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ON THE NONLINEAR WAVE EQUATION

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Abstract: In this paper, we study the nonlinear equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\triangle u(x,t) = f(x,t,u(x,t))$$

with the initial conditions

$$u(x,0) \quad and \quad \frac{\partial}{\partial t}u(x,0) = \delta(x),$$

where $u(x,t)$ is an unknown for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$,$\mathbb{R}^n$ is the dimension of the Euclidean space, $\triangle$ is the Laplacian operator defined by (1.3) and c is a positive constant, $\delta(x)$ is the Dirac delta distribution, f is the given function in nonlinear form depending on x, t and $u(x,t)$. By method of convolution in the distribution theory we obtain the solution of the nonlinear wave equation.

Key Words: ultra-hyperbolic, tempered distribution, Fourier transform

1. Introduction

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \tag{1.1}$$

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation where f

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and g are continuous.

Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(-\triangle)u(x,t) = 0, \tag{1.2}$$

with the initial condition $u(x,0) = f(x)$ and $\frac{\partial}{\partial t}u(x,0) = g(x)$ where $\triangle$ is defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}. \tag{1.3}$$

f and g are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(2\pi|\xi|)\,t + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|)\,t}{2\pi|\xi|},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ [see 1, p.177]. By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \tag{1.4}$$

where Φ_t is an inverse Fourier transform of $\widehat{\Phi}_t(\xi) = \dfrac{\sin(2\pi|\xi|)\,t}{2\pi|\xi|}$ and Ψ_t is an inverse Fourier transform of $\widehat{\Psi}_t(\xi) = \cos(2\pi|\xi|)\,t = \dfrac{\partial}{\partial t}\widehat{\Phi}(\xi)$.

In this paper, we study the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2(\triangle)u(x,t) = f(x,t,u(x,t)) \tag{1.5}$$

with the initial condition

$$u(x,0) \;\; and \;\; \frac{\partial}{\partial t}u(x,0) = \delta(x),$$

where $\triangle$ is defined by (1.3), c is a positive constant, f and g are continuous function function and absolute integrable.We consider the equation (1.5) with the following conditions on u and f as follows

(1) $u(x,t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$-derivatives.

(2) f satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A is constant with $0 < A < 1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0, \infty)$.

Under such conditions of f and u, we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{1.6}$$

as a unique solution of (1.5) where $E(x,t)$ is an elementary solution of (1.5).

Before going to that point the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1. *Let $f(x) \in L_1(\mathbb{R}^n)$ - the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by*

$$\widehat{f(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x) dx \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$. Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f(\xi)} d\xi. \tag{2.2}$$

If f is a distribution with compact supports by [3], Theorem 7.4-3, p.187 Eq.(2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle. \tag{2.3}$$

Definition 2.2. Given P is a hypersurface then the generalized function $(p + i0)^\lambda$,
$(p - i0)^\lambda$ can be expressed in term of P_+^λ and P_-^λ and is defined by

$$(P + i0)^\lambda = P^\lambda + e^{\pi\lambda i} P_-^\lambda$$

and

$$(P - i0)^\lambda = P^\lambda + e^{-\pi\lambda i} P_-^\lambda,$$

where

$$P_+^\lambda = \begin{cases} P^\lambda, & P > 0 \\ \\ 0, & p \leq 0 \end{cases}$$

and

$$P_-^\lambda = \begin{cases} P^\lambda, & P > 0 \\ \\ 0, & p \leq 0 \end{cases}$$

Lemma 2.1. *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t^2} - c^2 \triangle \tag{2.4}$$

where $\triangle$ is the Laplacian operator and is defined by(1.3). Then we obtain

$$E(x,t) = \begin{cases} \frac{\Gamma\left(\frac{(n+1)}{2}\right)(-1)^{\frac{n}{2}-1}}{2ci(n-1)\pi^{\frac{n+1}{2}}} P_-^{-\frac{(n-1)}{2}} & \text{for } n \text{ is even} \\ \\ 0, & \text{for } n \text{ is odd} \end{cases}$$

is the elementary solution of (2.4).

