ON THE NONLINEAR WAVE EQUATION

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Abstract: In this paper, we study the nonlinear equation of the form
\[
\frac{\partial^2}{\partial t^2} u(x, t) - c^2 \Delta u(x, t) = f(x, t, u(x, t))
\]
with the initial conditions
\[
u(x, 0) \text{ and } \frac{\partial}{\partial t} u(x, 0) = \delta(x),
\]
where \( u(x, t) \) is an unknown for \((x, t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty), \mathbb{R}^n \) is the dimension of the Euclidean space, \( \Delta \) is the Laplacian operator defined by (1.3) and \( c \) is a positive constant, \( \delta(x) \) is the Dirac delta distribution, \( f \) is the given function in nonlinear form depending on \( x, t \) and \( u(x, t) \). By method of convolution in the distribution theory we obtain the solution of the nonlinear wave equation.

Key Words: ultra-hyperbolic, tempered distribution, Fourier transform

1. Introduction

It is well known that for the 1-dimensional wave equation
\[
\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t), \quad (1.1)
\]
we obtain \( u(x, t) = f(x + ct) + g(x - ct) \) as a solution of the equation where \( f \)
and $g$ are continuous.

Also for the $n$-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) + c^2 (-\Delta) u(x, t) = 0,$$

(1.2)

with the initial condition $u(x, 0) = f(x)$ and $\frac{\partial}{\partial t} u(x, 0) = g(x)$ where $\Delta$ is defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

(1.3)

$f$ and $g$ are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cos (2\pi|\xi|) t + \hat{g}(\xi) \frac{\sin (2\pi|\xi|) t}{2\pi|\xi|},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$ [see 1, p.177]. By using the inverse Fourier transform, we obtain $u(x, t)$ in the convolution form, that is

$$u(x, t) = f(x) \ast \Psi_t(x) + g(x) \ast \Phi_t(x)$$

(1.4)

where $\Phi_t$ is an inverse Fourier transform of $\hat{\Phi}_t(\xi) = \frac{\sin (2\pi|\xi|) t}{2\pi|\xi|}$ and $\Psi_t$ is an inverse Fourier transform of $\hat{\Psi}_t(\xi) = \cos (2\pi|\xi|) t = \frac{\partial}{\partial t} \hat{\Phi}(\xi)$.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2} u(x, t) - c^2 (\Delta) u(x, t) = f(x, t, u(x, t))$$

(1.5)

with the initial condition

$$u(x, 0) \text{ and } \frac{\partial}{\partial t} u(x, 0) = \delta(x),$$

where $\Delta$ is defined by (1.3), $c$ is a positive constant, $f$ and $g$ are continuous function function and absolute integrable. We consider the equation (1.5) with the following conditions on $u$ and $f$ as follows

(1) $u(x, t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$-derivatives.

(2) $f$ satisfies the Lipchitz condition,

$$|f(x, t, u) - f(x, t, w)| \leq A|u - w|$$

where $A$ is constant with $0 < A < 1$. 
(3) \( \int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))| \, dx \, dt < \infty \) for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( 0 < t < \infty \) and \( u(x,t) \) is continuous function on \( \mathbb{R}^n \times (0, \infty) \).

Under such conditions of \( f \) and \( u \), we obtain the convolution

\[
u(x,t) = E(x,t) \ast f(x,t,u(x,t))
\]

as a unique solution of (1.5) where \( E(x,t) \) is an elementary solution of (1.5).

Before going to that point the following definitions and some concepts are needed.

