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WALLMAN COMPACTIFICATIONS ARE PRE-UNIFORM COMPLETIONS

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Abstract: We prove that every Wallman compactifications of a T_1-space (X, τ) may be viewed as the completion of certain pre-uniform space $(X, \mathcal{U})$, where $\mathcal{U}$ is a precompact pre-uniform basis compatible with τ. We also study perfect extensions of pre-uniform spaces.

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1. Introduction

It is a well known result that each normal Wallman basis $\mathcal{B}$ of a Tychonoff space X induces a T_2-compactification of X in two equivalent ways:

a) via the Wallman method which topologizes the family of ultrafilters of cobasic sets or

b) via the completion of the precompact uniformity basis of X with basic sets.

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If the Wallman basis is not normal, we cannot expect a T_2-compactification. However, the Wallman method of ultrafilters is still equivalent to the process of completing a precompact pre-uniformity basis. This is the principal result of this paper. We also give a necessary and sufficient condition on a pre-uniform space to have as a perfect pre-uniform extension, its canonical extension.

2. Wallman Bases and Pre-Uniform Spaces

Recall a Wallman basis for a topological space (X, τ) is a basis $\mathcal{B}$ for the topology τ satisfying two properties:

i) If $B_1, B_2 \in \mathcal{B}$, also $B_1 \cap B_2 \in \mathcal{B}$ and $B_1 \cup B_2 \in \mathcal{B}$.

ii) If $x \in B \in \mathcal{B}$, there exists an element $H \in C(\mathcal{B})$ such that $x \in H \subseteq B$, where $C(\mathcal{B}) = \{H \subseteq X \mid X - H \in \mathcal{B}\}$. (See [1] and [6]).

A topological space (X, τ) has a Wallman basis if and only if (X, τ) is R_0, i.e., every open set $V \subseteq X$ is a union of closed sets. In this case, the topology itself is a Wallman basis.

Each Wallman basis of $\mathcal{B}$ of a T_1-space (X, τ) induces a T_1-compactification $X(\mathcal{B})$ of X: the elements of $X(\mathcal{B})$ are precisely the ultrafilters in the family of cobasic sets $C(\mathcal{B})$. An element $\xi \in X(\mathcal{B})$ is a <u>fixed</u> <u>ultrafilter</u> if for some $x \in X$, $\xi = \{H \in C(\mathcal{B}) \mid x \in H\}$. $\xi \in X(\mathcal{B})$ is <u>free</u> if ξ is not fixed. Each set $A \subseteq X$ determines a subset $A^* \subseteq X(\mathcal{B})$, where $\xi \in A^*$ if and only if for some $H \in \xi$, we have $H \subseteq A$. We have the formulas (see [1], 3.45):

a) $(A_1 \cap A_2)^* = A_1^* \cap A_2^*$ for every $A_1, A_2 \subseteq X$;

b) $(A_1 \cup A_2)^* = A_1^* \cup A_2^*$ for every $A_1, A_2 \in \mathcal{B} \cup C(\mathcal{B})$.

The family $\mathcal{B}^* = \left\{B^* \mid B \in \mathcal{B}\right\}$ is a basis for a compact T_1-topology $\tau(\mathcal{B})$ on $X(\mathcal{B})$ and the mapping $x \to \xi_x$, where $\xi_x = \{H \in C(\mathcal{B}) \mid x \in H\}$, is a topological embedding. $(X(\mathcal{B}), \tau(\mathcal{B}))$ is the <u>Wallman</u> <u>compactification</u> of X corresponding to the Wallman basis $\mathcal{B}$.

We recall also some basic facts about pre-uniform spaces. (See [4]).

If X is any set, α is a cover of X and $p \in X$, we define

$$\mathrm{S_T}(p, \alpha) = \cup \{A \in \alpha \mid p \in A\}.$$

A non-empty family of covers $\mathcal{U}$ of a set X is <u>directed</u> if $\mathcal{U}$ satisfies the following property:

$*$) If $\alpha, \beta \in \mathcal{U}$, there exists a cover $\gamma \in \mathcal{U}$ which refines both covers α, β.

