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A SIMPSON'S-TYPE SECOND DERIVATIVE METHOD FOR STIFF SYSTEMS

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Abstract: In this paper, we show that a Simpson's type second derivative method (SSDM) can be adapted to cope with the integration of stiff systems in ordinary differential equations (ODEs). This is achieved by combining the SSDM with an additional method and implementing them as a block method. The block method is shown to be A-stable and of order 6. Numerical results produced by the block method show that the method is competitive with existing ones in the literature.

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Key Words: Simpson's method, stiff system, block method, second derivative method

1. Introduction

According to Lambert [20], if the Simpson's Rule is used to evaluate $\int_{x_n}^{x_{n+2}} f(x)dx$, then it is successfully applied on the sub-intervals $[x_0, x_2], [x_2, x_4] \ldots, [x_{N-2}, x_N]$ for $N > 0$, the error in integration over the whole interval is simply the sum of the errors over each sub-interval. In contrast, if the Simpson's Rule is used to integrate an initial value problem (IVP), then it is successively applied on

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the sub-intervals $[x_0, x_2], [x_1, x_3] \ldots, [x_{N-2}, x_N]$ which overlap and the accumulation of error is more complicated. Hence, the Simpson's Rule is an excellent method for quadrature, but a poor method for integrating IVPs. In this paper we show that a SSDM can be adapted using the logic behind the Simpson's Rule for quadrature to cope with the integration of stiff systems in ODEs. This is achieved by combining the SSDM with an additional method and implementing them as a block method. Thus, we consider the first order differential equation

$$y^{\prime} = f(x, y), \ y(a) = y_0 \ , \ x \ \epsilon \ [a, b] \tag{1}$$

where $f : \Re \times \Re^m \rightarrow \Re^m$, $y, y_0 \ \epsilon \ \Re^m$, f satisfies a Lipschitz condition (see Henrici [16]), and the Jacobian $(\frac{\partial f}{\partial y})$ whose eigenvalues have negative real parts varies slowly ([17]).

We recall that (1) is efficiently solved by A-stable methods and for high accuracy, higher order methods are preferable. However, for linear multistep methods (LMMs), the use of high order LMMs for (1) is restricted by the second Dahlquist [8] barrier theorem which stated that the order of an A-stable linear multistep method cannot exceed 2. Several methods have been proposed to overcome this barrier theorem, for instance, hybrid methods (see Gear [11],Gragg and Stetter [13], Butcher [5], Lambert [21], and Kohfeld and Thompson [19]), the second derivative methods (see Enright[9], Gupta [14], and Hairer and Wanner [15], and exponentially fitted methods (see Jackson and Kenue [17], Cash [6]).

In what follows, we derive a continuous SSDM through interpolation and collocation (see Lie and Norsett [22], Atkinson [2], Onumanyi et al [24], and Gladwell and Sayers [12]) which is used to obtain the main discrete SSDM and one additional method for solving (1). We note that this concept of combining the main and additional methods for solving (1) is due to Brugnano and Trigiante [4]. We emphasize that the main discrete SSDM and one additional method generated from the continuous representation are combined and used as a block method to simultaneously produce approximations $\{y_{n+1}, y_{n+2}\}$ to the exact solutions $\{y(x_{n+1}), y(x_{n+2})\}$, where the set of points $x_n = a + nh, 0(1)N$ belong to the partition

$$\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_N = b$$

$h = \frac{b-a}{N}$ is the constant step-size.

In order to apply the block method at the next block to obtain y_{n+3}, y_{n+4}, the only necessary starting value is y_{n+2}, and the loss of accuracy in y_{n+2}, does not affect subsequent points, thus the order of the algorithm is maintained.

It is unnecessary to make a function evaluation at the initial part of the new block. Thus, at all blocks except the first, the first function evaluation is already available from the previous block. The method preserves the Runge-kutta traditional advantage of being self-starting and is more accurate since it is implemented as a block method (see Milne [23], Sarafyan [26], Rosser [25],and Shampine and Watts [27]).

