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A NOTE ON SOLUTIONS OF SEMILINEAR EQUATIONS IN BANACH SPACES

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Abstract: In this article, we investigate the existence and uniqueness of mild solutions of semilinear equations by using the theory of analytic semigroups. We modify the main result obtained in [1] by imposing additional condition on the nonlinear term.

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1. Introduction

This article studies the existence and uniqueness of mild solutions to the semilinear equation

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \qquad t_0 < t < t_1 \\ u(t_0) &= x_0 \end{aligned} \tag{1.1}$$

where A generates a C_0 semigroup $T(t)$ [10], [11] on a Banach space X and f is a continuous function defined on an open subset U of $[0,\infty) \times X_\alpha$, $0 < \alpha < 1$, taking values in X. In [1], the existence and uniqueness of mild solutions to (1.1) was established under the following conditions:

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(i) A generates an analytic C_0 semigroup $T(t)$ on X satisfying

$$T(t) \leq Me^{\omega t} \tag{1.2}$$

for $\omega \geq 0$, $M \geq 1$ and $0 \leq t < \infty$.

(ii) $f(t,x)$ is locally H*ölder* continuous in t and locally Lipschitz continuous in x. That is, for each point (t,x) of U, there exist a neighbourhood $V \subset U$, constants $L = L(t,x,V) > 0$ and $0 < \gamma \leq 1$ such that

$$\|f(s_1,y_1) - f(s_2,y_2)\| \leq L\left(|s_1 - s_2|^\gamma + \|y_1 - y_2\|_\alpha\right),$$

for $(s_1,y_1) \in V, (s_2,y_2) \in V$, where X is a real or complex Banach space with norm $\|\cdot\|$, A^α is a closed linear, invertible operator with domain $D(A^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$ of A^α with

$$\|x\|_\alpha = \left(\|x\|^2 + \|A^\alpha x\|^2\right)^{1/2}, \qquad x \in D(A^\alpha).$$

In this note, we assume, in addition to f being Lipschitz and Hölder continuous that

$$\|f(t,x)\| \leq K(t)\left(1 + \|x\|_\alpha\right)$$

for all $(t,x) \in U$, where $K(\cdot)$ is a continuous function on (τ,∞).

McBride [10] proved the existence and uniqueness of solution to the nonhomogeneous initial value problem

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad 0 < t < \mathbf{T} \\ u(0) &= u_0 \end{aligned} \tag{1.3}$$

under the assumptions that A generates a C_0 semigroup $T(t)$ satisfying (1.2) and $f : [0,\mathbf{T}) \longrightarrow X$ in (1.3) is a continuously differentiable function.

The existence and uniqueness of solution of (1.3) was proved in [14] with $f : [0,\mathbf{T}) \longrightarrow X$ Lipschitz and Hölder continuous and A generates C_0 semigroups satisfying (1.2).

Our work in this paper is organized as follows: Section 2 reviews some results on fractional powers of operators with values in a Banach space. In Section 3, we state some technical lemmas and then state and prove our main result.

2. Preliminary Results

We define fractional powers of operators of certain unbounded linear operators [7], [8], [11] and we concentrate mainly on fractional powers of operators for which A is the infinitesimal generator of an analytic semigroup $T(t)$. The results obtained here will be used in Section 3.

For $\alpha > 0$ we define the fractional power $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt$$

with $A^\alpha = (A^{-\alpha})^{-1}$ since $A^{-\alpha}$ is one to one. For $0 < \alpha \leq 1$, A^α is a closed and densely defined linear operator (i.e., the domain $D(A^\alpha) \supset D(A)$ is dense in X) and $X_\alpha =$ the space $D(A^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$ of A^α with

$$\|x\|_\alpha = \left(\|x\|^2 + \|A^\alpha x\|^2\right)^{1/2}, \qquad x \in D(A^\alpha).$$

Proposition 2.1. *(i) The space X_α is a Banach space.*
(ii) The graph norm $\|x\|_\alpha$ is equivalent to the norm $\|A^\alpha x\|$.
(iii) If $0 < \alpha < \beta < 1$, then we have $X_\beta \subset X_\alpha$ with continuous injection.