The topology induced by a directed family of covers $\mathcal{U}$ is defined as follows: A subset V of X belongs to $\tau_{\mathcal{U}}$ if and only if for every $p \in V$, there exists a cover $\alpha_p \in \mathcal{U}$ such that $\mathrm{S_T}(p, \alpha_p) \subseteq V$.

A directed family of covers $\mathcal{U}$ on a set X is a pre-uniformity basis of X if for every $\alpha \in \mathcal{U}$, there exists a cover $\beta \in \mathcal{U}$ such that β refines the cover α°, where $\alpha^\circ = \{\mathrm{Int}_{\tau_{\mathsf{U}}} A \mid A \in \alpha\}$. We imply from here that for every $\alpha \in \mathcal{U}$, α° is also a cover of X. A pre-uniform space is a pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a pre-uniformity basis on X. It is easy to prove that if $L \subseteq X$ is an arbitrary subset of X, the family of restrictions:

$$\mathcal{U}_L = \{\alpha|_L \mid \alpha \in \mathcal{U}\}$$

is a pre-uniformity basis on L and the induced topology $\tau_{\mathcal{U}_L}$ coincides with the relative topology $\tau_{\mathcal{U}}|_L$. If $(X,\mathcal{U})$ is a pre-uniform space and if $\mathcal{F}$ is a filter on X, we say $\mathcal{F}$ is Cauchy in $(X,\mathcal{U})$ (or $\mathcal{U}$-Cauchy) if for every $\alpha \in \mathcal{U}$, we have $\mathcal{F} \cap \alpha \neq \varnothing$. A Cauchy filter $\mathcal{F}$ is minimal if it does not properly contain any other Cauchy filter.

A Cauchy filter $\mathcal{F}$ on a pre-uniform space $(X,\mathcal{U})$ is weakly round if for every $F \in \mathcal{F}$, we can find a cover $\alpha_F \in \mathcal{U}$ such that $\cup\{A \mid A \in \mathcal{F} \cap \alpha_F\} \subseteq F$. Weakly round filters are minimal and every neighborhood filter in the space $(X, \tau_{\mathcal{U}})$ is weakly round (see [4]).

A pre-uniform space $(X,\mathcal{U})$ is complete (resp., convergence complete) if every Cauchy filter $\mathcal{F}$ in $(X,\mathcal{U})$ has an adherence point(resp., a convergence point).

A pre-uniform space $(X,\mathcal{U})$ is totally bounded (resp., precompact) if every $\alpha \in \mathcal{U}$ has a finite subfamily which covers X (resp., if every $\alpha \in \mathcal{U}$ is finite).

Theorem 2.1. *Let $(X,\mathcal{U})$ be a pre-uniform space. Then the space $(X, \tau_{\mathcal{U}})$ is compact if and only if $(X,\mathcal{U})$ is complete and totally bounded. (See [5]).*

Proof. If $(X, \tau_{\mathcal{U}})$ is compact, every filter in X has an adherence point and hence $(X,\mathcal{U})$ is complete. If $\alpha \in \mathcal{U}$, then there exists a cover $\beta \in \mathcal{U}$ such that $\beta^\circ = \{\mathrm{Int}\,(\mathrm{B})\,|\,\mathrm{B} \in \beta\}$ refines α. By compactness, β° has a finite subcover. Hence $(X,\mathcal{U})$ is totally bounded.

Conversely, suppose $(X,\mathcal{U})$ is complete and totally bounded. To prove $(X, \tau_{\mathcal{U}})$ is compact, it is enough to prove that every ultrafilter in X is convergent. The hypothesis of total boundedness, implies that every ultrafilter in X has an adherence point. But every adherence point of an ultrafilter is a convergence point. Hence, $(X, \tau_{\mathcal{U}})$ is compact. $\square$

A map $\varphi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between pre-uniform spaces is uniformly continuous if for every cover $\beta \in \mathcal{V}$, there exists a cover $\alpha \in \mathcal{U}$ such that α refines the cover $\left\{\varphi^{-1}(B) \mid B \in \beta\right\}$. Clearly, if $\varphi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous, then $\varphi : (X,\tau_{\mathcal{U}}) \to (Y,\tau_{\mathcal{V}})$ is continuous, but the converse is not true in general.

A bijective map $\varphi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a unimorphism if both maps φ, φ^{-1} are uniformly continuous.