The paper is organized as follows. In Section 2, we obtain a continuous representation $u(x)$ for the exact solution $y(x)$ which is used to generate the main discrete SSDM and one additional method for solving (1). The analysis and implementation of the method are discussed in Section 3. Numerical examples are given in Section 4 to show the efficiency of the method. Finally, the conclusion of the paper is discussed in Section 5.

2. Development of Method

In this section, our objective is to derive the main SSDM and an additional method.

The main method is of the form

$$y_{n+2} - y_n = h \sum_{j=0}^{2} \beta_j f_{n+j} + h^2 \sum_{j=0}^{2} \gamma_j g_{n+j} \tag{2}$$

where β_j and γ_j are unknown constants. We note that y_{n+j} is the numerical approximation to the analytical solution $y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y(x_{n+j}))$, $j = 0, 1, 2$, and

$$g_{n+j} = \frac{df(x, y(x))}{dx}\Big|^{x_{\mathsf{n+j}}}_{y_{\mathsf{n+j}}}, \quad j = 0, 1, 2.$$

The additional method is of the form

$$y_{n+1} - y_n = h \sum_{j=0}^{2} \beta_j f_{n+j} + h^2 \sum_{j=0}^{2} \gamma_j g_{n+j} \tag{3}$$

Continuous approximation. In order to specify (2) and (3), we seek a continuous representation of the SSDM to approximate the exact solution $y(x)$. We assume that the solution of (1) is locally represented in the range $[x_n, x_{n+2}]$ by the interpolating function

$$u(x) = \sum_{j=0}^{6} \ell_j x^j \tag{4}$$

where ℓ_j are unknown coefficients to be determined. We then construct our continuous SSDM by imposing the following conditions.

- The interpolating function (4) coincides with the analytical solution at the point x_n
- The function (4) satisfies the differential equation (1) at the points $x_{n+j}, j = 0, 1, 2$
- The second derivative of (4) coincides with the second derivative of the analytical solution at the points $x_{n+j}, j = 0, 1, 2$

These conditions produce the following set of 7 equations

$$u\left(x_n\right) = y_n, \tag{5}$$

$$u^{\prime}\left(x_{n+j}\right) = f_{n+j}, \quad j = 0, 1, 2 \tag{6}$$

$$u^{\prime\prime}\left(x_{n+j}\right) = g_{n+j}, \quad j = 0, 1, 2, \tag{7}$$

which is solved to obtain ℓ_j. Our continuous SDAM is constructed by substituting the values of ℓ_j into equation (4). After some manipulation, our continuous approximation is expressed in the form

$$u(x) = y_n + h\sum_{j=0}^{2}\beta_j(x)f_{n+j} + h^2\sum_{j=0}^{2}\gamma_j(x)g_{n+j}, \tag{8}$$

where $\beta_j(x)$ and $\gamma_j(x)$ are continuous coefficients. Thus, evaluating (8) at $x = \{x_{n+2}, x_{n+1}\}$, (2) and (3) are specified as follows.

$$y_{n+2} = y_n + \frac{h}{15}(7f_n + 16f_{n+1} + 7f_{n+2}) + \frac{h^2}{15}(g_n - g_{n+2}), \tag{9}$$

$$y_{n+1} = y_n + \frac{h}{240}(101f_n + 128f_{n+1} + 11f_{n+2}) + \frac{h^2}{240}(13g_n - 40g_{n+1} - 3g_{n+2}). \tag{10}$$

