WALLMAN COMPACTIFICATIONS ARE PRE-UNIFORM COMPLETIONS

Adalberto García-Máynez Cervantes\textsuperscript{1}, Rubén Santos Mancio Toledo\textsuperscript{2} $\S$

\textsuperscript{1}Instituto de Matemáticas
Universidad Nacional Autónoma de México
Area de la Investigación Científica
Circuito Exterior, Ciudad Universitaria. México D.F. C.P. 04510

\textsuperscript{2}Escuela Superior de Física y Matemáticas del Instituto Politécnico Nacional
Edificio No. 9, Unidad Profesional Adolfo López Mateos
Colonia Lindavista, México D.F. C.P. 07738

Abstract: We prove that every Wallman compactifications of a $T_1$-space $(X, \tau)$ may be viewed as the completion of certain pre-uniform space $(X, \mathcal{U})$, where $\mathcal{U}$ is a precompact pre-uniform basis compatible with $\tau$. We also study perfect extensions of pre-uniform spaces.

AMS Subject Classification: 54A05, 54A20, 54E15

Key Words: Wallman compactification, pre-uniformity basis, weakly round filter, perfect extension, mingling filters

1. Introduction

It is a well known result that each normal Wallman basis $\mathcal{B}$ of a Tychonoff space $X$ induces a $T_2$-compactification of $X$ in two equivalent ways:

\begin{itemize}
  \item[a)] via the Wallman method which topologizes the family of ultrafilters of cobasic sets or
  \item[b)] via the completion of the precompact uniformity basis of $X$ with basic sets.
\end{itemize}
If the Wallman basis is not normal, we cannot expect a $T_2$-compactification. However, the Wallman method of ultrafilters is still equivalent to the process of completing a precompact pre-uniformity basis. This is the principal result of this paper. We also give a necessary and sufficient condition on a pre-uniform space to have as a perfect pre-uniform extension, its canonical extension.

2. Wallman Bases and Pre-Uniform Spaces

Recall a Wallman basis for a topological space $(X, \tau)$ is a basis $B$ for the topology $\tau$ satisfying two properties:

i) If $B_1, B_2 \in B$, also $B_1 \cap B_2 \in B$ and $B_1 \cup B_2 \in B$.

ii) If $x \in B \in B$, there exists an element $H \in C(B)$ such that $x \in H \subseteq B$, where $C(B) = \{H \subseteq X \mid X - H \in B\}$. (See [1] and [6]).

A topological space $(X, \tau)$ has a Wallman basis if and only if $(X, \tau)$ is $R_0$, i.e., every open set $V \subseteq X$ is a union of closed sets. In this case, the topology itself is a Wallman basis.

Each Wallman basis of $B$ of a $T_1$-space $(X, \tau)$ induces a $T_1$-compactification $X(B)$ of $X$: the elements of $X(B)$ are precisely the ultrafilters in the family of cobasic sets $C(B)$. An element $\xi \in X(B)$ is a fixed ultrafilter if for some $x \in X$, $\xi = \{H \in C(B) \mid x \in H\}$. $\xi \in X(B)$ is free if $\xi$ is not fixed. Each set $A \subseteq X$ determines a subset $A^* \subseteq X(B)$, where $\xi \in A^*$ if and only if for some $H \in \xi$, we have $H \subseteq A$. We have the formulas (see [1], 3.45):

$$a) \ (A_1 \cap A_2)^* = A_1^* \cap A_2^* \text{ for every } A_1, A_2 \subseteq X;$$

$$b) \ (A_1 \cup A_2)^* = A_1^* \cup A_2^* \text{ for every } A_1, A_2 \in B \cup C(B).$$

The family $B^* = \{B^* \mid B \in B\}$ is a basis for a compact $T_1$-topology $\tau(B)$ on $X(B)$ and the mapping $x \to \xi_x$, where $\xi_x = \{H \in C(B) \mid x \in H\}$, is a topological embedding. $(X(B), \tau(B))$ is the Wallman compactification of $X$ corresponding to the Wallman basis $B$.

We recall also some basic facts about pre-uniform spaces. (See [4]).

If $X$ is any set, $\alpha$ is a cover of $X$ and $p \in X$, we define

$$S_p(\alpha) = \cup \{A \in \alpha \mid p \in A\}.$$

A non-empty family of covers $U$ of a set $X$ is directed if $U$ satisfies the following property:

$$\ldots$$
*) If $\alpha, \beta \in U$, there exists a cover $\gamma \in U$ which refines both covers $\alpha, \beta$.

