frac{\partial^2 u}{\partial \nu^2}_{|\partial\Omega} = \tilde{\varphi}_2(s),\ s \in \partial\Omega, \tag{5}$$

where

$$\tilde{\varphi}_1(s) = \varphi_1(s) - \frac{\partial}{\partial \nu} V[V[f]], \quad \tilde{\varphi}_2(s) = \varphi_2(s) - \frac{\partial^2}{\partial \nu^2} V[V[f]].$$

From the results of the paper [6] it follows that the necessary and sufficient condition of solvability of the problem (4)-(5), and therefore of the problem (1)-(2) has the form

$$\int_{\partial\Omega} (\tilde{\varphi}_1(s) - \tilde{\varphi}_2(s))\, ds = 0.$$

Hence we can write this condition in terms of functions $f(x)$, $\varphi_1(s)$ and $\varphi_2(s)$:

$$\int_{\partial\Omega} (\varphi_1(s) - \varphi_2(s))\, ds = \int_{\partial\Omega} \Big(\frac{\partial}{\partial \nu} V[V[f]] - \frac{\partial^2}{\partial \nu^2} V[V[f]] \Big)\, ds.$$

Denote for simplicity $f_1(x) = V[f](x)$. Normal derivative on $\partial\Omega$ we can write in terms of homogeneous operator $\Lambda = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ in the form $\partial^k / \partial \nu^k (f) = \Lambda^{[k]}(f)$ [8], where $t^{[k]} = t(t-1)\dots(t-k+1)$ is a factorial power of t. Thus the solvability condition of the problem (1)-(2) can be written in the form

$$\int_{\partial\Omega} (\varphi_1(s) - \varphi_2(s))\, ds$$
$$= \int_{\partial\Omega} \big(\Lambda V[f_1] - \Lambda^{[2]} V[f_1] \big)\, ds = \int_{\partial\Omega} \Lambda(2-\Lambda) V[f_1]\, ds. \tag{6}$$

In the paper [3] the following statement was proved. Let $Q(t) = \sum_{i=0}^{m} a_i t^i$ be a polynomial, then denote

$$Q^{(\lambda)}(t) = \sum_{i=0}^{m-1} a_{i+1} (t-\lambda)^i - \frac{a_0}{2\lambda + n - 2}.$$

Theorem 2. *If $f(x)$ is a polynomial or $f \in C_0^{k-1}(\Omega)$, $\lambda \in \mathbb{N}_0 = N \cup \{0\}$ and $H_\lambda(x)$ is a homogeneous harmonic polynomial of degree λ, then the following equality holds*

$$\int_{\partial\Omega} H_\lambda(s) Q(\Lambda) V[f](s)\, ds = \int_{\Omega} H_\lambda(x) Q^{(\lambda)}(\Lambda + 2) f(x)\, dx. \tag{7}$$

Let $P_n(x)$ be a consequence of polynomials uniformly converges on $\bar\Omega$ together with all derivatives up to $(m-1)$th degree to function $f(x)$ i.e., $\forall\, k \le m-1,\ P_n^{(k)}(x) \underset{\bar\Omega}{\rightrightarrows} f^{(k)}(x),\ n\to\infty$. Then by virtue of properties of potentials $\forall\, k \le m,\ V^{(k)}[P_n](x) \underset{\bar\Omega}{\rightrightarrows} V^{(k)}[f](x)$. Passing to limit in the equality (7) (it holds true for polynomials $P_n(x)$) when $n\to\infty$ we obtain it correctness for $f\in C^{k-1}(\bar\Omega)$. We apply this equality to the integral from the right hand side of (6), with $f_1\in C^1(\bar\Omega)$ since $f\in C^1(\bar\Omega)$. Besides $Q(t)=2t-t^2$, $H_\lambda(x)=1$ and $\lambda=0$ is a root of the polynomial $Q(t)$. Therefore $Q^{(0)}(t)=(2t-t^2)/t=2-t$. Then

$$\begin{aligned}\int_{\partial\Omega}\Lambda(2-\Lambda)V[f_1]\,ds &= -\int_\Omega \Lambda f_1(x)\,dx\\ &= \frac{1}{\omega_n}\int_{|x|<1}\Big(\Lambda_x\int_{|\xi|<1}E(x,\xi)f(\xi)\,d\xi\Big)\,dx\\ &= \frac{1}{\omega_n}\int_{|\xi|<1}f(\xi)\Big(\int_{|x|<1}\Lambda_x E(x,\xi)\,dx\Big)\,d\xi.\end{aligned}$$

