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ON THE TOTAL SIGNED DOMINATION NUMBER OF $\mathbf{n} \cdot \mathbf{C}_m$

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Abstract: Let $\gamma_s^*(G)$ be the total signed domination number of G, C_m denotes the cycle of length m, $n \cdot C_m$ denotes the graph obtained from any n copies of C_m which have just one common vertex. In this paper, we obtain the total signed domination number of $n \cdot C_m$ for any positive integer n and m satisfying $m = 4$ *or* $m \equiv 0(mod\ 5)$.

AMS Subject Classification: 05C69
Key Words: graph, total signed domination function, total signed domination number

1. Introduction

We use Bondy and Murty [1] for terminology and nation not defined here and consider simple graphs only.

Let $G = (V, E)$ be a finite simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For any element $x \in V(G) \bigcup E(G)$, $N_T[x] = \{y | y \text{ is adjacent to } x, \text{ or } y \text{ is in- cident with } x, \text{ where } y \in V(G) \bigcup E(G)\} \bigcup \{x\}$ is called the closed neighborhood of x in G. If the graph is clear from the context, $N_T[x]$ can simply be denoted by $N[x]$.

In recent years, several kinds of domination in graphs have been investigated, such as signed domination [2], signed edge domination [3], signed edge total domination [4], minus domination [5], minus edge domination [6], signed star domination [7], etc. The concept of total signed domination was intro-

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duced in [8], [9]. Several papers have been published on lower bounds and upper bounds of the total signed domination number of graphs, for instance, see [8], [9], [10], [11], [12].

In [9], Lv introduced the total signed domination number as follows:

Definition. (see [9]) Let G be a graph and $f : V(G)\bigcup E(G) \to \{-1, +1\}$ be a function. The weight of f is $w(f) = \sum_{x\in V(G)\bigcup E(G)} f(x)$. A total signed domination function of G is a function $f : V(G)\bigcup E(G) \to \{-1, +1\}$ such that $f[x] \geqslant 1$ for all $x \in V(G)\bigcup E(G)$, where $f[x] = \sum_{y\in N_T[x]} f(y)$. The total signed domination number $\gamma_s^*(G)$ of G is defined as $\gamma_s^*(G) = \min\{w(f) \mid f$ is a total signed domination function of $G\}$.

Let C_m denotes the cycle of length m, $n \cdot C_m$ denotes the graph obtained from any n copies of C_m which have just one common vertex. In [13], we obtain the total signed domination number of $n \cdot C_3$ for any positive integer n. In this paper, we obtain the total signed domination number of $n \cdot C_m$ for any positive integer n and m satisfying $m = 4$ *or* $m \equiv 0(mod\ 5)$.

2. Main Results

Theorem 1 For any positive integer n, then $\gamma_s^*(n \cdot C_4) = n + 1$.

Proof. Let $C_4^{(i)}$ denotes the i-th cycle of length 4 in $n \cdot C_4$, u denotes a common vertex of $C_4^{(1)}, C_4^{(2)}, \cdots, C_4^{(n)}$, other three vertices of $C_4^{(i)}$ are $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}$ in proper order, and let $E(C_4^{(i)}) = \{e_1^{(i)} = v_1^{(i)}v_2^{(i)},\ e_2^{(i)} = v_2^{(i)}v_3^{(i)},\ e_3^{(i)} = v_3^{(i)}u,\ e_4^{(i)} = uv_1^{(i)}\}$, where $i = 1, 2, \cdots, n$.

Let f is a total signed domination function of $n \cdot C_4$, notice that:

$$f[u] = f(u) + f(v_1^{(1)}) + f(v_3^{(1)}) + \cdots + f(v_1^{(n)}) + f(v_3^{(n)}) + f(e_3^{(1)}) + f(e_4^{(1)}) + \cdots + f(e_3^{(n)}) + f(e_4^{(n)}),$$

$$f[v_1^{(i)}] = f(v_1^{(i)}) + f(v_2^{(i)}) + f(u) + f(e_1^{(i)}) + f(e_4^{(i)}),$$

$$f[v_2^{(i)}] = f(v_1^{(i)}) + f(v_2^{(i)}) + f(v_3^{(i)}) + f(e_1^{(i)}) + f(e_2^{(i)}),$$