The topology induced by a directed family of covers $U$ is defined as follows:
A subset $V$ of $X$ belongs to $\tau_U$ if and only if for every $p \in V$, there exists a cover $\alpha_p \in U$ such that $S_T(p, \alpha_p) \subseteq V$.

A directed family of covers $U$ on a set $X$ is a pre-uniformity basis of $X$ if for every $\alpha \in U$, there exists a cover $\beta \in U$ such that $\beta \circ \alpha \subseteq \alpha$. We imply from here that for every $\alpha \in U$, $\alpha$ is also a cover of $X$. A pre-uniform space is a pair $(X, U)$, where $X$ is a set and $U$ is a pre-uniformity basis on $X$. It is easy to prove that if $L \subseteq X$ is an arbitrary subset of $X$, the family of restrictions:
$$U_L = \{\alpha|_L \mid \alpha \in U\}$$
is a pre-uniformity basis on $L$ and the induced topology $\tau_{U_L}$ coincides with the relative topology $\tau_U|_L$. If $(X, U)$ is a pre-uniform space and if $F$ is a filter on $X$, we say $F$ is Cauchy in $(X, U)$ (or $U$-Cauchy) if for every $\alpha \in U$, we have $F \cap \alpha \neq \emptyset$. A Cauchy filter $F$ is minimal if it does not properly contain any other Cauchy filter.

A Cauchy filter $F$ on a pre-uniform space $(X, U)$ is weakly round if for every $F \in F$, we can find a cover $\alpha_F \in U$ such that $\cup \{A \mid A \in F \cap \alpha_F\} \subseteq F$. Weakly round filters are minimal and every neighborhood filter in the space $(X, \tau_U)$ is weakly round (see [4]).

A pre-uniform space $(X, U)$ is complete (resp., convergence complete) if every Cauchy filter $F$ in $(X, U)$ has an adherence point (resp., a convergence point).

A pre-uniform space $(X, U)$ is totally bounded (resp., precompact) if every $\alpha \in U$ has a finite subfamily which covers $X$ (resp., if every $\alpha \in U$ is finite).

**Theorem 2.1.** Let $(X, U)$ be a pre-uniform space. Then the space $(X, \tau_U)$ is compact if and only if $(X, U)$ is complete and totally bounded. (See [5]).

**Proof.** If $(X, \tau_U)$ is compact, every filter in $X$ has an adherence point and hence $(X, U)$ is complete. If $\alpha \in U$, then there exists a cover $\beta \in U$ such that $\beta^\circ = \{\text{Int}(B) \mid B \in \beta\}$ refines $\alpha$. By compactness, $\beta^\circ$ has a finite subcover. Hence $(X, U)$ is totally bounded.

Conversely, suppose $(X, U)$ is complete and totally bounded. To prove $(X, \tau_U)$ is compact, it is enough to prove that every ultrafilter in $X$ is convergent. The hypothesis of total boundedness, implies that every ultrafilter in $X$ has an adherence point. But every adherence point of an ultrafilter is a convergence point. Hence, $(X, \tau_U)$ is compact. □
A map \( \varphi : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) between pre-uniform spaces is **uniformly continuous** if for every cover \( \beta \in \mathcal{V} \), there exists a cover \( \alpha \in \mathcal{U} \) such that \( \alpha \) refines the cover \( \{ \varphi^{-1}(B) \mid B \in \beta \} \). Clearly, if \( \varphi : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is uniformly continuous, then \( \varphi : (X, \tau_U) \to (Y, \tau_V) \) is continuous, but the converse is not true in general.

A bijective map \( \varphi : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a **unimorphism** if both maps \( \varphi \), \( \varphi^{-1} \) are uniformly continuous.

\( \varphi : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a **uniform embedding** if the range set \( \varphi(X) \) is dense in \( Y \) and if \( \varphi : (X, \mathcal{U}) \to (\varphi(X), \mathcal{V}|_{\varphi(X)}) \) is a unimorphism. An important remark is the following:

If \( \varphi : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a uniform embedding and if \( \mathcal{G} \) is a weakly round filter in \( (Y, \mathcal{V}) \), then \( \varphi^{-1}(\mathcal{G}) \) is a weakly round filter in \( (X, \mathcal{U}) \).