Proof. Let

$$LE(x,t) = \delta(x,t),$$

where $E(x,t)$ is the elementary solution of the operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial^2}{\partial t^2} E(x,t) - c^2(\triangle)E(x,t) = \delta(x)\delta(t). \tag{2.5}$$

Taking the Fourier transform defined by (2.1) to both sides of the equation of (2.5) we obtain

$$\frac{\partial^2}{\partial t^2}\widehat{E(\xi,t)} + c^2|\xi|^2\widehat{E(\xi,t)} = \delta(t), \tag{2.6}$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$. The solution of (2.6) is

$$\widehat{E(\xi,t)} = H(t)\psi(t), \tag{2.7}$$

where $H(t)$ is the Heaviside function and $\psi(t)$ is a solution of homogeneous equation. Now, we are solving the solution of homogeneous equation. Given the homogeneous equation

$$\frac{\partial^2}{\partial t^2}\widehat{\psi(\xi,t)} + c^2|\xi|^2\widehat{\psi(\xi,t)} = 0, \tag{2.8}$$

with the initial condition

$$\widehat{u}(\xi,0) = 0 \quad , \quad \frac{\partial}{\partial t}\widehat{u}(\xi,0) = 1.$$

Then we obtain $\widehat{\psi}(\xi,t)$ as a solution of (2.8) and

$$\begin{aligned}\widehat{\psi}(\xi,t) &= \frac{sin(c|\xi|t)}{c|\xi|}\\ &= \frac{1}{2ic|\xi|}\left(e^{ic|\xi|t} - e^{-ic|\xi|t}\right)\end{aligned}$$

Thus the solution of (2.6) is

$$\widehat{E}(\xi,t) = H(t)\frac{sin(c|\xi|t)}{c|\xi|}, \tag{2.9}$$

where $H(t)$ is a heaviside function. By applying the inverse Fourier transform we obtain the solution$E(x,t)$. Since $\frac{sin(c|\xi|t)}{c|\xi|}$ is tempered distribution but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform directly.

Let us defined

$$\widehat{\psi}^{\epsilon}(\xi,t) = e^{-c\epsilon|\xi|}\widehat{\psi}(\xi,t).$$

We see that $\widehat{\psi}^{\epsilon}(\xi,t) \in L_1(\mathbb{R}^n)$ and $\widehat{\psi}^{\epsilon}(\xi,t) \to \widehat{\psi}(\xi,t)$ uniformly as $\epsilon \to 0$. Thus

$$\begin{aligned}\widehat{\psi}^{\epsilon}(\xi,t) &= \frac{1}{2ic|\xi|}\left(e^{-c(\epsilon-it)|\xi|} - (e^{-c(\epsilon+it)|\xi|}\right)\\ &= \frac{1}{2i}\int_{\epsilon-it}^{\epsilon+it} e^{-cs|\xi|}ds.\end{aligned} \tag{2.10}$$

Applying the inverse Fourier transform to (2.10), we obtain

$$\begin{aligned}\widehat{\psi}^{\epsilon}(x,t) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\psi}^{\epsilon}(\xi,t)d\xi\\ &= \frac{1}{2i(2\pi)^n}\int_{\epsilon-it}^{\epsilon+it}\int_{\mathbb{R}^n} e^{i\xi x - cs|\xi|}ds d\xi.\end{aligned} \tag{2.11}$$

By direct compute, we have

$$\widehat{\psi}^{\epsilon}(x,t)=\frac{\Gamma(\frac{n+1}{2})}{2ci(1-n)\pi^{\frac{(n-1)}{2}}} \cdot\left(\frac{1}{(c^2(\epsilon+it)^2+|x|^2)^{\frac{(n-1)}{2}}}-\frac{1}{(c^2(\epsilon-it)^2+|x|^2)^{\frac{(n-1)}{2}}}\right),$$

where $|x|^2=x_1^2+x_2^2+\ldots+x_n^2$. Let $\epsilon\to 0$, we obtain

$$\widehat{\psi}^{\epsilon}(x,t)=\frac{\Gamma(\frac{n+1}{2})}{2ci(1-n)\pi^{\frac{(n-1)}{2}}}\cdot\left((P+0i)^{-\frac{(n-1)}{2}}-(P-0i)^{-\frac{(n-1)}{2}}\right),$$

and $P=P(x,t)=x_1^2+x_2^2+\ldots+x_n^2-c^2t^2$. By definition 2.2, we obtain

$$\psi(x,t)=\frac{\Gamma(\frac{n+1}{2})}{2ci(1-n)\pi^{\frac{(n-1)}{2}}}(P)_-^{-\frac{(n-1)}{2}}sin\pi\frac{(n-1)}{2},$$

or

$$\psi(x,t)=\begin{cases}\frac{\Gamma\left(\frac{(n+1)}{2}\right)(-1)^{\frac{n}{2}-1}}{2ci(n-1)\pi^{\frac{n+1}{2}}}P_-^{-\frac{(n-1)}{2}} & \text{for } n \text{ is even}\\ 0, \quad \text{for } n \text{ is odd}\end{cases}$$