3. Block Method

In the spirit of Baker and Keech [3], a block-by-block method is a method for computing vectors $Y_0, Y_1, \ldots$ in sequence. Let the ν-vector (ν is the number of points within the block) Y_μ, F_μ, and G_μ, for $n = m\nu$, $m = 0, 1, \ldots$ be given as $Y_\mu = (y_{n+1}, \ldots, y_{n+\nu})^T$, $F_\mu = (f_{n+1}, \ldots, f_{n+\nu})^T$, $G_\mu = (g_{n+1}, \ldots, g_{n+\nu})^T$, then the l-block ν-point methods for (1) are given by

$$Y_\mu = \sum_{i=1}^{k} A^{(i)} Y_{\mu-i} + h \sum_{i=0}^{k} B^{(i)} F_{\mu-i} + h^2 \sum_{i=0}^{k} C^{(i)} G_{\mu-i} \qquad (11)$$

where $A^{(i)}$, $B^{(i)}$, $C^{(i)}$, $i = 0, \ldots, k$ are ν by ν matrices (see Fatunla[10]).

Let the theoretical solution of (1) be represented by

$$Z_\mu = \begin{pmatrix} y(x_{n+1}) \\ y(x_{n+2}) \\ \vdots \\ y(x_{n+\nu}) \end{pmatrix}$$

Definition 3.1. The local truncation error (LTE) of the block method (11) is given by the vector E_μ as follows:

$$E_\mu = Z_\mu - \sum_{i=1}^{k} A^{(i)} Z_{\mu-i} + h \sum_{i=0}^{k} B^{(i)} Z_{\mu-i} + h^2 \sum_{i=0}^{k} C^{(i)} Z_{\mu-i} \qquad (12)$$

Definition 3.2. The block method (11) has error order $p \geq 1$ provided there exists a constant C such that the LTE E_μ satisfies

$$\|E_\mu\| = Ch^{p+1} + O(h^{p+2}),$$

where $\| \cdot \|$ is the maximum norm.

Definition 3.3. The block method (11) is zero stable provided the roots R_j, $j = 1, \ldots, k$ of the first characteristic polynomial $\rho(R)$ specified by

$$\rho(R) = \det[\sum_{i=0}^{k} A^{(i)} R^{k-i}] = 0, A^{(0)} = -I \qquad (13)$$

satisfies $|R_j| \leq 1, j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1.

Definition 3.4. The block method (11) is said to be consistent if it has order at least one.

Block SSDM ($\nu = 2, l = 1$. The methods (9) and (10) can be rearranged in order to assume the form 13) as follows.

$$A^{(0)}Y_\mu = A^{(1)}Y_{\mu-1} + h[B^{(0)}F_\mu + B^{(1)}F_{\mu-1}] + h^2[C^{(0)}G_\mu + C^{(1)}G_{\mu-1}], \quad (14)$$

where:

$$Y_\mu = (y_{n+1}, y_{n+2})^T, \quad Y_{\mu-1} = (y_{n-1}, y_n)^T,$$
$$F_\mu = (f_{n+1}, f_{n+2})^T, \quad F_{\mu-1} = (f_{n-1}, f_n)^T,$$
$$G_\mu = (g_{n+1}, g_{n+2})^T, \quad G_{\mu-1} = (g_{n-1}, g_n)^T,$$

for $\mu = 1, \dots$ and $n = 0, 2, \dots, N-2$, and the matrices $A^{(0)}$, $A^{(1)}$, $B^{(0)}$, $B^{(1)}$,$C^{(0)}$, and $C^{(1)}$ are matrices of dimension 2 defined as follows: $A^{(0)}$ is an identity matrix of dimension 2,

$$A^{(1)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{(0)} = \begin{pmatrix} 8/15 & 11/240 \\ 16/15 & 7/15 \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} 0 & 101/240 \\ 0 & 7/15 \end{pmatrix},$$

$$C^{(0)} = \begin{pmatrix} -1/6 & -1/80 \\ 0 & -1/15 \end{pmatrix}, \quad C^{(1)} = \begin{pmatrix} 0 & 13/240 \\ 0 & 1/15 \end{pmatrix}$$

