International Journal of Pure and Applied Mathematics
Volume 81 No. 6 2012, 851-863
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

# ON THE GONALITY SEQUENCE AND THE BIRATIONAL GONALITY SEQUENCE OF SMOOTH CURVES

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**Abstract:** Let $C$ be a smooth curve of genus $g$. For any integer $r \geq 2$ the $r$-gonality of $C$ is a minimal degree of a line bundle $L$ such that $h^0(C,L) = r+1$. If we assume that the associated map $C \to \mathbb{P}^r$ is birational onto its image, then we get the $r$-birational gonality $s_r(C)$. For $g = 29, 30$ we prove the existence of $C$ with $d_2(C) = 10$ and $d_3(C) = 16$. For many genera we prove the existence of $C$ with $s_4(C)/4 > s_3(C)/3$ and $s_6(C)/6 > s_5(C)$ (both inequalities for the same curve).

**AMS Subject Classification:** 14H45, 14H50, 32L10
**Key Words:** gonality sequence, birational gonality sequence, slope inequality, smooth curve, nodal curve

## 1. Introduction

Let $C$ be a smooth and projective curve defined over an algebraically closed base field with characteristic zero. For each integer $r \geq 1$ the $r$-gonality $d_r(C)$ of $C$ is the minimal degree of a line bundle $L$ on $C$ such that $h^0(C,L) \geq r+1$ (see [15]). For many (but nor all) curves the slope inequality $d_{r+1}(C)/(r+1) \leq d_r(C)/r$ holds for all $r \geq 2$ (it is trivially true for $r = 1$) (see [15]). It is usually easier to study the following invariant $s_r(C)$ (see [9]). For each integer $r \geq 2$ let $s_r(C)$ denote the minimal degree of a line bundle $L$ such that $h^0(C,L) \geq r+1$ and the associated rational map $C \dashrightarrow \mathbb{P}^r$ is birational onto its image (see [9], §2). It is easy to check that each line bundle $L$ evincing $s_r(C)$ is spanned and

Received: October 8, 2012

$h^0(C,L) = r+1$. Obviously $s_r(C) \geq d_r(C)$. Equality holds if $d_r(C) < r \cdot d_1(C)$ and $C$ has no non-trivial morphisms onto a smooth curve of positive genus. The sequence $\{s_r(C)\}_{r\geq 2}$ is called the *birational gonality sequence* of $C$. It was introduced in [9] and it seems to be a natural invariant of $C$: ancient algebraic geometers often looked at birational model of an object, say the curve $C$, in a fixed projective space. Fix an integer $r \geq 2$. It is often technically easier to compute the integer $s_r(C)$ than the integer $d_r(C)$. Quite often $s_r(C) = d_r(C)$ (see Remark 1). Fix an integer $r \geq 3$. Another natural numerical invariant of $C$ is the minimal integer, $\beta_r(C)$, of a very ample line bundle $L$ on $C$ such that $h^0(C,L) = r+1$. We have nothing to say here for the sequence $\beta_r(C)$.

In Section 2 we prove the following result.

**Theorem 1.** *For each $g \in \{29, 30\}$ there is a smooth curve $C$ of genus $g$ with $d_2(C) = 10$ and $d_3(C) = 16$.*

In Section 3 we construct several examples of smooth curves $C$ for which two slopes inequalities for $s_r$ fails (in our case we have $s_4(C)/4 > s_3(C)/3$ and $s_6(C)/6 > s_5(C)/5$). We stress that both inequalities hold for the same curve $C$, i.e. in the birational gonality sequece of $C$ two non-consecutive slope inequalities fail. The examples give the following result.

**Theorem 2.** *Fix integers $a, x$ such that $a \geq 5$ and $x \leq a^2/9$. Let $F_0 \subset \mathbb{P}^3$ be a smooth quadric surface. Fix a general $S \subset F_0$ such that $\sharp(S) = x$ and a general $Y \in |\mathcal{I}_{2S}(ah + bf)|$. Then $Y$ is integral, nodal and $\mathrm{Sing}(Y) = S$. Let $C$ be the normalization of $Y$. The smooth curve $C$ has genus $(a-1)^2 - x$, $s_2(C) = 2a - 1 - \min\{1, x\}$, $s_3(C) = 2a$, $s_5(C) \leq 3a$ and $s_7(C) \leq 4a$.*

*(a) Fix an integer $z \leq 3a - 2 - \min\{1, x\}$. If $x < -1 + (a-1)^2 - (z-1)(z+3)/8$, then $z + 1 \leq s_4(C) \leq 3a - 1 - \min\{1, x\}$.*

*(b) If $x \leq a^2/9 - 2a + 1$, then $s_4(C)/4 > s_3(C)/3$.*

*(c) Assume $a \geq 108$. Fix an integer $z \leq 4a - 2 - \min\{1, x\}$. If $x < (a-1)^2 - (z-1)(z+15)/14$, then $z + 1 \leq s_4(C) \leq 3a - 1 - \min\{1, x\}$.*

*(d) If $a \geq 108$ and $x < (a-1)^2 - (18a-5)(18a+75)/350$, then $s_6(C)/6 > s_5(C)/5$.*

Notice that $(a-1)^2 - (18a-5)(18a+75)/350 \sim (13/175)a^2$ when $a \gg 0$.