Thus

$$\begin{aligned}E(x,t)&=H(t)\psi(x,t)\\&=H(t)\frac{\Gamma(\frac{n+1}{2})}{2ci(1-n)\pi^{\frac{(n-1)}{2}}}(P)_-^{-\frac{(n-1)}{2}}sin\pi\frac{(n-1)}{2}\\&=\frac{\Gamma(\frac{n+1}{2})}{2ci(1-n)\pi^{\frac{(n-1)}{2}}}(P)_-^{-\frac{(n-1)}{2}}sin\pi\frac{(n-1)}{2}\quad for\ \ t>0\end{aligned}\tag{2.12}$$

□

Definition 2.3. *We can extend* $E(x,t)$ *to* $\mathbb{R}^n\times\mathbb{R}$ *by setting*

$$E(x,t)=\begin{cases}\frac{\Gamma\left(\frac{(n+1)}{2}\right)(-1)^{\frac{n}{2}-1}}{2ci(n-1)\pi^{\frac{n+1}{2}}}P_-^{-\frac{(n-1)}{2}} & \text{for } n \text{ is even}\\ 0, \quad \text{for } n \text{ is odd}\end{cases}$$

3. Main Results

Theorem 3.1. *Given the nonlinear wave equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\triangle u(x,t) = f(x,t,u(x,t)) \tag{3.1}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$*,* k *is a positive number and with the following conditions on* u *and* f *as follows*

(1) $u(x,t) \in C^{(2k)}(\mathbb{R}^n)$ *for any* $t > 0$ *where* $C^{(2k)}(\mathbb{R}^n)$ *is the space of continuous function with* $2k$*-derivative.*

(2) f *satisfies the Lipchitz condition,*

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A *is constant with* $0 < A < 1$*.*

(3) $\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dxdt < \infty$ *for* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*,* $0 < t < \infty$ *and* $u(x,t)$ *is continuous function on* $\mathbb{R}^n \times (0,\infty)$*.*

Then we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.2}$$

as a unique solution of (3.1) for $x \in \Omega$ *where* Ω *is a compact subset of* $\mathbb{R}^n$ *and* $0 \leq t \leq T$ *with* T *is constant and* $E(x,t)$ *is an elementary solution defined by (2.12) and also* $u(x,t)$ *is bounded for any fixed* $t > 0$*.*

Proof. Convolving both sides of (3.1) with $E(x,t)$, that is

$$E(x,t) * \left[\frac{\partial}{\partial t}u(x,t) - c^2\triangle u(x,t)\right] = E(x,t) * f(x,t,u(x,t))$$

or

$$\left[\frac{\partial}{\partial t}E(x,t) - c^2\triangle E(x,t)\right] * u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

so

$$\delta(x,t) * u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Thus

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

$$= \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} E(r,s)f(x-r,t-s,u(x-r,t-s))drds$$

where $E(r,s)$ is given by definition (2.4). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned}|u(x,t)| &\leq \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))|drds\\ &\leq \frac{\Gamma(\frac{n+1}{2})N}{2c(1-n)\pi^{\frac{(n-1)}{2}}} \qquad \text{by condition (3) and (2.12)}\end{aligned}$$

where $N = \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |f(x-r,t-s,u(x-r,t-s))|drds$. Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. To show that $u(x,t)$ is unique.

Now, We next to show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (3.1), then

$$w(x,t) = E(x,t) * f(x,t,w(x,t))$$

for $(x,t) \in \Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $E(x,t)$ is defined by (2.6).

$$|Lu(x,t) - Lw(x,t)| \leq A|u(x,t) - w(x,t)|. \tag{3.3}$$

Let $\Omega_0 \times (0,T]$ the compact subset of $\mathbb{R}^n \times [0,\infty)$ and $L : C^{(2k)}(\Omega_0) \to C^{(2k)}(\Omega_0)$ for$0 \leq t \leq T$

Now $\left(C^{(2k)}(\Omega_0), ||..||\right)$ is a Banach space where $u(x,t) \in C^{(2k)}(\Omega_0)$ for $0 \leq t \leq T$and $||..||$ is given by

$$\|u(x,t)\| = \sup_{\substack{x\in\Omega_0\\ 0<t\leq T}} |u(x,t)|.$$

Then, from (2) with $0 < A < 1$, the operator L is a contraction mapping on $C^{(2k)}(\Omega_0)$. Since $\left(C^{(2k)}(\Omega_0), ||..||\right)$ is a Banach space and $L : C^{(2k)}(\Omega_0) \to C^{(2k)}(\Omega_0)$ is a contraction mapping on $C^{(2k)}(\Omega_0)$, by Contraction Theorem [2, p.300], we obtain the operator L has a fixe point and has uniqueness property. Thus $u(x,t) = w(x,t)$. It follows that the solution $u(x,t)$ of (3.1) is unique for $(x,t) \in \Omega_0 \times (0,T]$ where $u(x,t)$ is defined by (3.2). That is complete of proof.

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