The canonical extension \( (\hat{X}, \hat{\mathcal{U}}) \) of a T\(_1\) pre-uniform space \( (X, \mathcal{U}) \) is defined as follows: the points of \( \hat{X} \) are precisely the weakly round filters in \( (X, \mathcal{U}) \); the elements \( \hat{\alpha} \in \hat{\mathcal{U}} \) are defined for each \( \alpha \in \mathcal{U} \), where:

\[
\hat{\alpha} = \{ \hat{A} \mid A \in \alpha \},
\]

\[
\hat{A} = \{ \xi \in \hat{X} \mid A \in \xi \}, \quad A \subseteq X.
\]

\( \hat{\mathcal{U}} \) is in fact a pre-uniformity basis on \( \hat{X} \) and for each \( \alpha \in \mathcal{U} \), \( \hat{\alpha} \) is an open cover of the space \( (\hat{X}, \tau_{\hat{U}}) \). The map \( \nu : X \to \hat{X} \), where for each \( x \in X \), \( \nu(x) \) is the neighborhood filter of \( x \), is a uniform embedding of \( (X, \mathcal{U}) \) into \( (\hat{X}, \hat{\mathcal{U}}) \).

\((\hat{X}, \hat{\mathcal{U}})\) has the following additional property (see [4]):

Every weakly round filter \( \mathcal{G} \) in \( (\hat{X}, \hat{\mathcal{U}}) \) is convergent. In fact, \( \mathcal{G} \) converges to \( \nu^{-1}(\mathcal{G}) \).

Two filters \( \mathcal{F}, \mathcal{G} \) in a set \( X \) mingle (in symbols \( \mathcal{F} \leftrightarrow \mathcal{G} \)) if every element of \( \mathcal{F} \) intersects every element of \( \mathcal{G} \). For instance, if \( \mathcal{F} \) is a filter in a topological space \( X \) and if \( p \in X \), then \( p \) is an adherence point of \( \mathcal{F} \) if and only if \( \mathcal{F} \) mingles with the neighborhood filter of \( p \). It is also clear that two filters \( \mathcal{F}, \mathcal{G} \) on a set \( X \) mingle if and only if \( \mathcal{F} \) and \( \mathcal{G} \) are subfilters of a filter \( \mathcal{H} \) on \( X \).

A complete pre-uniform space \( (Y, \mathcal{V}) \) is a completion of a pre-uniform space \( (X, \mathcal{U}) \) if there exists a uniform embedding \( \varphi : (X, \mathcal{U}) \to (Y, \mathcal{V}) \).

Using previous remarks we conclude that \( (\hat{X}, \hat{\mathcal{U}}) \) is a completion of the T\(_1\)-pre-uniform space \( (X, \mathcal{U}) \) if and only if every Cauchy filter in \( (X, \mathcal{U}) \) mingles with a weakly round filter in \( (X, \mathcal{U}) \). Also, \( (\hat{X}, \hat{\mathcal{U}}) \) is a convergence complete extension of \( (X, \mathcal{U}) \) if and only if every Cauchy filter in \( (X, \mathcal{U}) \) contains a weakly round filter in \( (X, \mathcal{U}) \). This happens, for instance, if \( (X, \mathcal{U}) \) is uniform or semi-uniform (see [4]).
Going back to Wallman bases, we observe that each Wallman basis $\mathcal{B}$ on a $T_1$-topological space $(X, \tau)$ induces a pre-uniform basis $\mathcal{U}(\mathcal{B})$ in $X$, where $\mathcal{U}(\mathcal{B})$ is the family of finite covers of $X$ with elements of $\mathcal{B}$. Clearly $\tau_{\mathcal{U}(\mathcal{B})} = \tau$.

3. The Relation between $X(\mathcal{B})$ and $(\hat{X}, \hat{\mathcal{U}}(\mathcal{B}))$

**Theorem 3.1.** Let $\mathcal{B}$ be a Wallman basis of a $T_1$-topological space $(X, \tau)$. Then there exists a homeomorphism $\varphi$ between $X(\mathcal{B})$ and the canonical extension $(\hat{X}, \hat{\mathcal{U}}(\mathcal{B}))$ of the pre-uniform space $(X, \mathcal{U}(\mathcal{B}))$. $\varphi$ transforms the fixed ultrafilters of $C(\mathcal{B})$ onto the neighborhood filters of $(X, \tau)$.