From the proof of Theorem 2 we get the following result.

**Corollary 1.** *There is an integer $g_0$ such that for every integer $g \geq g_0$ there is a smooth curve $C$ of genus $g$ with $s_4(C)/4 > s_3(C)/3$ and $s_6(C)/6 > s_5(C)/5$.*

**Question 1.** (a) Are there a smooth curve $C$ and an integer $r \geq 3$ such that $s_{r+1}(C)/s_r(C) \geq (r+2)/r$? (or with $s_{r+1}(C)/s_r(C) > (r+2)/r$?

(b) Are there a smooth curve $C$ and an integer $r \geq 3$ such that $s_{r+2}(C)/(r+2) > s_{r+1}C)/(r+1) > s_r(C)/r$?

**Remark 1.** Fix a smooth curve $C$ of genus $\geq 3$ and an integer $r \geq 2$.Let $L \in \mathrm{Pic}^d(C)$, $d := s_r(C)$, be a line bundle evincing $d_r(C)$. Let $\phi : C \to \mathbb{P}^r$ the morphism associated to $|L|$. Let $F$ be the normalization of $\phi(C)$. If $s_r(C) \neq d_r(C)$, then $\deg(\phi) \geq 2$, i.e. $F$ has genus $q < g$. Assume that there is no non-constant map $C \to E$ with $E$ a smooth curve of genus $> 0$. We get $q = 0$. Since $\phi$ is induced by a complete linear system, we get that $\phi(C)$ is a rational normal curve. Hence $d_r(C) = r \cdot d_1(C)$ and any complete linear system evincing $d_r(C)$ is of the form $|R^{\otimes r}|$ with $R$ evincing the gonality $d_1(C)$ of $C$.

## 2. Proof of Theorem 1

**Notation 1.** For any smooth surface $M$ and any finite set $S \subset M$ let $2S$ denote the closed subscheme of $M$ with $(\mathcal{I}_S)^2$ as its ideal sheaf. The scheme $2S$ is a zero-dimensional scheme, $(2S)_{red} = S$ and $\deg(2S) = 3 \cdot \sharp(S)$. In this section we take $M = \mathbb{P}^2$. In the next section we will take $M = F_0$ (a smooth quadric surface).

**Lemma 1.** *Fix a general $S \subset \mathbb{P}^2$ such that $\sharp(S) \leq 7$. Fix any $P \in \mathbb{P}^2 \setminus S$. Then $h^1(\mathcal{I}_{S\cup\{P\}}(3)) = 0$.*

*Proof.* Set $S' := S \cup \{P\}$. First assume the existence of a line $D \subset \mathbb{P}^2$ such that $\sharp(D \cap S') \geq 3$. Since $S$ is general, we have $P \in D$ and $\sharp(S \cap D) = 2$. Since $\sharp(S' \cap D) \leq 4$, we have $h^1(D, \mathcal{I}_{S'\cap D,D}(3)) = 0$. Since $\sharp(S' \cap D) \leq 5$ and $S' \setminus S' \cap D$ is general, we have $h^1(\mathcal{I}_{S'\setminus S'\cap D}(2)) = 0$. The exact sequence

$$0 \to \mathcal{I}_{S'\setminus S'\cap D}(2) \to \mathcal{I}_{S'}(3) \to \mathcal{I}_{S'\cap D,D}(3) \to 0$$

gives $h^1(\mathcal{I}_{S'}(3)) = 0$. Now assume $\sharp(D \cap S') \leq 2$ for every line $D$. Fix a conic $E$ such that $\sharp(S' \cap E)$ is maximal. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, we have $\sharp(S' \cap E) \geq 5$. Hence $E$ is a smooth conic and $\sharp(S' \setminus S' \cap E) \leq 3$. Since no 3 of the points of $S'$ are collinear, we have $h^1(\mathcal{I}_{S'\setminus S'\cap E}(1)) = 0$. Since $S$ is general, $E$ contains at most 5 points of $S$. Hence $\sharp(S' \cap E) \leq 6$. Since $E$ is a smooth conic, $h^1(E, \mathcal{I}_{S'\cap E,E}(3)) = 0$. As above an obvious exact sequence gives $h^1(\mathcal{I}_{S\cup\{P\}}(3)) = 0$. □