Local truncation error. Following Fatunla [10] and Lambert [21] we define the local truncation error associated with normalized form of (14) to be the linear difference operator

$$L[Z(x);h] = \sum_{j=0}^{k} \{\alpha_j Z(x+jh) - hZ'\beta_j(x+jh) - h^2\gamma_j Z''(x+jh)\} \quad (15)$$

Assuming that $Z(x)$ is sufficiently differentiable, we can expand the terms in (15) as a Taylor series about the point x to obtain the expression

$$L[Z(x);h] = C_0 Z(x) + C_1 hZ'(x) + \ldots + C_q h^q Z^{(q)}(x) + \ldots, \quad (16)$$

where the constant coefficients C_q , $q = 0, 1, \dots$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j,$$

$$C_2 = \frac{1}{2}\sum_{j=1}^{k} j^2\alpha_j - \sum_{j=1}^{k} j\beta_j - \sum_{j=0}^{k}\gamma_j,$$

$$\vdots$$

$$C_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q\alpha_j - q\sum_{j=1}^{k} j^{q-1}\beta_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\gamma_j].$$

According to Henrici [16], we say that the method (14) has order p if

$$C_0 = C_1 = \ldots = C_p = 0, \quad C_{p+1} \neq 0.$$

Therefore, C_{p+1} is the error constant (EC) and $C_{p+1}h^{p+1}Z^{(p+1)}(x_n)$ the principal local truncation error at the point x_n.

Order and error constant. The block method (14) has order and error constant given by the vectors $p = (6,6)^T$ and $C_6 = (1/9450, 1/4725)^T$.

Zero-stability. It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (14) tends to the difference system

$$A^0Y_\mu - A^1Y_{\mu-1} = 0$$

whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(RA^0 - A^1) = R(R-1). \tag{17}$$

Following Fatunla[10], the block method (14) is zero-stable, since from (17), $\rho(R) = 0$ satisfy $|R_j| \leq 1$, $j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1. The block method (14) is consistent as it has order $p > 1$. According to Henrici[16], we can safely assert the convergence of the block method (14).

Linear-stability. A-stability is discussed in the spirit of [4], [15] where we consider the usual test equations

$$y' = \lambda y, \qquad y'' = \lambda^2 y$$

which are applied to (14) to yield

$$Y_\mu = M(q)Y_{\mu-1}\ ,\ q = \lambda h, \tag{18}$$

where the matrix $M(q)$ is given by

$$M(q) = (A^{(0)} - qB^{(0)} - q^2C^{(0)})^{-1}(A^{(1)} + qB^{(1)} + q^2C^{(1)}),$$

and $A^{(i)}, B^{(i)}, C^{(i)}$, $i = 0, 1$ are matrices.

We obtain the property of A-stability from (18) , which requires that for all $q \in \mathbb{C}^-$ and $Re(q) < 0$, $M(q)$ must have a dominant eigenvalue λ_2 such that

$$|\lambda_2| < 1.$$

Our calculations show that the matrix M(q) has eigenvalues $\{\lambda_1, \lambda_2\} = \{0, \lambda_2\}$, where the dominant eigenvalue λ_2 is a function of q given by

$$\lambda_2 = \frac{90 + 90q + 39q^2 + 9q^3 + q^4}{90 - 90q + 39q^2 - 9q^3 + q^4} \tag{19}$$

It is obvious from (19) that for $Re(q) < 0$, $|\lambda_2| < 1$.

Hence, the block method (14) is A-stable since its region of absolute stability contains the left half-complex plane $\{q \in \mathbb{C} | Re(q) < 0\}$ (see the unshaded region in Figure 1). Therefore, there is no restriction on λh, which makes (14) a viable candidate for stiff problems.