$$f[v_3^{(i)}] = f(v_2^{(i)}) + f(v_3^{(i)}) + f(u) + f(e_2^{(i)}) + f(e_3^{(i)}),$$

$$f[e_1^{(i)}] = f(v_1^{(i)}) + f(v_2^{(i)}) + f(e_1^{(i)}) + f(e_2^{(i)}) + f(e_4^{(i)}),$$

$$f[e_2^{(i)}] = f(v_2^{(i)}) + f(v_3^{(i)}) + f(e_1^{(i)}) + f(e_2^{(i)}) + f(e_3^{(i)}),$$

$$f[e_3^{(i)}] = f(v_3^{(i)}) + f(u) + f(e_2^{(i)}) + f(e_3^{(1)}) + f(e_4^{(1)}) + \cdots + f(e_3^{(n)}) + f(e_4^{(n)}),$$

$$f[e_4^{(i)}] = f(v_1^{(i)}) + f(u) + f(e_1^{(i)}) + f(e_3^{(1)}) + f(e_4^{(1)}) + \cdots + f(e_3^{(n)}) + f(e_4^{(n)}).$$

For convenience, for a given graph $G = (V, E)$, an element $x \in V(G) \bigcup E(G)$ is said to be $+1$ element of G if $f(x) = +1$, analogously, an element $x \in V(G) \bigcup E(G)$ is said to be -1 element of G if $f(x) = -1$.

We discuss the number of -1 elements of $C_4^{(i)}$ and the number of such $C_4^{(i)}$ in $n \cdot C_4$ by eight cases.

Case 1 If the number of C_4 which has eight -1 elements in $n \cdot C_4$ is exactly k_1. We may assume $C_4^{(i)}$ is the one of them. It is easy to verify that $f[v_1^{(i)}] = -5$, $f[v_2^{(i)}] = -5$, $f[v_3^{(i)}] = -5$, $f[e_1^{(i)}] = -5$, $f[e_2^{(i)}] = -5$, which is a contradiction.

Case 2 If the number of C_4 which has seven -1 elements in $n \cdot C_4$ is exactly k_2. We may assume $C_4^{(i)}$ is the one of them. It is easy to verify that $f[v_1^{(i)}] \leqslant -3$, $f[v_2^{(i)}] \leqslant -3$, $f[v_3^{(i)}] \leqslant -3$, $f[e_1^{(i)}] \leqslant -3$, $f[e_2^{(i)}] \leqslant -3$, which is a contradiction.

Case 3 If the number of C_4 which has six -1 elements in $n \cdot C_4$ is exactly k_3. We may assume $C_4^{(i)}$ is the one of them. It is easy to verify that $f[v_1^{(i)}] \leqslant -1$, $f[v_2^{(i)}] \leqslant -1$, $f[v_3^{(i)}] \leqslant -1$, $f[e_1^{(i)}] \leqslant -1$, $f[e_2^{(i)}] \leqslant -1$, which is a contradiction.

Case 4 If the number of C_4 which has five -1 elements in $n \cdot C_4$ is exactly k_4. We may assume $C_4^{(i)}$ is the one of them. Notice that $|N[v_1^{(i)}]| = |N[v_2^{(i)}]| = |N[v_3^{(i)}]| = |N[e_1^{(i)}]| = |N[e_2^{(i)}]| = 5$, there must exists an element $x \in \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, e_1^{(i)}, e_2^{(i)}\}$ such that $f[x] \leqslant -1$, which is a contradiction.

Case 5 If the number of C_4 which has four -1 elements in $n \cdot C_4$ is exactly k_5. We may assume $C_4^{(i)}$ is the one of them. We discuss $f(u) = -1$ or $+1$ as follows:

Case 5.1 When $f(u) = -1$, notice that $f[v_1^{(i)}] = f(v_1^{(i)}) + f(v_2^{(i)}) + f(u) + f(e_1^{(i)}) + f(e_4^{(i)}) \geqslant 1$, we have the number of -1 elements in $\{v_1^{(i)}, v_2^{(i)}, e_1^{(i)}, e_4^{(i)}\}$ is at most 1. Considering the symmetry of $C_4^{(i)}$, since $f[v_3^{(i)}] = f(v_2^{(i)}) + f(v_3^{(i)}) + f(u) + f(e_2^{(i)}) + f(e_3^{(i)}) \geqslant 1$, we have the number of -1 elements in $\{v_2^{(i)}, v_3^{(i)}, e_2^{(i)}, e_3^{(i)}\}$ is at most 1. So the number of -1 elements in $C_4^{(i)}$ is at most 3, which contradicts with the assumption. Analogously, when$f(u) = -1$, there doesn't exist four -1 elements in each one of the rest of $(k_5 - 1)$ C_4 in $n \cdot C_4$.

Case 5.2 When $f(u) = +1$, notice that $f[v_1^{(i)}] \geqslant 1$, $f[v_2^{(i)}] \geqslant 1$, $f[v_3^{(i)}] \geqslant 1$. We discuss $f(v_2^{(i)}) = -1$ or $+1$ as follows:

Case 5.2.1 When $f(v_2^{(i)}) = -1$, considering the symmetry of $C_4^{(i)}$, by a similar method as *Case 5.1*, which contradicts with the assumption.

Case 5.2.2 When $f(v_2^{(i)}) = +1$, notice that $f[v_1^{(i)}] = f(v_1^{(i)}) + f(v_2^{(i)}) + f(u) + f(e_1^{(i)}) + f(e_4^{(i)}) \geqslant 1$, we have the number of -1 elements in $\{v_1^{(i)}, e_1^{(i)}, e_4^{(i)}\}$ is at most 2. Analogously, notice that $f[v_3^{(i)}] = f(v_2^{(i)}) + f(v_3^{(i)}) + f(u) + f(e_2^{(i)}) + f(e_3^{(i)}) \geqslant 1$, we have the number of -1 elements in $\{v_3^{(i)}, e_2^{(i)}, e_3^{(i)}\}$ is at most 2. By the definition of the total signed domination number, we only need to consider that the number of -1 elements in $\{v_1^{(i)}, e_1^{(i)}, e_4^{(i)}\}$ is 2 and the number of -1 elements in $\{v_3^{(i)}, e_2^{(i)}, e_3^{(i)}\}$ is also 2 by the following nine cases.

Case 5.2.2.1 When $f(e_1^{(i)}) = f(v_1^{(i)}) = f(e_2^{(i)}) = f(v_3^{(i)}) = -1$, it is easy to verify that $f[v_2^{(i)}] = -3, f[e_1^{(i)}] = f[e_2^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.2 When $f(e_1^{(i)}) = f(v_1^{(i)}) = f(v_3^{(i)}) = f(e_3^{(i)}) = -1$, it is easy to verify that $f[v_2^{(i)}] = f[e_2^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.3 When $f(e_1^{(i)}) = f(v_1^{(i)}) = f(e_2^{(i)}) = f(e_3^{(i)}) = -1$, it is easy to verify that $f[v_2^{(i)}] = f[e_1^{(i)}] = f[e_2^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.4 When $f(e_1^{(i)}) = f(e_4^{(i)}) = f(e_2^{(i)}) = f(v_3^{(i)}) = -1$, it is easy to verify that $f[v_2^{(i)}] = f[e_1^{(i)}] = f[e_2^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.5 When $f(e_1^{(i)}) = f(e_4^{(i)}) = f(v_3^{(i)}) = f(e_3^{(i)}) = -1$, it is easy to verify that $f[e_2^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.6 When $f(e_1^{(i)}) = f(e_4^{(i)}) = f(e_2^{(i)}) = f(e_3^{(i)}) = -1$, it is easy to verify that $f[e_1^{(i)}] = f[e_2^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.7 When $f(v_1^{(i)}) = f(e_4^{(i)}) = f(e_2^{(i)}) = f(v_3^{(i)}) = -1$, it is easy to verify that $f[v_2^{(i)}] = f[e_1^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.8 When $f(v_1^{(i)}) = f(e_4^{(i)}) = f(e_2^{(i)}) = f(e_3^{(i)}) = -1$, it is easy to verify that $f[e_1^{(i)}] = -1$, which is a contradiction.