Fix an integer $x \in \{6, 7\}$. Let $S \subset \mathbb{P}^2$ be a general set with $\sharp(S) = x$. Let $Y \subset \mathbb{P}^2$ be a general element of $|\mathcal{I}_{2S}(10)|$. It is easy to check that $h^1(\mathcal{I}_{2S}(10)) = 0$ and that a general $Y \in |\mathcal{I}_{2S}(10)|$ is integral, nodal and with $\mathrm{Sing}(Y) = S$ (see

[1], Theorem 3.2). Hence the normalization $C$ of $Y$ is a smooth and connected curve of genus $36-x$. To prove Theorem 1 it is sufficient to prove that $d_2(C) = 10$ and $d_3(C) = 16$ for at least one $C$. We prove it when $Y$ is general. More precisely we assume that there is no order two automorphism $\sigma$ of $\mathbb{P}^2$ with $\sigma(Y) = Y$.

**Lemma 2.** *Assume $Y$ integral, nodal and with* $\mathrm{Sing}(Y) = S$. *Assume that there is no order two automorphism of $\mathbb{P}^2$ sending $Y$ into itself. Then $C$ has no order two automorphism.*

*Proof.* The line bundle $\mathcal{O}_Y(1)$ is the only line bundle $R$ on $Y$ with $\deg(R) \leq d$ and $h^0(Y,R) \geq 3$ by Noether's theorem for integral singular curves (see [13], Theorem 2.1). Hence $Y$ has no order 2 automorphism. The case $k = 1$ and $\delta = 1$ of [8], Theorem 2.3, gives $d_1(C) = 8$, that the only $g^1_8$'s on $C$ are the ones induced by $|\mathcal{I}_{\{P\}}(1)|$ for some $P \in S$. Every $g^1_9$ on $C$ is induced by a pencil of lines through some point of $Y \setminus S$ (part i) of [8], Theorem 2.4). Hence $u^*(\mathcal{O}_Y(1))$ is the only $g^2_{10}$ on $C$. Hence each automorphism of $C$ induces an automorphism of $Y$. Hence $C$ has no order two automorphism. $\square$

*Proof of Theorem 1.* Fix $x \in \{6,7\}$ and a general $S \subset \mathbb{P}^2$ such that $\sharp(S) = x$. Fix any $Y$ as in Lemma 2. Let $u : C \to Y$ be the normalization map. The smooth curve $C$ has genus $36 - x$ and we will check that $d_2(C) = 10$ and $d_3(C) = 16$. As in Lemma 2 we see that $d_2(C) = 10$. Fix $S' \subset S$ such that $\sharp(S') = 2$. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, the linear system $|\mathcal{I}_{S'}(2)|$ gives $d_3(C) \leq 16$. Assume $z := d_3(C) \leq 15$ and take $L \in \mathrm{Pic}^z(C)$ evincing $d_3(C)$. The line bundle $L$ is spanned (see [15], Lemma 3.1). Hence $|L|$ induces a morphism $v : C \to \mathbb{P}^4$.

(a) In this step we prove that $v$ is birational onto its image. Assume that $v$ is not birational onto its image. Set $w := \deg(v)$. Hence $\deg(v(C)) = z/w \in \mathbb{N}$. Since $C$ has no order two automorphism, we have $w \geq 3$. Since $v(C)$ spans $\mathbb{P}^3$ we have $z/w \geq 3$. Let $\nu : J \to v(C)$ denote the normalization map. Since $C$ is smooth, $v$ induces a morphism $\phi : C \to J$ such that $v = \nu \circ \phi$. Since $h^0(C,L) = 4$, we get $h^0(J, \nu^*(\mathcal{O}_{v(C)}(1))) = 4$. We have $d_1(C) \leq w \cdot d_1(J)$. If $J$ has genus 0, then we get $d_1(C) \leq z/3$ and hence $z \geq 3(d-2) > 2d-4$, a contradiction. Hence $z/w \geq 4$. Thus it is sufficient to check the case $(z,w) = (15,3)$. Assume $(z,w) = (15,3)$ and hence $\deg(v(C)) = 5$. Hence $g(J) \leq 2$. Hence $d_1(J) \leq 2$. Hence $8 = d_1(C) \leq 4$, a contradiction.