$\varphi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a uniform embedding if the range set $\varphi(X)$ is dense in Y and if $\varphi : (X,\mathcal{U}) \to \left(\varphi(X), \mathcal{V}|_{\varphi(X)}\right)$ is a unimorphism. An important remark is the following:

If $\varphi : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is a uniform embedding and if $\mathcal{G}$ is a weakly round filter in $(Y,\mathcal{V})$, then $\varphi^{-1}(\mathcal{G})$ is a weakly round filter in $(X,\mathcal{U})$.

The canonical extension $(\widehat{X},\widehat{\mathcal{U}})$ of a T_1 pre-uniform space $(X,\mathcal{U})$ is defined as follows: the points of $\widehat{X}$ are precisely the weakly round filters in $(X,\mathcal{U})$; the elements $\widehat{\alpha} \in \widehat{\mathcal{U}}$ are defined for each $\alpha \in \mathcal{U}$, where:

$$\widehat{\alpha} = \{\widehat{A} \mid A \in \alpha\}$$
$$\widehat{A} = \{\xi \in \widehat{X} \mid A \in \xi\}, \quad A \subseteq X.$$

$\widehat{\mathcal{U}}$ is in fact a pre-uniformity basis on $\widehat{X}$ and for each $\alpha \in \mathcal{U}$, $\widehat{\alpha}$ is an open cover of the space $(\widehat{X}, \tau_{\widehat{\mathcal{U}}})$. The map $\nu : X \to \widehat{X}$, where for each $x \in X$, $\nu(x)$ is the neighborhood filter of x, is a uniform embedding of $(X,\mathcal{U})$ into $(\widehat{X},\widehat{\mathcal{U}})$.

$(\widehat{X},\widehat{\mathcal{U}})$ has the following additional property (see [4]):

Every weakly round filter $\mathcal{G}$ in $(\widehat{X},\widehat{\mathcal{U}})$ is convergent. In fact, $\mathcal{G}$ converges to $\nu^{-1}(\mathcal{G})$.

Two filters $\mathcal{F}$, $\mathcal{G}$ in a set X mingle (in symbols $\mathcal{F} \leftrightarrow \mathcal{G}$) if every element of $\mathcal{F}$ intersects every element of $\mathcal{G}$. For instance, if $\mathcal{F}$ is a filter in a topological space X and if $p \in X$, then p is an adherence point of $\mathcal{F}$ if and only if $\mathcal{F}$ mingles with the neighborhood filter of p. It is also clear that two filters $\mathcal{F}$, $\mathcal{G}$ on a set X mingle if and only if $\mathcal{F}$ and $\mathcal{G}$ are subfilters of a filter $\mathcal{H}$ on X.

A complete pre-uniform space $(Y,\mathcal{V})$ is a completion of a pre-uniform space $(X,\mathcal{U})$ if there exists a uniform embedding $\varphi : (X,\mathcal{U}) \to (Y,\mathcal{V})$.

Using previous remarks we conclude that $(\widehat{X},\widehat{\mathcal{U}})$ is a completion of the T_1-pre-uniform space $(X,\mathcal{U})$ if and only if every Cauchy filter in $(X,\mathcal{U})$ mingles with a weakly round filter in $(X,\mathcal{U})$. Also, $(\widehat{X},\widehat{\mathcal{U}})$ is a convergence complete extension of $(X,\mathcal{U})$ if and only if every Cauchy filter in $(X,\mathcal{U})$ contains a weakly round filter in $(X,\mathcal{U})$. This happens, for instance, if $(X,\mathcal{U})$ is uniform or semi-uniform (see [4]).

Going back to Wallman bases, we observe that each Wallman basis $\mathcal{B}$ on a T_1-topological space (X, τ) induces a pre-uniform basis $\mathcal{U}(\mathcal{B})$ in X, where $\mathcal{U}(\mathcal{B})$ is the family of finite covers of X with elements of $\mathcal{B}$. Clearly $\tau_{\mathcal{U}(\mathcal{B})} = \tau$.

