International Journal of Pure and Applied Mathematics
Volume 81 No. 4 2012, 605-608
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

ON THE DIOPHANTINE EQUATION $3^x + 5^y = z^2$

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Abstract: In this paper, we show that the Diophantine equation $3^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is $(1, 0, 2)$.

AMS Subject Classification: 11D61
Key Words: exponential Diophantine equation

1. Introduction

In 1844, Catalan [2] posed a conjecture that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^{\mathsf{x}} - b^{\mathsf{y}} = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. In 2004, Mihailescu [4] proved the Catalan's conjecture.

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^{\mathsf{x}} + 5^{\mathsf{y}} = z^2$ where x, y and z are non-negative integers. The proof uses the Catalan's conjecture.

In 2011, Suvarnamani, Singta and Chotchaisthit [6] proved that the two Diophantine equations $4^{\mathsf{x}} + 7^{\mathsf{y}} = z^2$ and $4^{\mathsf{x}} + 11^{\mathsf{y}} = z^2$ have no non-negative integer solution. The proof uses the Catalan's conjecture.

In the same year, Suvarnamani [5] found some non-negative integer solutions for the Diophantine equation of type $2^{\mathsf{x}} + p^{\mathsf{y}} = z^2$ where p is a positive prime number. He showed that if $p = 2$, then there are infinitely many solutions for the equation. The proof uses the Catalan's conjecture.

Received: July 17, 2012

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In 2012, Chotchaisthit [3] found all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number. The proof uses the Catalan's conjecture.

In this paper, we will use the the Catalan's conjecture to show that $(1, 0, 2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

In this section, we use the Catalan's conjecture to prove the two Lemmas.

Proposition 2.1. [4] $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. $(1, 2)$ *is a unique solution* (x, z) *for the Diophantine equation* $3^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $3^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 3^x + 1 \geq 3^1 + 1 = 4$. Then $z \geq 2$. Now, we consider on the equation $z^2 - 3^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z = 2$. Hence, $(1, 2)$ is a unique solution (x, z) for the equation $3^x + 1 = z^2$ where x and z are non-negative integers. □

Lemma 2.3. *The Diophantine equation* $1 + 5^y = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 5^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 5^y \geq 1 + 5^1 = 6$. Then $z \geq 3$. Now, we consider on the equation $z^2 - 5^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 6$. This is a contradiction. Hence, the equation $1 + 5^y = z^2$ has no non-negative integer solution. □

3. Results

Theorem 3.1. $(1, 0, 2)$ *is a unique solution* (x, y, z) *for the Diophantine equation* $3^x + 5^y = z^2$ *where* x, y *and* z *are non-negative integers.*

Proof. Let x, y and z be non-negative integers such that $3^x + 5^y = z^2$. By Lemma 2.3, we have $x \geq 1$. Now, we divide the number y into two cases.

Case $y \geq 1$. Note that z is even. Then $z^2 \equiv 0 \pmod 4$. Since $5 \equiv 1 \pmod 4$, it follows that $3^x \equiv 3 \pmod 4$. Thus, x is odd. Then $3^x \equiv 2 \pmod 5$ or $3^x \equiv 3 \pmod 5$. We obtain that $z^2 \equiv 2 \pmod 5$ or $z^2 \equiv 3 \pmod 5$. This is impossible.

Case $y = 0$. By Lemma 2.2, we have $x = 1$ and $z = 2$.

Therefore, $(1, 0, 2)$ is a unique solution (x, y, z) for the equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. □

Corollary 3.2. *The Diophantine equation* $3^x + 5^y = w^4$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and w such that $3^x + 5^y = w^4$. Let $z = w^2$. Then $3^x + 5^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (1, 0, 2)$. Then $w^2 = z = 2$. This is a contradiction. Hence, the equation $3^x + 5^y = w^4$ has no non-negative integer solution. □

4. Open Problem

We note in our results that 3 and 5 are odd prime numbers such that $5 - 3 = 2$. Let p and q be positive odd prime numbers such that $q - p = 2$. We may ask for the set of all solutions (x, y, z) for the Diophantine equation $p^x + q^y = z^2$ where x, y and z are non-negative integers.

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