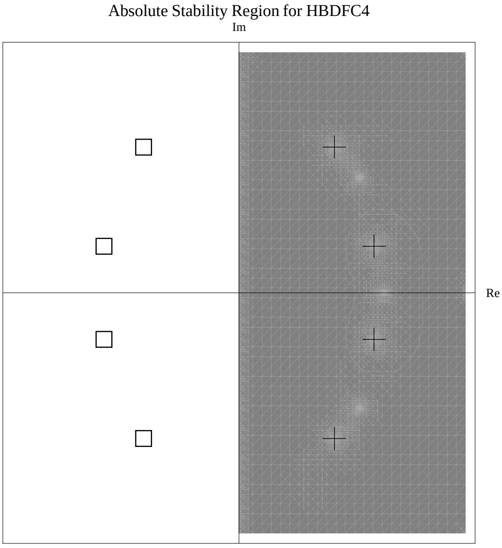

Figure 1: Stability region for the block SSMD

Implementation. Our method is implemented more efficiently by combining methods (9) and (10) as simultaneous integrators in the form (14) for IVPs without requiring starting values and predictors. We proceed by explicitly obtaining initial conditions at $x_{n+2}, n = 0, 2, \ldots, N-2$ using the computed values

$u(x_{n+2}) = y_{n+2}$ over sub-intervals $[x_0, x_2], \ldots, [x_{N-2}, x_N]$. For instance, $n = 0$, $\mu = 1$, $(y_1, y_2)^T$ are simultaneously obtained over the sub-interval $[x_0, x_2]$, as y_0 is known from the IVP, for $n = 2$, $\mu = 2$, $(y_3, y_4)^T$ are simultaneously obtained over the sub-interval $[x_2, x_4]$, as y_2 is known from the previous block, and so on. Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate than those obtained in the conventional way. We note that for linear problems, we solve (1) directly from the start with Gaussian elimination using partial pivoting and for nonlinear problems, we use the modified Newton-Raphson method.

4. Numerical Examples

In this section, we give four numerical examples to illustrate the accuracy of the method. We find absolute and relative errors of the approximate solution on the partition π_N as $|y - y(x)|$ and $(|y - y(x)|/(1 + |y(x)|))$ respectively. The rate of convergence is calculated using the formula $RC = \log_2(E^{2h}/E^h)$, where E^{2h} is the absolute maximum error $|y(x) - y|$ using the step size $2h$ and E^h is the absolute maximum error $|y(x) - y|$ using the step size h. All computations were carried out using our written Mathematica code in Mathematica 8.0.

Example 4.1. We consider the following IVP which was also solved by Cash [6] and Jackson and Kenue [17] on the range $0 \leq x \leq 1$.

$$y' = -y + 95z, \;\; y(0) = 1,$$

$$z' = -y - 97z, \;\; z(0) = 1.$$

$$\text{Exact} : y(x) = \frac{95}{47}e^{-2x} - \frac{48}{47}e^{-96x} \;,\; z(x) = \frac{48}{47}e^{-96x} - \frac{1}{47}e^{-2x}$$

The errors in the solution were obtained at $x = 1$ using our method for fixed step-sizes as shown in table 1. Similar results were obtained in [6] and [17] and are reproduced in table 1. It is seen that our method is more accurate than those in [6] and [17]. This is interesting to note that the Simpson's Rule which is a bad method for IVPs has been adapted via SSDM to be competitive with these methods in [6] and [17].

Example 4.2. As our second test example, we solve the given linear system on the range $0 \leq x \leq 1$