3. The Relation between $X(\mathcal{B})$ and $(\widehat{X}, \widehat{\mathcal{U}(\mathcal{B})})$

Theorem 3.1. *Let $\mathcal{B}$ be a Wallman basis of a T_1-topological space (X, τ). Then there exists a homeomorphism φ between $X(\mathcal{B})$ and the canonical extension $(\widehat{X}, \widehat{\mathcal{U}(\mathcal{B})})$ of the pre-uniform space $(X, \mathcal{U}(\mathcal{B}))$. φ transforms the fixed ultrafilters of $C(\mathcal{B})$ onto the neighborhood filters of (X, τ).*

Proof. For each $\mathcal{G} \subseteq \mathcal{P}(X)$, $\mathcal{G}^+$ denotes the family:

$$\mathcal{G}^+ = \{A \in \mathcal{P}(X) \mid G \subseteq A \text{ for some } G \in \mathcal{G}\}.$$

Define, for each $\xi \in X(\mathcal{B})$,

$$\varphi(\xi) = \{B \in \mathcal{B} \mid \exists H \in \xi,\ H \subseteq B\}^+.$$

Our first claim is that $\varphi(\xi)$ is Cauchy in $(X, \mathcal{U}(\mathcal{B}))$:

Let $\alpha = \{B_1, B_2, \ldots, B_n\} \in \mathcal{U}(\mathcal{B})$. Select an index $i \in \{1, 2, \ldots, n\}$. Clearly $B_i \notin \varphi(\xi)$ if and only if $X - B_i$ intersects each element of ξ. Since ξ is an ultrafilter in $C(\mathcal{B})$, we have $B_i \notin \varphi(\xi)$ if and only if $X - B_i \in \xi$. Hence, if $\alpha \cap \varphi(\xi) = \varnothing$, we would have $X - B_i \in \xi$ for each $i \in \{1, 2, \ldots, n\}$ and therefore:

$$\bigcap_{i=1}^{n} (X - B_i) = X - \bigcup_{i=1}^{n} B_i = \varnothing \in \xi,$$

a contradiction. Our second claim is that $\varphi(\xi)$ is weakly round in $(X, \mathcal{U}(\mathcal{B}))$: if $B \in \varphi(\xi)$, there exists an element $H \in \xi$ such that $H \subseteq B$. If $\alpha = \{B, X - H\}$, we have $\alpha \in \mathcal{U}(\mathcal{B})$ and $\cup\{L \in \alpha \mid L \in \xi\} = B$. ($X - H \in \varphi(\xi)$ would imply the existence of an element $H' \in \xi$ contained in $X - H$ and $H \cap H' = \varnothing$, impossible). This proves that $\varphi(\xi)$ is a weakly round filter in $(X, \mathcal{U}(\mathcal{B}))$ and hence φ is actually a map from $X(\mathcal{B})$ to $\widehat{X}$.

Our next claim is that φ is surjective.

Let η be a weakly round filter in $(X, \mathcal{U}(\mathcal{B}))$. For each $\beta \in \mathcal{U}(\mathcal{B})$, let $H_\beta = X - \cup\{B \in \beta \mid B \notin \eta\}$. Because η is Cauchy in $(X, \mathcal{U}(\mathcal{B}))$, we have $H_\beta \neq \varnothing$ for each $\beta \in \mathcal{U}(\mathcal{B})$. On the other hand, if $\beta_1, \beta_2 \in \mathcal{U}(\mathcal{B})$ and if $H_{\beta_1} \cap H_{\beta_2} = \varnothing$, we would have:

$$X = \cup\{B \in \beta_1 \mid B \notin \eta\} \cup \cup\{B \in \beta_2 \mid B \notin \eta\},$$

and $\{B \in \beta_1 \cup \beta_2 \mid B \notin \eta\}$ would be a cover in $\mathcal{U}(\mathcal{B})$ disjoint from η, contradicting the Cauchy property of η. Therefore, for every pair of covers $\beta_1, \beta_2 \in \mathcal{U}(\mathcal{B})$, we have $H_{\beta_1} \cap H_{\beta_2} \neq \varnothing$. We prove $\{H_\beta \mid \beta \in \mathcal{U}(\mathcal{B})\}$ in an ultrafilter in $C(\mathcal{B})$. Take an element $H \in C(\mathcal{B})$ such that $H \cap H_\beta \neq \varnothing$ for eve-ry $\beta \in \mathcal{U}(\mathcal{B})$. We have to prove that $H = H_\gamma$ for some $\gamma \in \mathcal{U}(\mathcal{B})$. Take any $B \in \beta$ such that $B \supseteq H$ and consider the cover $\gamma_B = \{B, X - H\} \in \mathcal{U}(\mathcal{B})$. If $B \notin \eta$, then $X - H \in \eta$. Because η is weakly round, there exists a cover $\alpha \in \mathcal{U}(\mathcal{B})$ such that:

$$\cup\{A \in \alpha \mid A \in \eta\} \subseteq X - H.$$

Hence, $H \subseteq \cup\{A \in \alpha \mid A \notin \eta\} = X - H_\alpha$ and $H \cap H_\alpha = \varnothing$, a contradiction. Therefore, $B \in \eta$, $X - H \notin \eta$ and $H = H_{\gamma_B}$.

We have proved that $\xi = \{H_\beta \mid \beta \in \mathcal{U}(\mathcal{B})\}$ is an ultrafilter in $C(\mathcal{B})$. It remains to prove that $\varphi(\xi) = \eta$. If $H \in \xi$, $B \in \beta$ and $H \subseteq B$, necessarily $B \in \eta$, because otherwise $X - H \in \eta$ and, as before, we could find a cover $\alpha \in \mathcal{U}(\mathcal{B})$ such that $H \cap H_\alpha = \varnothing$, a contradiction. Hence $\varphi(\xi) \subseteq \eta$. Conversely, if $N \in \eta$, there exists a cover $\alpha \in \mathcal{U}(\mathcal{B})$ such that $\cup\{A \in \alpha \mid A \in \eta\} \subseteq N$. Therefore, $X - N \subseteq X - \cup\{A \in \alpha \mid A \in \eta\} \subseteq \cup\{A \in \alpha \mid A \notin \eta\} = X - H_\alpha$. But $H_\alpha \in \xi$ and $H_\alpha \subseteq N$. Therefore, $N \in \varphi(\xi)$.

We prove next that φ is injective. Let $\xi, \xi' \in X(\mathcal{B})$ be different. Therefore, there exist cobasic sets $H \in \xi$ and $H' \in \xi'$ such that $H \cap H' = \varnothing$. Therefore, $X - H' \in \varphi(\xi)$ but $X - H' \notin \varphi(\xi')$. Hence, $\varphi(\xi) \neq \varphi(\xi')$.

We prove finally that φ is continuous and open. For this, it will be enough to show that for each $B \in \mathcal{B}$, $\varphi\left(B^*\right) = \widehat{B}$. If $\eta \in \widehat{B}$, we have $B \in \eta$ and $\varphi^{-1}(\eta) = \{H_\beta \mid \beta \in \mathcal{U}(\mathcal{B})\}$. Since η is weakly round, there exists a cover $\beta \in \mathcal{U}(\mathcal{B})$ such that $\cup\{L \in \beta \mid L \in \eta\} \subseteq \beta$. But $H_\beta \in \varphi^{-1}(\eta)$ and $H_\beta \subseteq \cup\{L \in \beta \mid L \in \eta\}$. By previous remarks, $\varphi^{-1}(\eta) \in B^*$. On the other hand, if $\xi \in B^*$, there exists a cobasic set $H \in \xi$ such that $H \subseteq B$. By the definition of φ, we have $B \in \varphi(\xi)$ and hence $\varphi(\xi) \in \widehat{B}$. □

Corollary 3.2. *Let $\mathcal{B}$ be a Wallman basis on a T_1-topological space (X, τ). Then there exists a homeomorphism $\varphi : X(\mathcal{B}) \to (\widehat{X}, \widehat{\mathcal{U}(\mathcal{B})})$ where for each fixed ultrafilter $\xi_x \in X(\mathcal{B})$, $\varphi(\xi_x)$ is the neighborhood filter of x.*

Corollary 3.3. *Every Cauchy filter $\mathcal{F}$ in $(X, \mathcal{U}(\mathcal{B}))$ mingles with a weakly round filter in $(X, \mathcal{U}(\mathcal{B}))$.*