2. Preliminaries

**Definition 2.1.** Let \( f(x) \in L_1(\mathbb{R}^n) \) - the space of integrable function in \( \mathbb{R}^n \). The Fourier transform of \( f(x) \) is defined by

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x) \, dx
\]

where \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \), \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( (\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \) is the usual inner product in \( \mathbb{R}^n \) and \( dx = dx_1 \, dx_2 \cdots dx_n \). Also, the inverse of Fourier transform is defined by

\[
f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{f}(\xi) \, d\xi.
\]

If \( f \) is a distribution with compact supports by [3], Theorem 7.4-3, p.187 Eq.(2.1) can be written as

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle.
\]

**Definition 2.2.** Given \( P \) is a hypersurface then the generalized function \((p + i0)^{\lambda}\), \((p - i0)^{\lambda}\) can be expressed in term of \( P^{\lambda}_{+} \) and \( P^{\lambda}_{-} \) and is defined by

\[
(P + i0)^{\lambda} = P^{\lambda} + e^{\pi \lambda i} P^{\lambda}_{-}
\]

and

\[
(P - i0)^{\lambda} = P^{\lambda} + e^{-\pi \lambda i} P^{\lambda}_{+},
\]
where

\[
P_+^\lambda = \begin{cases} 
P^\lambda, & P > 0 \\
0, & p \leq 0
\end{cases}
\]

and

\[
P_-^\lambda = \begin{cases} 
P^\lambda, & P > 0 \\
0, & p \leq 0
\end{cases}
\]

**Lemma 2.1.** Let \( L \) be the operator defined by

\[
L = \frac{\partial}{\partial t^2} - c^2 \Delta
\]

where \( \Delta \) is the Laplacian operator and is defined by (1.3). Then we obtain

\[
E(x, t) = \begin{cases} 
\Gamma\left(\left(\frac{n+1}{2}\right)\right)(-1)^{\frac{n-1}{2}} c^{(n-1)}_\mu P_-^{\frac{(n-1)}{2}} \text{ for } n \text{ is even} \\
0, & \text{for } n \text{ is odd}
\end{cases}
\]

is the elementary solution of (2.4).

**Proof.** Let

\[
LE(x, t) = \delta(x, t),
\]

where \( E(x, t) \) is the elementary solution of the operator \( L \) and \( \delta \) is the Dirac-delta distribution. Thus

\[
\frac{\partial^2}{\partial t^2} E(x, t) - c^2 (\Delta) E(x, t) = \delta(x) \delta(t).
\]

(2.5)

Taking the Fourier transform defined by (2.1) to both sides of the equation of (2.5) we obtain

\[
\frac{\partial^2}{\partial t^2} \hat{E}(\xi, t) + c^2 |\xi|^2 \hat{E}(\xi, t) = \delta(t),
\]

(2.6)

where \( |\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 \). The solution of (2.6) is

\[
\hat{E}(\xi, t) = H(t) \psi(t),
\]