$$y_1' = -21y_1 + 19y_2 - 20y_3, \;\; y_1(0) = 1,$$

$$y_2' = 19y_1 - 21y_2 + 20y_3, \;\; y_2 = 0,$$

		$y(1)$	$z(1) \times 10^2$
Step	Method	$(\lvert error \rvert)$	$(\lvert error \rvert) \times 10^2$
0.0625	Jackson-Kenue	0.2735503	-0.2879477
		(3×10^{-7})	(4×10^{-7})
	Cash $(p = 4)$	0.2735498	-0.2879471
		(3×10^{-7})	(3×10^{-7})
	Cash $(p = 5)$	0.27355005	-0.28794742
		(1×10^{-8})	(1×10^{-8})
	SSDM $(p = 6)$	0.27355004	-0.28794740
		(9×10^{-11})	(1×10^{-8})
0.03125	Jackson-Kenue	0.27355005	-0.28794742
		(1×10^{-8})	(1×10^{-8})
	Cash $(p = 4)$	0.27355003	-0.28794740
		(1×10^{-8})	(1×10^{-8})
	SSDM $(p = 6)$	0.27355004	-0.28794741
		(4×10^{-12})	(4×10^{-12})
	True solution	0.27355004	$-0.28794741 \times 10^{-2}$

Table 1: A comparison of methods for Example 4.1

$$y_3' = 40y_1 - 40y_2 + 40y_3, \;\; y_3 = -1.$$

The exact solution of the system is given by

$$y_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x))),$$

$$y_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x))),$$

$$y_3(x) = \frac{1}{2}(2e^{-40x}(\sin(40x) - \cos(40x))).$$

This problem has also been solved by Amodio and Mazzia [1] using the Boundary Value Methods (BVMs), implicit Adams methods (Ad-IVMs), and Backward Differentiation Formulas (BDFs). The results for their order 6 methods are reproduced in table 2 and compared with the results given by the SSDM which is also of order 6. It is seen from table 2 that our method performs better than those in [1]. In all cases the rate of convergence is consistent with the order of the methods. Thus, for this example, our method is superior in terms of accuracy.We note that the maximum relative errors displayed in Table 2 are computed as max $(|y - y(x)|/(1 + |y(x)|))$.

Steps	SSDM	Rate	BVM	Rate	Ad-IVM	Rate	BDF	rate
20	2.9×10^{-3}		5.7×10^{-2}		1.6×10^{2}		2.0×10^{-1}	
40	6.8×10^{-5}	5.4	8.7×10^{-3}	2.7	1.8×10^{-1}		2.6×10^{-1}	
80	1.8×10^{-6}	5.2	4.9×10^{-4}	4.2	3.5×10^{-4}		2.6×10^{-3}	
160	2.9×10^{-8}	5.9	1.2×10^{-5}	5.4	9.9×10^{-6}	5.1	9.1×10^{-5}	4.8
320	4.6×10^{-10}	6.0	2.2×10^{-7}	5.8	2.0×10^{-7}	5.7	1.8×10^{-6}	5.6
640	7.4×10^{-12}	6.0	3.7×10^{-9}	5.9	3.4×10^{-9}	5.8	3.3×10^{-8}	5.8

Table 2: A comparison of methods for Example 4.2

Example 4.3. We consider the following IVP which was solved by Wu and Xia [28].

$$y_1' = -1002y_1 + 1000y_2^2,\ y_1(0) = 1,$$

$$y_2' = y_1 - y_2(1 + y_2),\ y_2(0)) = 1.$$

$$\text{Exact} : y_1(x) = e^{-2x}\ ,\ y_2(x) = e^{-x}$$

	Wu and Xia [28]				ASDM			
x	h	N	$Err(y_1)$	$Err(y_2)$	h	N	$Err(y_1)$	$Err(y_2)$
1	0.002	500	2.5606×10^{-7}	8.0150×10^{-8}	0.04	250	1.3112×10^{-13}	1.7186×10^{-13}
10	0.001	10000	5.5468×10^{-16}	6.0936×10^{-12}	0.02	500	1.3235×10^{-22}	1.4162×10^{-18}

Table 3: Absolute Errors,($Err(y_1) = |y(x) - y_1|$, $Err(y_2) = |y(x) - y_2|$), for Example 4.3

It is obvious from the numerical results in table 3 that our method performs very well for smaller step sizes $h = \{0.04, 0.02\}$ compared with the method in Wu and Xia [28] where step sizes $h = \{0.002, 0.001\}$ were used.