Proof. Since $(\widehat{X}, \widehat{\mathcal{U}(\mathcal{B})})$ is compact, $(\widehat{X}, \widehat{\mathcal{U}(\mathcal{B})})$ is complete. Hence every Cauchy filter in $(X, \mathcal{U}(\mathcal{B}))$ mingles with a weakly round filter in $(X, \mathcal{U}(\mathcal{B}))$. □

4. Perfect Pre-Uniform Extensions

Let $\varphi : X \to Y$ be a topological embedding. We say that Y is a perfect extension of X if whenever a closed set $C \subseteq X$ separates two sets $A, B \subseteq X$, then $C\ell_Y \varphi(C)$ separates in Y the two sets $\varphi(A)$, $\varphi(B)$.

A useful and simpler formulation of this property is the following (see [3]):

$\star$) If $H, K \subseteq X$ are closed and $X = H \cup K$, then

$$C\ell_Y(\varphi(H) \cap \varphi(K)) = C\ell_Y \varphi(H) \cap C\ell_Y \varphi(K).$$

Using the operator $*$ from $\mathcal{P}(X)$ to the topology of Y defined by the formula:

$$A^* = Y - C\ell_Y \varphi(X - A),$$

we see that ($\star$) is equivalent to:

$\star\star$) If $V, W \subseteq X$ are open and disjoint, then $(V \cup W)^* = V^* \cup W^*$. (Observe the formula $(A \cap B)^* = A^* \cap B^*$ is true for arbitrary sets $A, B \subseteq X$).

Some well known perfect extension are the following:

1) The Stone-Čech compactification βX for any Tychonoff space X.
2) The Freudenthal compactification FX for any rim compact Hausdorff space. (See [1]).
3) The metric completion $(\widetilde{X}, \widetilde{d})$ of (X, d), when d is a metric on X with property S. (See [2]).
4) The Wallman compactification $X(\mathcal{B})$ of X, where $\mathcal{B}$ is a Wallman basis on X and having the property that every clopen subset of a cobasic set is also a cobasic set and every clopen subset of a basic set is also a basic set. (See [3]).

We study now when the canonical extension $(\widehat{X}, \widehat{\mathcal{U}})$ of a T_1 pre-uniform space $(X, \mathcal{U})$ is a perfect extension of X.

Theorem 4.1. *Let a* T_1 *be a pre-uniform space. Then* $(\widehat{X}, \tau_{\widehat{\mathcal{U}}})$ *is a perfect extension of* $(X, \tau_{\mathcal{U}})$ *if and only if we have the following condition:*

$\circ$) *If* ξ *is a weakly round filter on* $(X, \mathcal{U})$ *and* L, M *are disjoint open sets on* $(X, \tau_{\mathcal{U}})$, *then* $L \cup M \in \xi$ *if and only if* $L \in \xi$ *or* $M \in \xi$.

Proof. We first show that for every open set $L \subseteq X$, we have $\widehat{L} = L^*$. We also assume, without loss of generality, that each $\alpha \in \mathcal{U}$ is an open cover of X. Let ξ be a weakly round filter on $(X,\mathcal{U})$ such that $\xi \in \widehat{L}$. Then $L \in \xi$ and there exists a cover $\alpha \in \mathcal{U}$ such that $\cup\{A \in \alpha \mid A \in \xi\} \subseteq L$. Take any element $A \in \alpha \cap \xi$. Then $A \subseteq L$ and $\xi \in \widehat{A}$. Then $\widehat{A}$ is an open set in $\widehat{X}$ containing $\varphi(A)$ and disjoint from $\varphi(X-L)$. Hence $\xi \in \widehat{X} - C\ell_{\widehat{X}}\varphi(X-L) = L^*$. Conversely, suppose $\xi \in L^*$. Since the sets $\widehat{A}$, where $A \in \alpha$ for some $\alpha \in \mathcal{U}$, form a basis for $\tau_{\widehat{\mathcal{U}}}$, we have $\xi \in \widehat{A}$ and $\widehat{A} \cap \varphi(X-L) = \varnothing$ for some such set. Then $\varphi^{-1}(\widehat{A} \cap \varphi(X-L)) = \varphi^{-1}(\widehat{A}) \cap (X-L) = (\mathrm{int}A) \cap (X-L) = \varnothing$, i.e., $\mathrm{int}A \subseteq L$ and $\xi \in \widehat{A} \subseteq \widehat{L}$.