**Proof.** For each $G \subseteq \mathcal{P}(X)$, $G^+$ denotes the family:

$G^+ = \{ A \in \mathcal{P}(X) \mid G \subseteq A \text{ for some } G \in G\}.$

Define, for each $\xi \in X(\mathcal{B})$, 

$\varphi(\xi) = \{ B \in \mathcal{B} \mid \exists H \in \xi, \ H \subseteq B \}^+.$

Our first claim is that $\varphi(\xi)$ is Cauchy in $(X, \mathcal{U}(\mathcal{B}))$:

Let $\alpha = \{B_1, B_2, \ldots, B_n\} \in \mathcal{U}(\mathcal{B})$. Select an index $i \in \{1, 2, \ldots, n\}$. Clearly $B_i \notin \varphi(\xi)$ if and only if $X - B_i$ intersects each element of $\xi$. Since $\xi$ is an ultrafilter in $C(\mathcal{B})$, we have $B_i \notin \varphi(\xi)$ if and only if $X - B_i \in \xi$. Hence, if $\alpha \cap \varphi(\xi) = \emptyset$, we would have $X - B_i \in \xi$ for each $i \in \{1, 2, \ldots, n\}$ and therefore:

$$\bigcap_{i=1}^{n} (X - B_i) = X - \bigcup_{i=1}^{n} B_i = \emptyset \in \xi,$$

a contradiction. Our second claim is that $\varphi(\xi)$ is weakly round in $(X, \mathcal{U}(\mathcal{B}))$: if $B \in \varphi(\xi)$, there exists an element $H \in \xi$ such that $H \subseteq B$. If $\alpha = \{B, X - H\}$, we have $\alpha \in \mathcal{U}(\mathcal{B})$ and $\cup \{L \in \alpha \mid L \in \xi\} = B$. $(X - H \in \varphi(\xi)$ would imply the existence of an element $H' \in \xi$ contained in $X - H$ and $H \cap H' = \emptyset$, impossible). This proves that $\varphi(\xi)$ is a weakly round filter in $(X, \mathcal{U}(\mathcal{B}))$ and hence $\varphi$ is actually a map from $X(\mathcal{B})$ to $\hat{X}$.

Our next claim is that $\varphi$ is surjective.

Let $\eta$ be a weakly round filter in $(X, \mathcal{U}(\mathcal{B}))$. For each $\beta \in \mathcal{U}(\mathcal{B})$, let $H_\beta = X - \cup \{B \in \beta \mid B \notin \eta\}$. Because $\eta$ is Cauchy in $(X, \mathcal{U}(\mathcal{B}))$, we have $H_\beta \neq \emptyset$ for each $\beta \in \mathcal{U}(\mathcal{B})$. On the other hand, if $\beta_1, \beta_2 \in \mathcal{U}(\mathcal{B})$ and if $H_{\beta_1} \cap H_{\beta_2} = \emptyset$, we would have:

$$X = \cup \{B \in \beta_1 \mid B \notin \eta\} \cup \cup \{B \in \beta_2 \mid B \notin \eta\},$$
and \( \{B \in \beta_1 \cup \beta_2 \mid B \notin \eta\} \) would be a cover in \( \mathcal{U}(\mathcal{B}) \) disjoint from \( \eta \), contradicting the Cauchy property of \( \eta \). Therefore, for every pair of covers \( \beta_1, \beta_2 \in \mathcal{U}(\mathcal{B}) \), we have \( H_{\beta_1} \cap H_{\beta_2} \neq \emptyset \). We prove \( \{H_{\beta} \mid \beta \in \mathcal{U}(\mathcal{B})\} \) in an ultrafilter in \( \mathcal{C}(\mathcal{B}) \).

Take an element \( H \in \mathcal{C}(\mathcal{B}) \) such that \( H \cap H_{\beta} \neq \emptyset \) for every \( \beta \in \mathcal{U}(\mathcal{B}) \). We have to prove that \( H = H_{\gamma} \) for some \( \gamma \in \mathcal{U}(\mathcal{B}) \). Take any \( B \in \beta \) such that \( B \supseteq H \) and consider the cover \( \gamma_B = \{B, X - H\} \in \mathcal{U}(\mathcal{B}) \). If \( B \notin \eta \), then \( X - H \notin \eta \). Because \( \eta \) is weakly round, there exists a cover \( \alpha \in \mathcal{U}(\mathcal{B}) \) such that:

\[
\bigcup \{A \in \alpha \mid A \in \eta\} \subseteq X - H.
\]

Hence, \( H \subseteq \bigcup \{A \in \alpha \mid A \notin \eta\} = X - H_\alpha \) and \( H \cap H_\alpha = \emptyset \), a contradiction. Therefore, \( B \notin \eta \) and \( H = H_{\gamma_B} \).