Example 4.4. We consider the given non-linear system on the range $0 \leq x \leq 48$.

$$y_1' = -0.013y_2 - 1000y_1y_2 - 2500y_1y_3,\ y_1(0) = 0,$$

$$y_2' = -0.013y_2 - 1000y_1y_2,\ y_2(0) = 1,$$

$$y_3' = -2500y_1y_3,\ y_3(0) = 1.$$

Step	$y_1(2)$ (Rel Err)	$y_2(2)$ (Rel Err)	$y_3(2)$ (Rel Err)
1/8	$-9.837251127012 \times 10^{-7}$ (2.63320×10^{-6})	0.981800332370 (1.50057×10^{-4})	1.018198684005 (1.46002×10^{-4})
1/16	$-2.631974292087 \times 10^{-6}$ (9.84955×10^{-7})	0.981552379782 (2.49230×10^{-5})	1.018444988244 (2.39783×10^{-5})
1/32	$-3.597666669215 \times 10^{-6}$ (1.92664×10^{-8})	0.981507192987 (2.11868×10^{-6})	1.018489209346 (2.07031×10^{-6})
1/64	$-3.616934539598 \times 10^{-6}$ $(1.37030 \times 10^{-12})$	0.981503257729 (1.32680×10^{-7})	1.018493125336 (1.30250×10^{-7})
True solution	$-3.616933169289 \times 10^{-6}$	0.9815029948230	1.018493388244

Step	$y_1(48)$ (Rel Err)	$y_2(48)$ (Rel Err)	$y_3(48)$ (Rel Err)
1/8	$-1.946451478612 \times 10^{-6}$ (1.11252×10^{-9})	0.611815992635 (4.77025×10^{-4})	1.388182060913 (3.21694×10^{-4})
1/16	$-1.945530753856 \times 10^{-6}$ $(1.91797 \times 10^{-10})$	0.611096683455 (3.05393×10^{-5})	1.388901371015 (2.05950×10^{-5})
1/32	$-1.9453510082 \times 10^{-6}$ $(1.20513 \times 10^{-11})$	0.611050574647 (1.91894×10^{-6})	1.388947480002 (1.29409×10^{-6})
1/64	$-1.945339708518 \times 10^{-6}$ $(7.51709 \times 10^{-13})$	0.611047675979 (1.19695×10^{-7})	1.388950378680 (8.07196×10^{-8})
True solution	$-1.945338956808 \times 10^{-6}$	0.6110474831446	1.388950571516

Table 4: Relative errors computed as Rel Err= $(|y - y(x)|/(1 + |y(x)|))$ for Example 4.4

For this example, we give the relative errors at $x = 2$ and $x = 48$. Our method performs well on this example as shown by the results in table 4. The exact solutions at $x = 2$ and $x = 48$ were taken from Jeltsch[18]. Although, Jeltsch[18] also gives the relative errors for this problem, we chose not to make a comparison with the methods because the results were presented graphically with a different emphasis.

5. Conclusion

In this paper we show that the SSDM can be adapted using the logic behind the Simpson's Rule for quadrature to cope with the integration of stiff systems in ODEs. This is achieved by combining the SSDM with an additional method and implementing them as a block method. We have demonstrated the accuracy of the methods on a linear stiff 2 by 2 system, a linear stiff 3 by 3 system, a nonlinear 2 by 2 system, and a nonlinear 3 by 3 system (see tables 1 - 4). The numerical results show that our method is highly competitive with the existing methods cited in this paper.

References

[1] P. Amodio and F. Mazzia, Boundary value methods based on Adams, *Applied Numerical Mathematics*, **18** (1995), 23-35.

[2] K. E. Atkinson, *An introduction to numerical analysis*, 2nd edition John Wiley and Sons, New York (1989).

