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ON THE FEKETE-SZEGÖ PROBLEM FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract: In this paper, a class of analytic functions $\delta(r, \lambda, A, B)$ is introduced. The Fekete-Szegö problem for $\delta(r, \lambda, A, B)$ is discussed with the properties of analytic functions and skill of fundamental inequalities. The accurate value is obtained, and thus,some related result s are derived.

Key Words: analytic functions, subordinate, Fekete-Szegö problem

1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disc $U := \{z : |z| < 1\}$(for details, see [123]). Let T be the family of analytic functions $w(z)$ in Usatisfying the conditions $w(0) = 0, |w(z)| < 1$ for $z \in U$.Note that $f(z) \prec g(z)$ if there is a function such that $f(z) = g(w(z))$. A classical theorem of Fekete and Szegö [4] states that for $f(z) \in S$ given by (1),

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, & \mu \leq 0, \\ 1 + 2e^{-\frac{2\mu}{1-\mu}}, & 0 \leq \mu \leq 1, \\ 4\mu - 3, & \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each real μ there exists a function

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in S such that equality holds (see [5]-[7]).

In this paper,we reseach the Fekete-Szegö problem of a subclass of analytic functions $\delta(r,\lambda,A,B)$ in U which defined as following:

$$1+\frac{1}{r}\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right\}\prec\frac{1+Az}{1+Bz},$$

for some $0\le\lambda\le 1$, $-1\le B<A\le 1$, and r is given as real (see [8]).

Let $k-SP^{\alpha}$ denote the subclass of analytic function which satisfy the inequalities:

$$Re\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}\right\}>\left|k\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\right|.$$

for some α $(0\le\alpha\le 1)$ (for details, see [9], [10]). It is easy to check that $k-SP^{\alpha}\subset\delta(1,\lambda,A,B)$.

2. Main Results

To prove the main result, we need the following lemma.

Lemma. (see [4]) *Let $w(z)=d_1z+d_2z^2+\cdots$ be analytic in U and satisfy $|w(z)|\le|z|$ for $z\in U$, then*

$$|d_1|\le 1,|d_2|\le 1-|d_1|^2. \tag{2}$$

With the help of this lemma, we derive the following result.

Theorem. *Let $f(z)\in\delta(r,\lambda,A,B)$ and be given by (1). Then for real number μ,*

$$|a_3-ua_2^2|\le\begin{cases}\frac{(A-B)|r|}{2+4\lambda}\left[\frac{r(1+\lambda)^2-u(2+4\lambda)}{(1+\lambda)^2}(A-B)-B\right], & 0\le u\le\delta_1,\\ \frac{(A-B)|r|}{2+4\lambda}, & \delta_1<u\le\delta_3,\\ \frac{(A-B)|r|}{2+4\lambda}\left[B-\frac{r(1+\lambda)^2-u(2+4\lambda)}{(1+\lambda)^2}(A-B)\right], & u>\delta_3,\end{cases}$$

where $\delta_1=\frac{(1+\lambda)^2[r(A-B)-(B+1)]}{r(A-B)(2+4\lambda)}$ and $\delta_3=\frac{(1+\lambda)^2[r(A-B)-(B-1)]}{r(A-B)(2+4\lambda)}$.

Proof. Suppose that $f(z)\in\delta(r,\lambda,A,B)$ and there exists a analytic function $w(z)$ in U such that $h(z)\prec\frac{1+Az}{1+Bz}$, where

$$h(z)=1+\frac{1}{r}\{\frac{zf'(z)+\lambda z^2 f''(z)}{\lambda z f'(z)+(1-\lambda)f(z)}-1\}=1+c_1z+c_2z^2+\cdots.$$

Equating coefficents, we obtain that

$$(1+\lambda)a_2 = rc_1, (2+4\lambda)a_3 = r^2c_1^2 + rc_2. \tag{3}$$

Because $h(z) \prec \frac{1+Az}{1+Bz}$ there exists a function analytic in U satisfy $|w(z)| \leq |z|$ with $w(z) = d_1z + d_2z^2 + \cdots$ and $h(z) = \frac{1+Aw(z)}{1+Bw(z)}$.

Equating coefficents, we obtain that

$$c_1 = (A-B)d_1, \quad c_2 = (A-B)(d_2 - Bd_1^2). \tag{4}$$

Then with the help of the lemma, (3), and (4), we obtain

$$\begin{aligned}
|a_3 - ua_2^2| =& \left|\frac{(A-B)^2d_1^2r^2 + r(A-B)(d_2-Bd_1^2)}{2+4\lambda} - \frac{ur^2(A-B)^2d_1^2}{(1+\lambda)^2}\right| \\
=& \frac{(A-B)|r|}{2+4\lambda}\left|(A-B)d_1^2r + (d_2 - Bd_1^2) - \frac{ur(2+4\lambda)(A-B)d_1^2}{(1+\lambda)^2}\right| \\
\leq& \frac{(A-B)|r|}{2+4\lambda}\left[|d_2| + \left|\frac{r(1+\lambda)^2 - ur(2+4\lambda)}{(1+\lambda)^2}(A-B) - B\right||d_1|^2\right] \\
\leq& \frac{(A-B)|r|}{2+4\lambda}\left[1 - |d_1|^2 + \left|\frac{r(1+\lambda)^2 - ur(2+4\lambda)}{(1+\lambda)^2}(A-B) - B\right||d_1|^2\right].
\end{aligned}$$

Next, we will consider four cases to obtain the bound of $|a_3 - ua_2^2|$.

(1) If $0 \leq u \leq \delta_1 = \frac{(1+\lambda)^2[r(A-B)-(B+1)]}{r(A-B)(2+4\lambda)}$, then we have

$$|a_3 - ua_2^2| \leq \frac{(A-B)|r|}{2+4\lambda}\left[\frac{r(1+\lambda)^2 - u(2+4\lambda)}{(1+\lambda)^2}(A-B) - B\right].$$

The results are sharp by choosing $|d_1| = 1$, $|d_2| = 0$.

(2) If $\delta_1 < u \leq \delta_2 = \frac{(1+\lambda)^2[r(A-B)-B]}{r(A-B)(2+4\lambda)}$, then

$$|a_3 - ua_2^2| \leq \frac{(A-B)|r|}{2+4\lambda}.$$

The results are sharp by choosing $|d_1| = 0$, $|d_2| = 1$.

(3) If $\delta_2 < u \leq \delta_3 = \frac{(1+\lambda)^2[r(A-B)-(B-1)]}{r(A-B)(2+4\lambda)}$, then we have

$$|a_3 - ua_2^2| \leq \frac{(A-B)|r|}{2+4\lambda}.$$

The results are sharp by choosing $|d_1| = 0$, $|d_2| = 1$.

(4) If $u > \delta_3 = \frac{(1+\lambda)^2[r(A-B)-(B-1)]}{r(A-B)(2+4\lambda)}$, then we obtain that

$$|a_3 - ua_2^2| \leq \frac{(A-B)|r|}{2+4\lambda}\left[B - \frac{r(1+\lambda)^2 - u(2+4\lambda)}{(1+\lambda)^2}(A-B)\right].$$

The results are sharp by choosing $|d_1| = 1$, $|d_2| = 0$.

Therefore, the proof is complete. □

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