We have proved that \( \xi = \{H_{\beta} \mid \beta \in \mathcal{U}(\mathcal{B})\} \) is an ultrafilter in \( \mathcal{C}(\mathcal{B}) \). It remains to prove that \( \varphi(\xi) = \eta \). If \( H \in \xi \), \( B \in \beta \) and \( H \subseteq B \), necessarily \( B \notin \eta \), because otherwise \( X - H \notin \eta \) and, as before, we could find a cover \( \alpha \in \mathcal{U}(\mathcal{B}) \) such that \( H \cap H_\alpha = \emptyset \), a contradiction. Hence \( \varphi(\xi) \subseteq \eta \). Conversely, if \( N \in \eta \), there exists a cover \( \alpha \in \mathcal{U}(\mathcal{B}) \) such that \( \bigcup \{A \in \alpha \mid A \notin \eta\} \subseteq N \). Therefore, \( X - N \subseteq X - \bigcup \{A \in \alpha \mid A \notin \eta\} \subseteq \bigcup \{A \in \alpha \mid A \notin \eta\} = X - H_\alpha \). But \( H_\alpha \in \xi \) and \( H_\alpha \subseteq N \). Therefore, \( N \in \varphi(\xi) \).

We prove next that \( \varphi \) is injective. Let \( \xi, \xi' \in \mathcal{X}(\mathcal{B}) \) be different. Therefore, there exist cobasic sets \( H \in \xi \) and \( H' \in \xi' \) such that \( H \cap H' = \emptyset \). Therefore, \( X - H' \in \varphi(\xi) \) but \( X - H' \notin \varphi(\xi') \). Hence, \( \varphi(\xi) \neq \varphi(\xi') \).

We prove finally that \( \varphi \) is continuous and open. For this, it will be enough to show that for each \( B \in \mathcal{B} \), \( \varphi(B^*) = \hat{B} \). If \( \eta \in \hat{B} \), we have \( B \in \eta \) and \( \varphi^{-1}(\eta) = \{H_{\beta} \mid \beta \in \mathcal{U}(\mathcal{B})\} \). Since \( \eta \) is weakly round, there exists a cover \( \beta \in \mathcal{U}(\mathcal{B}) \) such that \( \bigcup \{L \in \beta \mid L \in \eta\} \subseteq \beta \). But \( H_{\beta} \in \varphi^{-1}(\eta) \) and \( H_{\beta} \subseteq \bigcup \{L \in \beta \mid L \in \eta\} \). By previous remarks, \( \varphi^{-1}(\eta) \in \mathcal{B}^* \). On the other hand, if \( \xi \in \mathcal{B}^* \), there exists a cobasic set \( H \in \xi \) such that \( H \subseteq B \). By the definition of \( \varphi \), we have \( B \in \varphi(\xi) \) and hence \( \varphi(\xi) \in \hat{B} \).

**Corollary 3.2.** Let \( \mathcal{B} \) be a Wallman basis on a \( T_1 \)-topological space \((X, \tau)\). Then there exists a homeomorphism \( \varphi : X(\mathcal{B}) \to (\hat{X}, \mathcal{U}(\mathcal{B})) \) where for each fixed ultrafilter \( \xi_x \in X(\mathcal{B}) \), \( \varphi(\xi_x) \) is the neighborhood filter of \( x \).

**Corollary 3.3.** Every Cauchy filter \( \mathcal{F} \) in \((X, \mathcal{U}(\mathcal{B}))\) mingles with a weakly round filter in \((X, \mathcal{U}(\mathcal{B}))\).