(2.7)
where $H(t)$ is the Heaviside function and $\psi(t)$ is a solution of homogeneous equation. Now, we are solving the solution of homogeneous equation. Given the homogeneous equation
\begin{equation}
\frac{\partial^2}{\partial t^2} \widehat{\psi}(\xi, t) + c^2 |\xi|^2 \widehat{\psi}(\xi, t) = 0, \tag{2.8}
\end{equation}
with the initial condition
\[ \widehat{u}(\xi, 0) = 0, \quad \frac{\partial}{\partial t} \widehat{u}(\xi, 0) = 1. \]
Then we obtain $\widehat{\psi}(\xi, t)$ as a solution of (2.8) and
\[ \widehat{\psi}(\xi, t) = \frac{\sin(c|\xi|t)}{c|\xi|} = \frac{1}{2ic|\xi|} \left( e^{ic|\xi|t} - e^{-ic|\xi|t} \right) \]
Thus the solution of (2.6) is
\[ \widehat{E}(\xi, t) = H(t) \frac{\sin(c|\xi|t)}{c|\xi|}, \tag{2.9} \]
where $H(t)$ is a heaviside function. By applying the inverse Fourier transform we obtain the solution $E(x, t)$. Since $\frac{\sin(c|\xi|t)}{c|\xi|}$ is tempered distribution but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform directly.
Let us defined
\[ \widehat{\psi^\epsilon}(\xi, t) = e^{-\epsilon|\xi|} \widehat{\psi}(\xi, t). \]
We see that $\widehat{\psi^\epsilon}(\xi, t) \in L_1(\mathbb{R}^n)$ and $\widehat{\psi^\epsilon}(\xi, t) \to \widehat{\psi}(\xi, t)$ uniformly as $\epsilon \to 0$. Thus
\begin{align*}
\widehat{\psi^\epsilon}(\xi, t) &= \frac{1}{2ic|\xi|} \left( e^{-c(\epsilon-it)|\xi|} - e^{-c(\epsilon+it)|\xi|} \right) \\
&= \frac{1}{2i} \int_{\epsilon-it}^{\epsilon+it} e^{-cs|\xi|} ds.
\end{align*} \tag{2.10}
Applying the inverse Fourier transform to (2.10), we obtain
\begin{align*}
\widehat{\psi^\epsilon}(x, t) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\psi^\epsilon}(\xi, t) d\xi \\
&= \frac{1}{2i(2\pi)^n} \int_{\epsilon-it}^{\epsilon+it} \int_{\mathbb{R}^n} e^{i\xi x - cs|\xi|} ds d\xi. \tag{2.11}
\end{align*}
By direct compute, we have
\[ \hat{\psi}^\epsilon(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2ci(1-n)\pi^{\frac{n-1}{2}}} \cdot \left( \frac{1}{(c^2(\epsilon + it)^2 + |x|^2)^{\frac{n-1}{2}}} - \frac{1}{(c^2(\epsilon - it)^2 + |x|^2)^{\frac{n-1}{2}}} \right), \]
where \(|x|^2 = x_1^2 + x_2^2 + \ldots + x_n^2\). Let \(\epsilon \to 0\), we obtain
\[ \hat{\psi}^\epsilon(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2ci(1-n)\pi^{\frac{n-1}{2}}} \cdot \left( (P + 0i)^{-\frac{n-1}{2}} - (P - 0i)^{-\frac{n-1}{2}} \right), \]
and \(P = P(x,t) = x_1^2 + x_2^2 + \ldots + x_n^2 - c^2t^2\). By definition 2.2, we obtain
\[ \psi(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2ci(1-n)\pi^{\frac{n-1}{2}}} (P)^{-\frac{n-1}{2}} \sin\pi \frac{(n-1)}{2}, \]
or
\[ \psi(x,t) = \begin{cases} 
\Gamma\left(\frac{n+1}{2}\right)(-1)^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}}}{P^{-\frac{n-1}{2}}} & \text{for } n \text{ is even} \\
0, & \text{for } n \text{ is odd}
\end{cases} \]
Thus
\[ E(x,t) = H(t)\psi(x,t) \]
\[ = H(t) \frac{\Gamma\left(\frac{n+1}{2}\right)}{2ci(1-n)\pi^{\frac{n-1}{2}}} (P)^{-\frac{n-1}{2}} \sin\pi \frac{(n-1)}{2} \]
\[ = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2ci(1-n)\pi^{\frac{n-1}{2}}} (P)^{-\frac{n-1}{2}} \sin\pi \frac{(n-1)}{2} \quad \text{for } t > 0 \quad (2.12) \]

\[ \square \]

**Definition 2.3.** We can extend \(E(x,t)\) to \(\mathbb{R}^n \times \mathbb{R}\) by setting
\[ E(x,t) = \begin{cases} 
\Gamma\left(\frac{n+1}{2}\right)(-1)^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}}}{P^{-\frac{n-1}{2}}} & \text{for } n \text{ is even} \\
0, & \text{for } n \text{ is odd}
\end{cases} \]
3. Main Results

**Theorem 3.1.** Given the nonlinear wave equation

\[
\frac{\partial^2}{\partial t^2} u(x, t) - c^2 \triangle u(x, t) = f(x, t, u(x, t)) \tag{3.1}
\]

for \((x, t) \in \mathbb{R}^n \times (0, \infty)\), \(k\) is a positive number and with the following conditions on \(u\) and \(f\) as follows:

1. \(u(x, t) \in C^{(2k)}(\mathbb{R}^n)\) for any \(t > 0\) where \(C^{(2k)}(\mathbb{R}^n)\) is the space of continuous function with 2\(k\)-derivative.

2. \(f\) satisfies the Lipchitz condition,

\[
|f(x, t, u) - f(x, t, w)| \leq A|u - w|
\]

where \(A\) is constant with \(0 < A < 1\).

