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ON THE DIOPHANTINE EQUATION $31^x + 32^y = z^2$

Banyat Sroysang

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University
Rangsit Center, Pathumthani 12121, THAILAND

Abstract: In this paper, we prove that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

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1. Introduction

In 2007, Acu [1] proved that $(3, 0, 3)$ and $(2, 1, 3)$ are only two solutions (x, y, z) for the Diophantine equation $2^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2011, Suvarnamani [6] studied some non-negative integer solutions for the Diophantine equation of type $2^x + p^y = z^2$ where p is a positive prime number. He showed that $(3, 0, 3)$ is a solution for the Diophantine equation $2^x + 19^y = z^2$ where x, y and z are non-negative integers. In 2012, Peker and Cenberci [5] suggested that $(3, 0, 3)$ is a unique solution for the Diophantine equation $2^x + 19^y = z^2$ where x, y and z are non-negative integers. But the proof is not completed. In the same year, Chotchaisthit [3] found all non-negative integer solutions for the Diophantine equation of type $4^x + p^y = z^2$ where p is a positive prime number.

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In 2011, Suvarnamani, Singta and Chotchaisthit [7] proved that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no non-negative integer solution. In this paper, we will show that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

2. Preliminaries

In 1844, Catalan [2] conjectures that $(3, 2, 2, 3)$ is a unique solution (a, b, x, y) for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$. In 2004, Mihailescu [4] showed that the Catalan's conjecture is true. In this section, we will use the Catalan's conjecture to prove two Lemmas.

Proposition 2.1. [4] $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers with* $\min\{a, b, x, y\} > 1$.

Lemma 2.2. *The Diophantine equation* $31^x + 1 = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x and z such that $31^x + 1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $z^2 = 31^x + 1 \geq 31^1 + 1 = 32$. Then $z \geq 6$. Now, we consider on the equation $z^2 - 31^x = 1$. By Proposition 2.1, we have $x = 1$. Then $z^2 = 32$. This is a contradiction. Hence, the equation $31^x + 1 = z^2$ has no non-negative integer solution. □

Lemma 2.3. *The Diophantine equation* $1 + 32^y = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers y and z such that $1 + 32^y = z^2$. If $y = 0$, then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus, $z^2 = 1 + 32^y \geq 1 + 32^1 = 33$. Then $z \geq 6$. Now, we consider on the equation $z^2 - 32^y = 1$. By Proposition 2.1, we have $y = 1$. Then $z^2 = 33$. This is a contradiction. Hence, the equation $1 + 32^y = z^2$ has no non-negative integer solution. □

3. Results

In this section, we prove that the Diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

Theorem 3.1. *The Diophantine equation* $31^x + 32^y = z^2$ *has no non-negative integer solution.*

Proof. Suppose that there are non-negative integers x, y and z such that $31^x + 32^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Thus, z is odd. Then there is a non-negative integer t such that $z = 2t+1$. Thus, $31^x + 32^y = 4(t^2+t)+1$. This implies that $31^x \equiv 1 \pmod 4$. Since $31 \equiv 3 \pmod 4$, we have $3^x \equiv 1 \pmod 4$. Then x is even. By Lemma 2.3, we have $x \geq 2$. Then there is a positive integer k such that $x = 2k$. Then $32^y = z^2 - 31^{2k}$. Then $2^{5y} = (z-31^k)(z+31^k)$. Thus, $z - 31^k = 2^u$ where u is a non-negative integer. Then $z + 31^k = 2^{5y-u}$. Thus, $2^{5y-u} - 2^u = 2(31^k)$. Then $2^u(2^{5y-2u} - 1) = 2(31^k)$. We divide the number u into two cases.

Case $u = 0$. Then $z - 31^k = 1$. Thus, z is even. This is a contradiction.

Case $u = 1$. Then $2^{5y-2} - 1 = 31^k$. Then $2^{5y-2} - 31^k = 1$. Note that $5y - 2 \geq 3$. By Proposition 2.1, we have $k = 1$. Then $2^{5y-2} = 32$. Then $y = \frac{7}{5}$. This is a contradiction.

Therefore, the equation $31^x + 32^y = z^2$ has no non-negative integer solution. $\square$

Corollary 3.2. *The Diophantine equation* $31^x + 32^y = w^4$ *has no non-negative integer solution.*

Proof. We set $z = w^2$. By Theorem 3.1, the equation $31^x + 32^y = z^2$ has no non-negative integer solution. Hence, the equation $31^x + 32^y = w^4$ has no non-negative integer solution. $\square$

4. Open Problem

We can write the equation $31^x + 32^y = z^2$ into $2^a + 31^b = c^2$ where a, b and c are non-negative integers. Now, for any non-negative integer z, we know that $(5, 1, z)$ is not a solution (a, b, c) for the equation $2^a + 31^b = c^2$ where a, b and c are non-negative integers. We may ask what is the set of all solutions (x, y, z) for the Diophantine equation $2^x + 31^y = z^2$ where x, y and z are non-negative integers.

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