**Proof.** Since \((\hat{X}, \mathcal{U}(\mathcal{B}))\) is compact, \((\hat{X}, \mathcal{U}(\mathcal{B}))\) is complete. Hence every Cauchy filter in \((X, \mathcal{U}(\mathcal{B}))\) mingles with a weakly round filter in \((X, \mathcal{U}(\mathcal{B}))\).
4. Perfect Pre-Uniform Extensions

Let \( \varphi : X \to Y \) be a topological embedding. We say that \( Y \) is a perfect extension of \( X \) if whenever a closed set \( C \subseteq X \) separates two sets \( A, B \subseteq X \), then \( \text{Cl}_Y \varphi (C) \) separates in \( Y \) the two sets \( \varphi (A), \varphi (B) \).

A useful and simpler formulation of this property is the following (see [3]):

\( \star \) If \( H, K \subseteq X \) are closed and \( X = H \cup K \), then

\[
\text{Cl}_Y (\varphi (H) \cap \varphi (K)) = \text{Cl}_Y \varphi (H) \cap \text{Cl}_Y \varphi (K).
\]

Using the operator \( * \) from \( \mathcal{P} (X) \) to the topology of \( Y \) defined by the formula:

\[
A^* = Y - \text{Cl}_Y \varphi (X - A),
\]

we see that (\( \star \)) is equivalent to:

\( \star \star \) If \( V, W \subseteq X \) are open and disjoint, then \( (V \cup W)^* = V^* \cup W^* \). (Observe the formula \( (A \cap B)^* = A^* \cap B^* \) is true for arbitrary sets \( A, B \subseteq X \)).

Some well known perfect extension are the following:

1) The Stone-Čech compactification \( \beta X \) for any Tychonoff space \( X \).

2) The Freudenthal compactification \( F X \) for any rim compact Hausdorff space. (See [1]).

3) The metric completion \( (\tilde{X}, \tilde{d}) \) of \( (X, d) \), when \( d \) is a metric on \( X \) with property S. (See [2]).

4) The Wallman compactification \( X (B) \) of \( X \), where \( B \) is a Wallman basis on \( X \) and having the property that every clopen subset of a cobasic set is also a cobasic set and every clopen subset of a basic set is also a basic set. (See [3]).

We study now when the canonical extension \( (\hat{X}, \hat{U}) \) of a \( T_1 \) pre-uniform space \( (X, U) \) is a perfect extension of \( X \).

**Theorem 4.1.** Let a \( T_1 \) be a pre-uniform space. Then \( (\hat{X}, \tau_{\hat{U}}) \) is a perfect extension of \( (X, \tau_{\downarrow}) \) if and only if we have the following condition:

\( \circ \) If \( \xi \) is a weakly round filter on \( (X, \mathcal{U}) \) and \( L, M \) are disjoint open sets on \( (X, \tau_{\downarrow}) \), then \( L \cup M \in \xi \) if and only if \( L \in \xi \) or \( M \in \xi \).
Proof. We first show that for every open set \( L \subseteq X \), we have \( \widehat{L} = L^* \). We also assume, without loss of generality, that each \( \alpha \in \mathcal{U} \) is an open cover of \( X \).

Let \( \xi \) be a weakly round filter on \((X, \mathcal{U})\) such that \( \xi \in \widehat{L} \). Then \( L \in \xi \) and there exists a cover \( \alpha \in \mathcal{U} \) such that \( \cup \{ A \in \alpha \mid A \in \xi \} \subseteq L \). Take any element \( A \in \alpha \cap \xi \). Then \( A \subseteq L \) and \( \xi \in \widehat{A} \). Then \( \widehat{A} \) is an open set in \( \widehat{X} \) containing \( \varphi(A) \) and disjoint from \( \varphi(X - L) \). Hence \( \xi \in \widehat{X} - C\ell_{\widehat{X}}\varphi(X - L) = L^* \). Conversely, suppose \( \xi \in L^* \). Since the sets \( \widehat{A} \), where \( A \in \alpha \) for some \( \alpha \in \mathcal{U} \), form a basis for \( \tau_{\mathcal{U}} \), we have \( \xi \in \widehat{A} \) and \( \widehat{A} \cap \varphi(X - L) = \emptyset \) for some such set. Then \( \varphi^{-1}(\widehat{A} \cap \varphi(X - L)) = \varphi^{-1}(\widehat{A} \cap (X - L)) = (\text{int}A) \cap (X - L) = \emptyset \), i.e., \( \text{int}A \subseteq L \) and \( \xi \in \widehat{A} \subseteq \widehat{L} \).