Case 5.2.2.9 When $f(v_1^{(i)}) = f(e_4^{(i)}) = f(v_3^{(i)}) = f(e_3^{(i)}) = -1$.

Considering the symmetry of $n \cdot C_4$, by a similar method as the above eight cases of *Case 5.2.2*, we only need to consider *Case 5.2.2.9* in each one of the rest of $(k_5 - 1)$ C_4 in $n \cdot C_4$. Notice that $f[e_3^{(i)}] = f(v_3^{(i)}) + f(u) + f(e_2^{(i)}) + f(e_3^{(1)}) + f(e_4^{(1)}) + \cdots + f(e_3^{(n)}) + f(e_4^{(n)}) \geqslant 1$, $f[e_4^{(i)}] = f(v_1^{(i)}) + f(u) + f(e_1^{(i)}) + f(e_3^{(1)}) + f(e_4^{(1)}) + \cdots + f(e_3^{(n)}) + f(e_4^{(n)}) \geqslant 1$, thus the number of -1 elements in $\{e_3^{(1)}, e_4^{(1)}, e_3^{(2)}, e_4^{(2)} \cdots e_3^{(n)}, e_4^{(n)}\}$ is at most n, at this time, the number of C_4, in which the edges incident with u are all -1 elements, is at most $\lfloor \frac{n}{2} \rfloor (\geqslant k_5)$. When $k_5 = \lfloor \frac{n}{2} \rfloor$, we consider the parity of n in following two cases.

Case 5.2.2.9.1 When n is even, $v_1^{(i)}$ and $v_3^{(i)}$ must be $+1$ elements in each one of the rest of $(n-k_5)$ C_4, and the number of -1 elements in $\{e_1^{(i)}, e_2^{(i)}, v_2^{(i)}\}$ is at most 2. By the definition of the total signed domination number, we only need to consider that the number of -1 elements in $\{e_1^{(i)}, e_2^{(i)}, v_2^{(i)}\}$ is 2, and we give a total signed domination function as follows:

$$f(x)=\begin{cases} +1, & x \in v_2^{(i)}, e_1^{(i)}, e_2^{(i)} (i=1,2,\cdots,k_5), \\ -1, & x \in v_1^{(i)}, v_3^{(i)}, e_3^{(i)}, e_4^{(i)} (i=1,2,\cdots,k_5), \\ +1, & x \in v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, e_3^{(i)}, e_4^{(i)} (i=k_5+1,\cdots,n), \\ -1, & x \in e_1^{(i)}, e_2^{(i)} (i=k_5+1,\cdots,n). \end{cases}$$

It is easy to verify that $f[x] \geqslant 1$ for all $x \in V(G) \bigcup E(G)$(which implies that the case of $k_5 < \lfloor \frac{n}{2} \rfloor$ is not considered), we have $w(f) = 2n+1$.

Case 5.2.2.9.2 When n is odd, considering the symmetry of $n \cdot C_4$ and $f[e_3^{(i)}] \geqslant 1$, the number of -1 elements in each one of $(n-k_5-1)$ C_4 is at most 2, and the the number of -1 elements in last C_4 is at most 3. We only need to consider that the number of -1 elements in each one of $(n-k_5-1)$ C_4 is 2 and the the number of -1 elements in last C_4 is 3, and we give a total signed domination function as follows:

$$f(x)=\begin{cases} +1, & x \in v_2^{(i)}, e_1^{(i)}, e_2^{(i)} (i=1,2,\cdots,k_5), \\ -1, & x \in v_1^{(i)}, v_3^{(i)}, e_3^{(i)}, e_4^{(i)} (i=1,2,\cdots,k_5), \\ +1, & x \in v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, e_3^{(i)}, e_4^{(i)} (i=k_5+1,\cdots,n-1), \\ -1, & x \in e_1^{(i)}, e_2^{(i)} (i=k_5+1,\cdots,n-1), \\ +1, & x \in v_1^{(i)}, v_2^{(i)}, e_2^{(i)}, e_3^{(i)} (i=n), \\ -1, & x \in v_3^{(i)}, e_1^{(i)}, e_4^{(i)} (i=n). \end{cases}$$

It is easy to verify that $f[x] \geqslant 1$ for all $x \in V(G) \bigcup E(G)$(which implies that the case of $k_5 < \lfloor \frac{n}{2} \rfloor$ is not considered), we have $w(f) = n+1$.