Proof. (i) Assume that $\{x_n\}$ is a Cauchy sequence in X_α, that is,

$$\{x_n\} \text{ is a Cauchy sequence in } X, \quad \{A^\alpha x_n\} \text{ is a Cauchy sequence in } X.$$

Then there exist elements $x, y \in X$ such that

$$x_n \to x \text{ in } X, \quad A^\alpha x_n \to y \text{ in } X.$$

Hence, by the closedness of A^α, we have

$$x \in D(A^\alpha), \quad A^\alpha x = y.$$

This proves that

$$x \in X_\alpha, \quad x_n \to x \text{ in } X_\alpha.$$

(ii) Recall that

$$\left\|A^{-\alpha}\right\| \leq M, \qquad 0 < \alpha < 1. \tag{2.1}$$

Hence we have for all $x \in A^\alpha$

$$\begin{aligned}\|A^{\alpha}x\|^2 \le \|x\|^2 + \|A^{\alpha}x\|^2 &= \left(\|x\|^2 \|A^{\alpha}x\|^{-2} + 1\right) \|A^{\alpha}x\|^2 \\ &\le \left(\|x\|^2 \left\|A^{-\alpha}\right\|^2 \|x\|^{-2} + 1\right) \|A^{\alpha}x\|^2 \\ &\le (M^2+1) \|A^{\alpha}x\|^2 ,\end{aligned}$$

so that

$$\|A^{\alpha}x\| \le \|x\|_{\alpha} \le (M^2+1)^{1/2} \|A^{\alpha}x\| , \qquad x \in X_{\alpha}. \tag{2.2}$$

This proves part (ii).

(iii) We remark that

$$A^{\alpha} = A^{-(\beta-\alpha)} A^{\beta}.$$

Hence we have $D(A^{\beta}) \subset D(A^{\alpha})$. Furthermore, in view of inequaliites (2.1) and (2.2), it follows that, for all $x \in X_{\alpha} = D(A^{\alpha})$,

$$\begin{aligned}\|x\|_{\alpha} &\le (M^2+1)^{1/2} \|A^{\alpha}x\| \\ &\le (M^2+1)^{1/2} \left\|A^{-(\beta-\alpha)} A^{\beta}x\right\| \\ &\le M(M^2+1)^{1/2} \left\|A^{\beta}x\right\| \\ &\le M(M^2+1)^{1/2} \|x\|_{\beta} \qquad \left(\left\|A^{\beta}x\right\| \le \|x\|_{\beta}\right).\end{aligned}$$

This proves part (iii). □

Lemma 2.2. *Let A be the infinitesimal generator of an analytic semigroup $T(t)$. If $0 \in \rho(A)$ then*

(i) for $t > 0$, $\alpha \ge 0$ the operator $A^{\alpha}T(t)$ is bounded and

$$\|A^{\alpha}T(t)\| \le C_{\alpha} t^{-\alpha} e^{-\delta t}$$

(ii) for $0 < \alpha \le 1$ and $x \in D(A^{\alpha})$, we have

$$\|(T(t) - I)x\| \le C_{\alpha} t^{\alpha} \|A^{\alpha}x\| .$$

Proof. See [1] □

3. Main Results

Definition 3.1. A function $u(t) : [t_0, t_1) \longrightarrow X$ is a (classical) solution of (1.1) on $[t_0, t_1)$ if u is continuous on $[t_0, t_1)$, continuously differentiable on (t_0, t_1), $u(t) \in D(A)$ for $t_0 < t < t_1$ and (1.1) is satisfied.

We state the following lemmas before proving our main result.

Lemma 3.2. *Let A be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying* (1.2) *and suppose f is a locally Hölder continuous function on $[0, \mathbf{T})$, with exponent $0 < \gamma \leq 1$, which satisfies the condition*

$$\int_0^{\mathbf{T}} \|f(s)\| \, ds < \infty. \tag{3.1}$$

Then for every $u_0 \in X$, the function

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \tag{3.2}$$

belongs to the space $C\left([0, \mathbf{T}], X\right) \cap C^1\left((0, \mathbf{T}), X\right)$, and is a unique solution of (1.3).