[3] C. T. H. Baker and M. S. Keech, Stability regions in the numerical treatment of Volterra integral equations, *SIAM J. Numer. Anal.*, **15** (1978), 394-417.

[4] L. Brugnano and D. Trigiante, *Solving Differential Problems by Multistep Initial and Boundary Value Methods*, Gordon and Breach Science Publishers, Amsterdam (1998).

[5] J. C. Butcher, A modified multistep method for the numerical integration of ordinary differential equations, *J. Assoc. Comput. Mach.*, **12** (1965), 124-135.

[6] J. R. Cash, On the exponential fitting of composite multiderivative linear multistep methods, *SIAM J. Numer. Anal.*, **18** (1981), 808-821.

[7] G. G. Dahlquist, Numerical integration of ordinary differential equations, *Math. Scand.*, **4** (1956), 69-86.

[8] G. Dahlquist, A special stability problem for linear multistep methods, *BIT.*, **3** (1963), 27-43.

[9] W. H. Enright, Second Derivative Multistep Methods for Stiff ordinary differential equations, *SIAM J. Numer. Anal.*, **11** (1974), 321-331.

[10] S. O. Fatunla, Block methods for second order IVPs, *Intern. J. Comput. Math.*, **41** (1991), 55-63.

[11] C. W. Gear, Hybrid methods for initial value problems in ordinary differential equations, *SIAM J. Numer. Anal.*, **2** (1965), 69-86.

[12] I. Gladwell and D. K. Sayers, Eds., *Computational techniques for ordinary differential equations*, Academic Press, New York (1976).

[13] W. Gragg and H. J. Stetter, Generalized multistep predictor-corrector methods, *J. Assoc. Comput. Mach.*, **11** (1964), 188-209.

[14] G. K. Gupta, Implementing second-derivative multistep methods using Nordsieck polynomial representation, *Math. Comp.*, **32** (1978), 13-18.

[15] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II*, Springer, New York (1996), .

[16] P. Henrici, *Discrete Variable Methods in ODEs* , John Wiley, New York (1962).

[17] L. W. Jackson and S. K. Kenue, A fourth order exponentially fitted method, *SIAM J. Numer. Anal.*, **11** (1974), 965-978.

[18] R. Jeltsch, Multistep Methods Using Higher Derivatives and Damping at Infinity, *Math Comp.*, **31** (1977), 124-138.

[19] J. J. Kohfeld and G. T. Thompson, Multistep methods with modified predictors and correctors, *J. Assoc. Comput. Mach.*, **14** (1967), 155-166.

[20] J. D. Lambert, *Numerical methods for ordinary differential systems*, John Wiley, New York (1991).

[21] J. D. Lambert, *Computational methods in ordinary differential equations*, John Wiley, New York (1973).

[22] I. Lie and S. P. Norsett,1989, Superconvergence for Multistep Collocation, *Math Comp.*, **52** (1989), 65-79.

[23] W. E. Milne, *Numerical solution of differential equations*, John Wiley and Sons (1953).

[24] P. Onumanyi, D. O. Awoyemi, S. N. Jator, and U. W. Sirisena, 1994, New linear mutlistep methods with continuous coefficients for first order initial value problems, *J. Nig. Math. Soc.*, **13** (1994), 37-51.

[25] J. D. Rosser, A Runge-kutta for all seasons, *SIAM*, **Rev. 9** (1967), 417-452.

[26] D.Sarafyan, Multistep methods for the numerical solution of ordinary differential equations made self-starting, Tech. Report 495, *Math. Res. Center*, Madison (1965).

[27] L. F. Shampine and H. A. Watts, Block implicit one-step methods, *Math. Comp.*, **23** (1969), 731-740.

[28] X.-.Y Wu. and J.-L. Xia, Two low accuracy methods for stiff systems, *Appl. Math. Comp.*, **123** (2001), 141-153.