Making use of the following representation from [3]

$$E(x,\xi)=\sum_{m=0}^{\infty}\frac{|\xi|^{-(2m+n-2)}}{2m+n-2}\sum_{i=1}^{h_m}H_m^{(i)}(x)H_m^{(i)}(\xi),\quad |x|<|\xi|,$$

where $\{H_m^{(i)}(x) : i=1,2,\dots,h_m\}$ is the full system of orthogonal harmonic polynomials of degree m normalized so that $\int_{\partial\Omega}(H_m^{(i)}(\xi))^2\,d\xi=\omega_n$ we can write

$$\begin{aligned}&\int_{|x|<1}\Lambda_x E(x,\xi)\,dx\\ &=\lim_{\varepsilon\to+0}\Big(\int_{|x|<|\xi|-\varepsilon}+\int_{|\xi|-\varepsilon<|x|<|\xi|+\varepsilon}+\int_{|\xi|+\varepsilon<|x|<1}\Big)\Lambda_x E(x,\xi)\,dx\\ &=\lim_{\varepsilon\to+0}\int_{|x|<|\xi|-\varepsilon}\Lambda_x\sum_{m=0}^{\infty}\frac{|\xi|^{-(2m+n-2)}}{2m+n-2}\sum_{i=1}^{h_m}H_m^{(i)}(x)H_m^{(i)}(\xi)\,dx\\ &+\lim_{\varepsilon\to+0}\int_{|\xi|+\varepsilon<|x|<1}\Lambda_x\sum_{m=0}^{\infty}\frac{|x|^{-(2m+n-2)}}{2m+n-2}\sum_{i=1}^{h_m}H_m^{(i)}(x)H_m^{(i)}(\xi)\,dx\\ &=\lim_{\varepsilon\to+0}\sum_{m=0}^{\infty}\frac{m}{2m+n-2}\sum_{i=1}^{h_m}\int_{|x|<|\xi|-\varepsilon}H_m^{(i)}(x)\,dx H_m^{(i)}(\xi)|\xi|^{-(2m+n-2)}\end{aligned}$$

$$+\lim_{\varepsilon\to+0}\sum_{m=0}^{\infty}\frac{-(2m+n-2)+m}{2m+n-2}$$
$$\times\sum_{i=1}^{h_m}\int_{|\xi|+\varepsilon<|x|<1}H_m^{(i)}(x)|x|^{-(2m+n-2)}\,dx H_m^{(i)}(\xi).$$

We were able to interchange the summation and integration because the functional series in the integration domain are uniformly converged. Because of orthogonality of the harmonic polynomials $\{H_m^{(i)}(x)\,:\,i=1,2,\dots,h_m\}$ on the sphere $\partial\Omega$ in each sum it remains only one term with $m=0$. The first sum at $m=0$ gives zero but the second sum at $m=0$ gives the item $-\int_{|\xi|<|x|<1}|x|^{-n+2}\,dx$. Thus we have

$$\int_{|x|<1}\Lambda_x E(x,\xi)\,dx=-\int_{|\xi|<|x|<1}|x|^{-n+2}\,dx$$
$$=-\int_{|\xi|}^{1}r^{n-1}\,dr\int_{|x|=1}r^{-n+2}\,ds=-\int_{|\xi|}^{1}r\omega_n\,dr=\frac{\omega_n}{2}(|\xi|^2-1).$$

Hence the solvability conditions of the boundary value problem (1)-(2) has the form

$$\int_{\partial\Omega}(\varphi_1(s)-\varphi_2(s))\,ds$$
$$=-\int_{\Omega}\Lambda f_1(x)\,dx=\frac{1}{\omega_n}\int_{|\xi|<1}f(\xi)\Big(\int_{|x|<1}\Lambda_x E(x,\xi)\,dx\Big)\,d\xi$$
$$=\frac{1}{\omega_n}\frac{\omega_n}{2}\int_{|\xi|<1}(|\xi|^2-1)f(\xi)\,d\xi=\frac{1}{2}\int_{\Omega}(|\xi|^2-1)f(\xi)\,d\xi.$$

Proof is complete. □

3. Examples

The necessary and sufficient solvability condition (3) of the Neumann boundary value problem (1)-(2) obtained in Theorem 1 is easy to verify. For example, for $f(x)=1$ we have

$$\frac{1}{2}\int_{|x|<1}(|x|^2-1)\,d\xi=\frac{\omega_n}{2}\Big(\frac{1}{n+2}-\frac{1}{n}\Big)=-\frac{\omega_n}{n(n+2)}$$

and therefore the solvability condition (3) of the Neumann boundary value problem with $f(x)=1$ has the form

$$\frac{1}{\omega_n}\int_{\partial\Omega}(\varphi_1(s)-\varphi_2(s))\,ds=-\frac{1}{n(n+2)}. \tag{8}$$