Proof. By [10], it suffices to consider the function

$$v(t) = \int_0^t T(t-s)f(s)ds, \qquad 0 < t < \mathbf{T}. \tag{3.3}$$

So by (1.2) we have

$$\|v(t)\| \leq \int_0^t \|T(t-s)\| \, \|f(s)\| \, ds \leq M \int_0^t \|f(s)\| \, ds,$$

so that

$$\lim_{t \downarrow 0} v(t) = 0.$$

(a) We show the continuity of $v(t)$. Now, for each small $\tau > 0$, we let

$$v_\tau(t) = \begin{cases} 0 & 0 \leq t \leq \tau, \\ \int_0^{t-\tau} T(t-s)f(s)ds & \tau < t < \mathbf{T}. \end{cases}$$

Then we have

$$\|v_\tau(t) - v(t)\| \le \begin{cases} \int_0^t \|T(t-s)\| \, \|f(s)\| \, ds & 0 \le t \le \tau, \\ \\ \int_{t-\tau}^t \|T(t-s)\| \, \|f(s)\| \, ds & \tau < t < \mathbf{T}. \end{cases}$$

So

$$\begin{aligned} \|v_\tau(t) - v(t)\| &\le \int_0^\tau \|T(t-s)\| \, \|f(s)\| \, ds + \int_{t-\tau}^t \|T(t-s)\| \, \|f(s)\| \, ds \\ &\le M \left(\int_0^\tau \|f(s)\| \, ds + \int_{t-\tau}^t \|f(s)\| \, ds \right). \end{aligned}$$

Thus, it follows from condition (3.3) that

$$\lim_{\tau \downarrow 0} v_\tau(t) = v(t) \qquad \text{uniformly in } t \in [0, \mathbf{T}]. \tag{3.4}$$

But we have for $0 \le t < t+h \le \mathbf{T}$ (setting $f(s) = 0$ for $s < 0$)

$$\begin{aligned} \|v_\tau(t+h) - v_\tau(t)\| &= \int_0^{t+h-\tau} T(t+h-s) f(s) ds - \int_0^{t-\tau} T(t-s) f(s) ds \\ &= \int_{t-\tau}^{t+h-\tau} T(t+h-s) f(s) ds + \int_0^{t-\tau} T(t+h-s) f(s) ds \\ &\quad - \int_0^{t-\tau} T(t-s) f(s) ds \\ &= \int_{t-\tau}^{t+h-\tau} T(t+h-s) f(s) ds \\ &\quad + \int_0^{t-\tau} \left[T(t+h-s) - T(t-s) \right] f(s) ds \\ &= \int_{t-\tau}^{t+h-\tau} T(t+h-s) f(s) ds \\ &\quad + \int_0^{t-\tau} \left[T(h) T(t-s) - T(t-s) \right] f(s) ds \\ \|v_\tau(t+h) - v_\tau(t)\| &= \int_{t-\tau}^{t+h-\tau} T(t+h-s) f(s) ds \end{aligned}$$

$$+ [T(h) - I] \int_0^{t-\tau} T(t-s)f(s)ds.$$

So that as $h \downarrow 0$

$$\begin{aligned}\|v_\tau(t+h) - v_\tau(t)\| \leq M \int_{t-\tau}^{t+h-\tau} \|f(s)\| \, ds \\ + \left\| [T(h) - I] \int_0^{t-\tau} T(t-s)f(s)ds \right\| \\ \longrightarrow 0.\end{aligned}$$

This proves that

$$v_\tau(t) \in C([0, \mathbf{T}], X).$$

Hence, in view of (3.4), it follows that

$$v(t) \in C([0, \mathbf{T}], X).$$

(b) Next we show that

$$v(t) \in D(A) \qquad \text{for } 0 < t < \mathbf{T} \tag{3.5}$$

and

$$Av(t) = \int_0^t AT(t-s)(f(s) - f(t))ds + (T(t) - I)f(t). \tag{3.6}$$

Since we have by

$$\|AT(t)\| \leq M_1 \frac{e^{-\delta t}}{t} \tag{3.7}$$

and condition (3.3)

$$\begin{aligned}\int_0^{t-\tau} \|AT(t-s)(f(s))\|\, ds &\le M_1 \int_0^{t-\tau} \frac{1}{t-s}\, \|f(s)\|\, ds \\ &\le \frac{M_1}{\tau} \int_0^{\mathbf{T}} \|f(s)\|\, ds \\ &\le \infty,\end{aligned}$$

in view of the closedness of A it follows that

$$\begin{aligned} v_\tau(t) &= \int_0^{t-\tau} T(t-s)f(s)ds \in D(A) \qquad \text{for} \qquad \tau < t < \mathbf{T}, \qquad \text{and} \\ Av_\tau(t) &= \int_0^{t-\tau} AT(t-s)(f(s)ds.\end{aligned}$$