3. \(\int_0^\infty \int_{\mathbb{R}^n} |f(x, t, u(x, t))| \, dx \, dt < \infty\) for \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\), \(0 < t < \infty\) and \(u(x, t)\) is continuous function on \(\mathbb{R}^n \times (0, \infty)\).

Then we obtain the convolution

\[
u(x, t) = E(x, t) \ast f(x, t, u(x, t)) \tag{3.2}
\]

as a unique solution of (3.1) for \(x \in \Omega\) where \(\Omega\) is a compact subset of \(\mathbb{R}^n\) and \(0 \leq t \leq T\) with \(T\) is constant and \(E(x, t)\) is an elementary solution defined by (2.12) and also \(u(x, t)\) is bounded for any fixed \(t > 0\).

**Proof.** Convolving both sides of (3.1) with \(E(x, t)\), that is

\[
E(x, t) \ast \left[ \frac{\partial}{\partial t} u(x, t) - c^2 \triangle u(x, t) \right] = E(x, t) \ast f(x, t, u(x, t))
\]

or

\[
\left[ \frac{\partial}{\partial t} E(x, t) - c^2 \triangle E(x, t) \right] \ast u(x, t) = E(x, t) \ast f(x, t, u(x, t)),
\]

so

\[
\delta(x, t) \ast u(x, t) = E(x, t) \ast f(x, t, u(x, t)).
\]

Thus

\[
u(x, t) = E(x, t) \ast f(x, t, u(x, t))
\]
\[
= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r, s) f(x - r, t - s, u(x - r, t - s)) dr ds
\]
where \( E(r, s) \) is given by definition (2.4). We next show that \( u(x, t) \) is bounded on \( \mathbb{R}^n \times (0, \infty) \). We have
\[
|u(x, t)| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r, s)||f(x - r, t - s, u(x - r, t - s))| dr ds
\]
by condition (3) and (2.12)
\[
N = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |f(x - r, t - s, u(x - r, t - s))| dr ds.
\]
Thus \( u(x, t) \) is bounded on \( \mathbb{R}^n \times (0, \infty) \). To show that \( u(x, t) \) is unique.

Now, We next to show that \( u(x, t) \) is unique. Let \( w(x, t) \) be another solution of (3.1), then
\[
w(x, t) = E(x, t) * f(x, t, w(x, t))
\]
for \((x, t) \in \Omega_0 \times (0, T]\) the compact subset of \( \mathbb{R}^n \times [0, \infty) \) and \( E(x, t) \) is defined by (2.6).
\[
|Lu(x, t) - Lw(x, t)| \leq A|u(x, t) - w(x, t)|. \quad (3.3)
\]
Let \( \Omega_0 \times (0, T] \) the compact subset of \( \mathbb{R}^n \times [0, \infty) \) and \( L : C^{(2k)}(\Omega_0) \to C^{(2k)}(\Omega_0) \)
for \( 0 \leq t \leq T \)

Now \( (C^{(2k)}(\Omega_0), \|\|) \) is a Banach space where \( u(x, t) \in C^{(2k)}(\Omega_0) \) for \( 0 \leq t \leq T \) and \( \|\| \) is given by
\[
\|u(x, t)\| = \sup_{x \in \Omega_0, 0 \leq t \leq T} |u(x, t)|.
\]
Then, from (2) with \( 0 < A < 1 \), the operator \( L \) is a contraction mapping on \( C^{(2k)}(\Omega_0) \). Since \( (C^{(2k)}(\Omega_0), \|\|) \) is a Banach space and \( L : C^{(2k)}(\Omega_0) \to C^{(2k)}(\Omega_0) \) is a contraction mapping on \( C^{(2k)}(\Omega_0) \), by Contraction Theorem [2, p.300], we obtain the operator \( L \) has a fixe point and has uniqueness property. Thus \( u(x, t) = w(x, t) \). It follows that the solution \( u(x, t) \) of (3.1) is unique for \((x, t) \in \Omega_0 \times (0, T]\) where \( u(x, t) \) is defined by (3.2). That is complete of proof.

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References