To prove the theorem, suppose first that $\widehat{X}$ is a perfect extension of X. Let L, M be disjoint open sets in X and let ξ be a weakly round filter such that $L \cup M \in \xi$. Then $\xi \in \widehat{L \cup M} = (L \cup M)^* = L^* \cup M^* = \widehat{L} \cup \widehat{M}$. Therefore, $\xi \in \widehat{L}$ or $\xi \in \widehat{M}$, i.e., $L \in \xi$ or $M \in \xi$.

Suppose now that whenever L, M are disjoint open sets in X, $\xi \in \widehat{X}$ and $L \cup M \in \xi$, we have $L \in \xi$ or $M \in \xi$. This implies that $\widehat{L \cup M} = \widehat{L} \cup \widehat{M}$, or equivalently, $(L \cup M)^* = L^* \cup M^*$. By ($\star$), $\widehat{X}$ is a perfect extension of X. □

Corollary 4.2. *Let $\mathcal{B}$ a a Wallman basis of a T_1-space X satisfying the properties of example 4. Then $X(\mathcal{B})$ is a perfect compactification of X.*

Proof. Let $\xi \in (\widehat{X}, \widehat{\mathcal{U}(\mathcal{B})})$ and let L, M be disjoint open sets in X such that $L \cup M \in \xi$. Then there exists an ultrafilter η in $C(\mathcal{B})$ such that:

$$\xi = \{B \in \mathcal{B} \mid H \subseteq B \text{ for some } H \in \eta\}^+ .$$

Select $H_0 \in \eta$, $B \in \mathcal{B}$ such that $H_0 \subseteq B \subseteq L \cup M$. At least one of sets L, M intersects every member of η, because otherwise we could find elements $H_1, H_2 \in \eta$ such that $L \cap H_1 = \varnothing = M \cap H_2$. Then $(L \cup M) \cap H_1 \cap H_2 = \varnothing$, contradicting the fact that every member of η intersects H_0 and, hence, intersects $L \cup M$. Assuming that $L \cap H \neq \varnothing$ for every $H \in \eta$, we have $L \cap H_0 \in \eta$, $L \cap B \in \mathcal{B}$ and hence $L \in \xi$. □

Corollary 4.3. *Suppose $(X,\mathcal{U})$ is unimorphic to a pre-uniform space $(X,\mathcal{V})$ where every cover $\beta \in \mathcal{V}$ is formed with connected elements of $(X,\tau_{\mathcal{U}})$. Then $(\widehat{X},\widehat{\mathcal{U}})$ is a perfect extension of $(X,\mathcal{U})$.*

Proof. Let L, M be disjoint open sets in X, let $\xi \in \widehat{X}$ and suppose $L \cup M \in \xi$. Since ξ is weakly round, there exists a cover $\alpha \in \mathcal{U}$ such that:

$$\cup\{A \in \alpha \mid A \in \xi\} \subseteq L \cup M.$$

Let $\beta \in \mathcal{V}$ be a refinement of α. Then:

$$\cup\{B \in \beta \mid B \in \xi\} \subseteq \cup\{A \in \alpha \mid A \in \xi\} \subseteq L \cup M.$$

But $\cup\{B \in \beta \mid B \in \xi\}$ is a connected set. Then $\cup\{B \in \beta \mid B \in \xi\} \subseteq L$ or $\cup\{B \in \beta \mid B \in \xi\} \subseteq M$. Therefore, $L \in \xi$ or $M \in \xi$ and $\widehat{X}$ is a perfect extension of X. □

Corollary 4.4. *Let $\mathcal{B}$ be a Wallman basis of a* T_1*-topological space X. Then $X(\mathcal{B})$ is a perfect compactification of X if and only if for every $B \in \mathcal{B}$, for every separation $B = L \cup M$ and for every $\xi \in B^*$, there exist elements $H \in \xi$ and $B' \in \mathcal{B}$ such that $H \subseteq B'$ and $B' \subseteq L$ or $B' \subseteq M$. (Compare with Theorem 3.1 in [3]).*

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