We remark that

$$\begin{aligned} Av_\tau(t) &= \int_0^{t-\tau} AT(t-s)f(s)ds - \int_0^{t-\tau} AT(t-s)f(t)ds \\ &\quad + \int_0^{t-\tau} AT(t-s)f(t)ds \\ &= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + \int_0^{t-\tau} AT(t-s)f(t)ds \\ &= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds - \int_0^{t-\tau} \frac{d}{ds}\left(T(t-s)f(t)\right) ds \\ &= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds - \left(T(\tau)-T(t)\right) f(t) \\ &= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + \left(T(t)-T(\tau)\right) f(t).\end{aligned}$$

Let $[a,b]$ be an arbitrary closed interval of $(0,\mathbf{T})$, and let L be a H*ölder* constant for the function f on the interval $[a/2,b]$:

$$\|f(t)-f(s)\| \le L\,|t-s|^\gamma\,, \qquad t,s \in [a/2,b].$$

Then, by estimate (3.7), we have for $0 < \tau' < \tau < a/2$

$$\begin{aligned}\left\|\int_{t-\tau}^{t-\tau'} AT(t-s)(f(s)-f(t))ds\right\| &\leq M_1 L \int_{t-\tau}^{t-\tau'} (t-s)^{\gamma-1}ds \\ &= \frac{M_1 L}{\gamma}(\tau^\gamma - \tau'^\gamma), \qquad t \in [a,b].\end{aligned}$$

This proves that the improper integral

$$\int_0^t AT(t-s)(f(s)-f(t))ds = \lim_{\tau\downarrow 0}\int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds \tag{3.8}$$

exists, and the convergence is uniform in $t \in [a,b] \subset (0, \mathbf{T})$.

Hence, as $\tau \downarrow 0$ we have

$$\begin{aligned}Av_\tau(t) &= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + (T(t)-T(\tau))f(t) \\ &\longrightarrow \int_0^t AT(t-s)(f(s)-f(t))ds \\ &+ (T(t)-I)f(t) \\ &= Av(t), \qquad 0 < t < \mathbf{T}.\end{aligned} \tag{3.9}$$

Therefore, (3.5)-(3.6) follows from (3.4) and (3.9).

(c) Finally we show that

$$\begin{cases} v(t) \in C^1((0,\mathbf{T}), X), \\ v'(t) = Av(t) + f(t), \quad 0 < t < \mathbf{T}. \end{cases}$$

First we remark that the function $v_\tau(t)$ is continuously differentiable on the interval $(\tau, \mathbf{T})$ and satisfies

$$v'_\tau(t) = \int_0^{t-\tau} \frac{d}{dt}(T(t-s)f(s))ds + T(\tau)f(t-\tau)$$

$$= \int_0^{t-\tau} AT(t-s)f(s)ds + T(\tau)f(t-\tau),$$

since the semigroup $T(t)$ is real analytic for $t > 0$. Thus we have for $\tau < t < \mathbf{T}$

$$\begin{aligned}
v'_\tau(t) = &\int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + T(\tau)f(t-\tau) \\
&+ \int_0^{t-\tau} AT(t-s)f(t)ds \\
&= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + T(\tau)f(t-\tau) \\
&- \int_0^{t-\tau} \frac{d}{ds}(T(t-s)f(t))ds \\
&= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + T(\tau)f(t-\tau) - (T(\tau)-T(t))f(t) \\
&= \int_0^{t-\tau} AT(t-s)(f(s)-f(t))ds + T(\tau)(f(t-\tau)-f(t)) + T(t)f(t).
\end{aligned}$$

But, by (1.2), we have for any closed interval $[a,b]$ of $(0,\mathbf{T})$

$$\|T(\tau)(f(t-\tau)-f(t))\| \le ML\tau^\gamma, \qquad t \in [a,b], \tag{3.10}$$

where $0 < \tau < a/2$ and $L > 0$ is a Hölder constant for the function f on the interval $[a/2,b]$.