To prove the theorem, suppose first that \( \widehat{X} \) is a perfect extension of \( X \). Let \( L, M \) be disjoint open sets in \( X \) and let \( \xi \) be a weakly round filter such that \( L \cup M \in \xi \). Then \( \xi \in \widehat{L} \cup \widehat{M} = (L \cup M)^* = L^* \cup M^* = \widehat{L} \cup \widehat{M} \). Therefore, \( \xi \in \widehat{L} \) or \( \xi \in \widehat{M} \), i.e., \( L \in \xi \) or \( M \in \xi \).

Suppose now that whenever \( L, M \) are disjoint open sets in \( X \), \( \xi \in \widehat{X} \) and \( L \cup M \in \xi \), we have \( L \in \xi \) or \( M \in \xi \). This implies that \( \widehat{L} \cup \widehat{M} = \widehat{L} \cup \widehat{M} \), or equivalently, \((L \cup M)^* = L^* \cup M^* \). By (\ref{eq:4理解和证明}), \( \widehat{X} \) is a perfect extension of \( X \).

Corollary 4.2. Let \( B \) a a Wallman basis of a \( T_1 \)-space \( X \) satisfying the properties of example 4. Then \( X(\mathcal{B}) \) is a perfect compactification of \( X \).

Proof. Let \( \xi \in (\widehat{X}, \mathcal{U}(\mathcal{B})) \) and let \( L, M \) be disjoint open sets in \( X \) such that \( L \cup M \in \xi \). Then there exists an ultrafilter \( \eta \) in \( C(\mathcal{B}) \) such that:

\[ \xi = \{ B \in \mathcal{B} \mid H \subseteq B \text{ for some } H \in \eta \}^+ \cdot \]

Select \( H_0 \in \eta \), \( B \in \mathcal{B} \) such that \( H_0 \subseteq B \subseteq L \cup M \). At least one of sets \( L, M \) intersects every member of \( \eta \), because otherwise we could find elements \( H_1, H_2 \in \eta \) such that \( L \cap H_1 = \emptyset = M \cap H_2 \). Then \( (L \cup M) \cap H_1 \cap H_2 = \emptyset \), contradicting the fact that every member of \( \eta \) intersects \( H_0 \) and, hence, intersects \( L \cup M \). Assuming that \( L \cap H \neq \emptyset \) for every \( H \in \eta \), we have \( L \cap H_0 \in \eta \), \( L \cap B \in \mathcal{B} \) and hence \( L \in \xi \).

Corollary 4.3. Suppose \( (X, \mathcal{U}) \) is unimorphic to a pre-uniform space \( (X, \mathcal{V}) \) where every cover \( \beta \in \mathcal{V} \) is formed with connected elements of \( (X, \tau_{\mathcal{U}}) \). Then \( (\widehat{X}, \mathcal{U}) \) is a perfect extension of \( (X, \mathcal{U}) \).

Proof. Let \( L, M \) be disjoint open sets in \( X \), let \( \xi \in \widehat{X} \) and suppose \( L \cup M \in \xi \). Since \( \xi \) is weakly round, there exists a cover \( \alpha \in \mathcal{U} \) such that:

\[ \cup \{ A \in \alpha \mid A \in \xi \} \subseteq L \cup M. \]
Let $\beta \in \mathcal{V}$ be a refinement of $\alpha$. Then:

$$\bigcup \{B \in \beta \mid B \in \xi\} \subseteq \bigcup \{A \in \alpha \mid A \in \xi\} \subseteq L \cup M.$$  

But $\bigcup \{B \in \beta \mid B \in \xi\}$ is a connected set. Then $\bigcup \{B \in \beta \mid B \in \xi\} \subseteq L$ or $\bigcup \{B \in \beta \mid B \in \xi\} \subseteq M$. Therefore, $L \in \xi$ or $M \in \xi$ and $\hat{X}$ is a perfect extension of $X$.  

Corollary 4.4. Let $\mathcal{B}$ be a Wallman basis of a $T_1$-topological space $X$. Then $X(\mathcal{B})$ is a perfect compactification of $X$ if and only if for every $B \in \mathcal{B}$, for every separation $B = L \cup M$ and for every $\xi \in \mathcal{B}^*$, there exist elements $H \in \xi$ and $B' \in \mathcal{B}$ such that $H \subseteq B'$ and $B' \subseteq L$ or $B' \subseteq M$. (Compare with Theorem 3.1 in [3]).

References