Case 6 If the number of C_4 which has three -1 elements in $n \cdot C_4$ is exactly k_6.

In[9], Lv proved that $\gamma_s^*(C_4) = 2$, which implies there are five $+1$ elements and three -1 elements in C_4. When $k_6 = n$, we give a total signed domination function in following two cases:

Case 6.1 When $f(u) = -1$, let

$$f(x)=\begin{cases} +1, & x \in v_2^{(i)}, v_3^{(i)}, e_1^{(i)}, e_3^{(i)}, e_4^{(i)} (i=1,2,\cdots,n), \\ -1, & x \in v_1^{(i)}, e_2^{(i)} (i=1,2,\cdots,n). \end{cases}$$

It is easy to verify that $f[x] \geqslant 1$ for all $x \in V(G) \bigcup E(G)$(which implies that the case of $k_6 < n$ is not considered), we have $w(f) = 3n - 1$.

Case 6.2 When $f(u) = +1$, let

$$f(x) = \begin{cases} +1, & x \in v_3^{(i)}, e_1^{(i)}, e_2^{(i)}, e_4^{(i)} (i = 1, 2, \cdots, n), \\ -1, & x \in v_1^{(i)}, v_2^{(i)}, e_3^{(i)} (i = 1, 2, \cdots, n). \end{cases}$$

It is easy to verify that $f[x] \geqslant 1$ for all $x \in V(G) \bigcup E(G)$(which implies that the case of $k_6 < n$ is not considered), we have $w(f) = n + 1$.

Case 7 If the number of C_4 which has two -1 elements in $n \cdot C_4$ is exactly k_7.

Since Case 6 holds, which implies *Case 7* is not considered.

Case 8 If the number of C_4 which has one -1 element in $n \cdot C_4$ is exactly k_8.

Since Case 6 holds, which implies *Case 8* is not considered.

Combining *Case 5* and *Case 6*, we have $\gamma_s^*(n \cdot C_4) = n + 1$.

Combining the above eight cases, we have completed the proof of Theorem 1.

Theorem 2 For any positive integer n and m satisfying $m \equiv 0(mod\ 5)$, then $\gamma_s^*(n \cdot C_m) = 2n\lceil \frac{m}{5} \rceil - (n - 1)$.

Proof. Let $C_m^{(i)}$ denotes the i-th cycle of length m of $n \cdot C_m$, u denotes a common vertex of $C_m^{(1)}, C_m^{(2)}, \cdots, C_m^{(n)}$, other $m - 1$ vertices of $C_m^{(i)}$ are $v_1^{(i)}$, $v_2^{(i)}$, $\cdots, v_{m-1}^{(i)}$ in proper order, and let $E(C_4^{(i)}) = \{e_1^{(i)} = v_1^{(i)} v_2^{(i)},\ e_2^{(i)} = v_2^{(i)} v_3^{(i)}, \cdots,\ e_{m-1}^{(i)} = v_{m-1}^{(i)} u,\ e_m^{(i)} = u v_1^{(i)}\}$, where $i = 1, 2, \cdots, n$. We give a total signed domination function of $n \cdot C_m$ for $m \equiv 0(mod\ 5)$ as follows:

$$\begin{cases} f(u) = +1, & \\ f(v_j^{(i)}) = -1, & \text{if } j \equiv 1, 2\ (mod\ 5), \\ f(v_j^{(i)}) = +1, & \text{if } j \equiv 0, 3, 4\ (mod\ 5), \\ f(e_j^{(i)}) = -1, & \text{if } j \equiv 3, 4\ (mod\ 5), \\ f(e_j^{(i)}) = +1, & \text{if } j \equiv 0, 1, 2\ (mod\ 5). \end{cases}$$

It is easy to verify that $f[x] = 1$ for all $x \in V(G) \bigcup E(G)$, we have $\gamma_s^*(n \cdot C_m) = 2n\lceil \frac{m}{5} \rceil - (n - 1)$ for $m \equiv 0(mod\ 5)$. We have completed the proof of Theorem 2.

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