Therefore, combining (3.8) and (3.10), we find that as $\tau \downarrow 0$

$$v'_\tau(t) \longrightarrow \int_0^t AT(t-s)(f(s)-f(t))ds + T(t)f(t) = Av(t) + f(t),$$

uniformly in t over closed intervals of $(0,\mathbf{T})$.

Thus one can let $\tau \downarrow 0$ in the formula

$$v'_\tau(t) = \int_\epsilon^t v'_\tau(t)d\tau + v'_\tau(\epsilon), \qquad 0 < \tau < \epsilon,$$

to obtain that

$$v_\tau(t) = \int_\epsilon^t (Av(t) + f(\tau))d\tau + v(\epsilon), \qquad 0 < \epsilon \leq t < \mathbf{T}.$$

Since ϵ is arbitrary, this proves that

$$v(t) \in C^1((0, \mathbf{T}), X),$$

and also

$$v'(t) = Av(t) + f(t), \qquad 0 < t < \mathbf{T}.$$

□

Lemma 3.3. *If $u(t)$ is a solution of (1.1) on (t_0, t_1), then*

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^t T(t - s)f(s, u(s))ds. \tag{3.11}$$

Conversely, if $u(t) : [t_0, t_1) \longrightarrow X_\alpha$ is a continuous function, and

$$\int_{t_0}^{t_0+\tau} \|f(s, u(s))\| \, ds < \infty$$

for some $\tau > 0$, and if the integral equation (3.11) holds for $t_0 < t < t_1$, then $u(t)$ is a solution of (1.1) on (t_0, t_1).

Proof. See [1] □

Definition 3.4. A continuous solution u of the integral equation (3.11) is called a *mild solution* of the initial value problem (1.1).

Lemma 3.5. ***Gronwall's inequality (integral form.)*** *Let $\Phi(t)$ be a nonnegative, summable function on $[0, \mathbf{T}]$ which satisfies for almost every t the integral inequality*

$$\Phi(t) \leq C_1 + C_2 \int_0^t \Phi(s)ds \tag{3.12}$$

for constants $C_1, C_2 \geq 0$. Then

$$\Phi(t) \leq C_1(1 + C_2 t e^{C_2 t}) \tag{3.13}$$

for almost every $0 \leq t \leq \mathbf{T}$.

Proof. See [6] $\square$

Lemma 3.6. *Let* $\eta(t,s) \geq 0$ *be continuous on* $0 \leq s < t \leq \mathbf{T}$. *If there are positive constants* N_1, N_2, α *such that*

$$\eta(t,s) \leq N_1 + N_2 \int_s^t (t-\rho)^{\alpha-1}\eta(\rho,s)d\rho \qquad \text{for } 0 \leq s < t \leq \mathbf{T} \tag{3.14}$$

then there is a constant C *such that*

$$\eta(t,s) \leq C \qquad \text{for } 0 \leq s < t \leq \mathbf{T}. \tag{3.15}$$

Proof. Iterating (3.14) $n-1$ times using the well known identity

$$\int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{\beta-1}d\tau = (t-s)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3.16}$$

and estimating $(t-s)$ by $\mathbf{T}$ we find

$$\eta(t,s) \leq N_1 \sum_{i=0}^{n-1}\left(\frac{N_2\mathbf{T}^\alpha}{\alpha}\right)^i + \frac{(N_2\Gamma(\alpha))^n}{\Gamma(n\alpha)}\int_s^t (t-\rho)^{n\alpha-1}\eta(\rho,s)d\rho.$$

Choosing n sufficiently large so that $n\alpha > 1$ and estimating $(t-\rho)^{n\alpha-1}$ by $\mathbf{T}^{n\alpha-1}$ we get

$$\eta(t,s) \leq C_1 + C_2\int_s^t \eta(\rho,s)d\rho$$

which by Lemma 3.5 implies

$$\eta(t,s) \leq C_1 e^{C_2(t-s)} \leq C_1 e^{C_2\mathbf{T}} \leq C.$$

Since C_1 and C_2 do not depend on s this estimate holds for $0 \leq s < t \leq \mathbf{T}$.