For example, the following Neumann boundary value problem

$$\Delta^2 u(x)=1,\ x\in\Omega; \tag{9}$$

$$\frac{\partial u}{\partial\nu}_{|\partial\Omega}=\frac{1}{2n+4},\quad \frac{\partial^2 u}{\partial\nu^2}_{|\partial\Omega}=\frac{1}{2n} \tag{10}$$

is solvable because the condition (8) is fulfilled. A solution, as it is not difficult to verify, has the form

$$u(x)=\frac{|x|^4}{8n(n+2)}+\frac{(n-1)|x|^2}{4n(n+2)}+C. \tag{11}$$

Indeed, $\Delta^2 u=1$ and

$$\frac{\partial u}{\partial\nu}_{|x|=1}=\frac{1}{2n(n+2)}+\frac{n-1}{2n(n+2)}=\frac{1}{2(n+2)},$$

$$\frac{\partial^2 u}{\partial\nu^2}_{|x|=1}=\frac{3+n-1}{2n(n+2)}=\frac{1}{2n}.$$

Another Neumann boundary value problem

$$\Delta^2 u(x)=1,\ x\in\Omega;$$

$$\frac{\partial u}{\partial\nu}_{|\partial\Omega}=0,\quad \frac{\partial^2 u}{\partial\nu^2}_{|\partial\Omega}=0$$

is not solvable because it do not satisfy to condition (8). Indeed, let this problem is solvable and $u(x)$ is its solution. Then according to the equality $\Delta\Lambda u=(\Lambda+2)\Delta u$ (formula (50) from [8]) the function $v=\Lambda u$ is a solution of the following Dirichlet boundary value problem

$$\Delta^2 v(x)=4,\ x\in\Omega;$$

$$v_{|\partial\Omega}=0,\quad \Lambda v_{|\partial\Omega}=0.$$

Solution of this problem (see [9], formula (17)) is a polynomial

$$v(x)=\frac{(|x|^2-1)^2}{2n(n+2)}.$$

However, it is impossible to find a biharmonic function $u(x)$ from the equation

$$\Lambda u = \frac{1}{2n(n+2)}(|x|^4 - 2|x|^2 + 1), \quad x \in \Omega$$

because analytic in Ω function Λu do not have a term of the form cx^0, but the polynomial on the right hand side of this equation has such a term: $1/(2n(n+2))$. Therefore, the original Neumann boundary value problem is not solvable.

4. Polynomial Solutions

Let the functions $f(x)$, $\varphi_1(s)$ and $\varphi_2(s)$ from the Neumann boundary value problem (1)-(2) be polynomials. Acting by the operator $\Lambda+4$ on the biharmonic operator we obtain: $(\Lambda+4)\Delta^2 u = \Delta^2\Lambda u = \Delta^2 v$, where $v = \Lambda u$. Therefore the problem (1)-(2) can be reduced to the Dirichlet problem

$$\Delta^2 v(x) = (\Lambda+4)f(x),\ x \in \Omega; \tag{12}$$

$$v_{|\partial\Omega} = \varphi_1(s), \quad \frac{\partial v}{\partial \nu}_{|\partial\Omega} = \varphi_1(s) + \varphi_2(s),\ s \in \partial\Omega, \tag{13}$$

with polynomial data. Using Theorem 6 from [9] we can write a polynomial solution $v(x)$ of this problem in the form

$$\begin{aligned} v(x) = \varphi_1(x) &+ \frac{|x|^2-1}{2}(\varphi_1(x) + \varphi_2(x) - \Lambda\varphi_1(x)) \\ &+ \frac{(|x|^2-1)^2}{4}\int_0^1 \sum_{s=0}^{\infty} \frac{(1-\alpha|x|^2)^s(1-\alpha)^s}{(2s)!!(2s+2)!!}\Delta^s\Big(\Delta(\Lambda\varphi_1 \\ &- \varphi_1 - \varphi_2) + \frac{1-\alpha}{2s+4}((\Lambda+4)f - \Delta^2\varphi_1)\Big)(\alpha x)\,\alpha^{n/2-1}\,d\alpha. \end{aligned} \tag{14}$$

Then we need to solve the equation $v = \Lambda u$ in $\bar{\Omega}$. Obviously, the condition $v(0) = 0$ must be fulfilled. Our polynomial solution has the form

$$u(x) = \int_0^1 v(tx)\frac{dt}{t} + C. \tag{15}$$

Let us check the solution (15) on the problem (9)-(10). We have

$$v(x) = \frac{1}{2n+4} + \frac{|x|^2-1}{2}\Big(\frac{1}{2n+4} + \frac{1}{2n}\Big) + \frac{(|x|^2-1)^2}{8}$$

$$\times \int_0^1 (1-\alpha)\alpha^{n/2-1}\,d\alpha = \frac{1}{2(n+2)} + (|x|^2-1)\frac{n+1}{2n(n+2)} + \frac{(|x|^2-1)^2}{2n(n+2)} = \frac{1}{2n(n+2)}\Big((n-1)|x|^2 + |x|^4\Big).$$

Using (15) we obtain

$$u(x) = \frac{1}{2n(n+2)}\Big(\frac{n-1}{2}|x|^2 + \frac{1}{4}|x|^4\Big) + C.$$

This coincides with the solution (11).

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