$\square$

The main result of this paper is the following theorem.

Theorem 3.7. *Let A be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$. Assume also that $U = (\tau, \infty) \times X_\alpha$, $f(t,x)$ is locally Hölder continuous in t, locally Lipschitz continuous in x for $(t,x) \in U$, and also*

$$\|f(t,x)\| \leq K(t)\left(1 + \|x\|_\alpha\right) \tag{3.17}$$

for all $(t,x) \in U$, where $K(\cdot)$ is continuous on (τ, ∞). If $t_0 > \tau$, $x_0 \in X_\alpha$, the initial value problem (1.1) has a unique solution u which exists for all $t \geq t_0$.

Proof. Applying Lemma 3.3 we can continue the solution (3.11) of the initial value problem (1.1) as long as $\|u(t)\|_\alpha$ remains bounded. It is therefore sufficient to show that if u exists on $[0, \mathbf{T})$ then $\|u(t)\|_\alpha$ is bounded as $t \uparrow \mathbf{T}$.

Since by (3.11)

$$\|u(t)\|_\alpha \leq \|T(t-t_0)x_0\|_\alpha + \int_{t_0}^{t} \|T(t-s)\|_\alpha \|f(s,u(s))\| \, ds.$$

It follows that

$$\begin{aligned}
\|u(t)\|_\alpha &\leq \|T(t-t_0)x_0\|_\alpha + \int_{t_0}^{t} \|T(t-s)\|_\alpha \cdot K(s)\left(1 + \|u(s)\|_\alpha\right) ds \\
&\leq M\|A^\alpha x_0\| + \int_{t_0}^{t} K(s)\,\|T(t-s)\|_\alpha \, ds \\
&+ \int_{t_0}^{t} K(s)\,\|T(t-s)\|_\alpha \,\|u(s)\|_\alpha \, ds \\
&\leq M\|A^\alpha x_0\| + \int_{t_0}^{t} K(s)\,\|A^\alpha T(t-s)\| \, ds \\
&+ \int_{t_0}^{t} K(s)\,\|A^\alpha T(t-s)\|\,\|u(s)\|_\alpha \, ds \\
&\leq M\|A^\alpha x_0\| + \frac{K(\mathbf{T})M_\alpha \mathbf{T}^{1-\alpha}}{1-\alpha} + K(\mathbf{T})M_\alpha \int_{t_0}^{t} (t-s)^{-\alpha}\,\|u(s)\|_\alpha \, ds
\end{aligned}$$

which by Lemma 3.6 implies that $\|u(t)\|_\alpha \leq C$ on $[0, \mathbf{T})$ and the proof is complete.

□

Remark 3.8. By the continuity of $u(t)$ and inequality

$$\|f(t,x) - f(s,y)\| \leq L\left(|t-s|^\gamma + \|x-y\|_\alpha\right),$$

for $(t,x) \in V, (s,y) \in V$, $L = L(t_0, x_0, V) > 0$ a local Hölder constant for the function f, it follows that $f(t,u(t))$ is continuous on $[t_0, t_0+\tau]$. Hence there exists a constant $N > 0$ such that

$$\|f(t,u(t))\|_\alpha \le N, \qquad t \in [t_0, t_0+\tau].$$

and therefore

$$\begin{aligned}
\left\| \int_{t_0}^t T(t-s) f(s,u(s)) ds \right\|_\alpha &\le \int_{t_0}^t \|T(t-s)\|_\alpha \|f(s,u(s))\|_\alpha ds \\
&= \int_{t_0}^t \|A^\alpha T(t-s)\| \|f(s,u(s))\|_\alpha ds \\
&\le N \int_{t_0}^t \|A^\alpha T(t-s)\| ds \\
&\le N M_\alpha \int_{t_0}^t (t-s)^{-\alpha} ds \\
&= \left(\frac{N M_\alpha}{1-\alpha} \right) (t-t_0)^{1-\alpha}.
\end{aligned}$$

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