(b) Fix a general $A \in |L|$ and set $B := u(A)$. Since $L$ has no base points and $A$ is general, we have $B \cap S = \emptyset$. Since $v$ is birational onto its image, the monodromy group of a general hyperplane section of $v(C)$ is the full symmetric group. Hence for every $t \in \{1,2,3\}$ either $h^0(\mathcal{I}_B(t)) > 0$ or $h^0(\mathcal{I}_{B_1}(t)) = \max\{0, \binom{t+2}{2} - \sharp(B_1)\}$ for all $B_1 \subset B$. Since $Y$ is a degree 10 nodal curve with $S$ as its singular locus, adjunction theory gives $H^0(C, \omega_C) \cong H^0(\mathcal{I}_S(7))$. Since $L$ is spanned and $A \in |L|$, Rieman-Roch gives $h^0(C, \omega_C \otimes \mathcal{O}_C(A \setminus \{Q\}) = h^0(C, \omega_C \otimes \mathcal{O}_C(A))$ for every $Q \in A$. Hence $h^0(\mathcal{I}_{S\cup(B\setminus\{O\})}(7)) = h^0(\mathcal{I}_{S\cup B}(7))$ for each $O \in B$. Hence $h^1(\mathcal{I}_{S\cup B}(7)) > 0$.

(b1) Assume $h^0(\mathcal{I}_B(4)) = 0$. Hence $h^0(\mathcal{I}_B(t)) = 0$ for all $t \le 3$. Hence for every $t \in \{1,2,3,4\}$ we have $\sharp(B \cap D_t) \le \binom{t+2}{2} - 1$ for every degree $t$ curve $D_t$. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(4)) = 15$, there is a degree 4 curve $W$ containing 14 points of $B$. Since $h^0(\mathcal{I}_B(4)) = 0$, we have $z = 15$ and $B \setminus B \cap W$ is a unique point, $P$. Set $S' := S \setminus S \cap W$. First assume $h^1(W, \mathcal{I}_{W\cap(S\cup B),W}(7)) = 0$. The exact sequence

$$0 \to \mathcal{I}_{S'\cup\{P\}}(3)) \to \mathcal{I}_{S\cup B}(7) \to \mathcal{I}_{W\cap(S\cup B),W}(7) \to 0$$

gives $h^1(\mathcal{I}_{S'\cup\{P\}}(3)) > 0$. Lemma 1 gives a contradiction.

Now assume $h^1(W, \mathcal{I}_{W\cap(S\cup B),W}(7)) = 0$. Since for every $t \in \{1,2,3\}$ we have $\sharp(B \cap D_t) \le \binom{t+2}{2} - 1$ for every degree $t$ curve $D_t$, $W$ is irreducible. We have $p_a(W) = 3$ and $\deg(\mathcal{O}_W(7)) = 28$. Since $\deg(\omega_W) = 4$ and the torsion free sheaf $\mathcal{I}_{W\cap(S\cup B),W}$ satisfies $h^1(W, \mathcal{I}_{W\cap(S\cup B),W}(7)) > 0$, we have $z + \sharp(W \cap S) \ge 24$, contradicting the inequality $x + z \le 22$.

(b2) Assume $h^0(\mathcal{I}_B(4)) > 0$ and $h^0(\mathcal{I}_B(3)) = 0$. Let $G \subset \mathbb{P}^2$ be a quartic curve containing $B$. Since $S$ is general, we have $h^1(\mathcal{I}_{S\setminus S\cap G}(3)) = 0$. Hence the exact sequence

$$0 \to \mathcal{I}_{S\setminus S\cap G}(3)) \to \mathcal{I}_{S\cup B}(7) \to \mathcal{I}_{G\cap(S\cup B),G}(7) \to 0$$

gives $h^1(G, \mathcal{I}_{G\cap(S\cup B),G}(7)) > 0$. Since $h^0(\mathcal{I}_B(t)) = 0$ for $t = 2, 3$, we have $\sharp(B \cap D) \le 5$ for all conics $D$ and $\sharp(B \cap D') \le 9$ for all cubics $D'$. Hence $G$ is irreducible. We have $p_a(G) = 3$ and $\deg(\mathcal{O}_G(7)) = 28$. Since $\deg(\omega_G) = 4$ and the torsion free sheaf $\mathcal{I}_{G\cap(S\cup B),G}$ satisfies $h^1(G, \mathcal{I}_{G\cap(S\cup B),G}(7)) > 0$, we have $z + \sharp(G \cap S) \ge 24$, contradicting the inequality $x + z \le 22$.

(b3) Assume $h^0(\mathcal{I}_B(3)) > 0$ and $h^0(\mathcal{I}_B(2)) = 0$. Let $E \subset \mathbb{P}^2$ be a cubic curve containing $B$. Since $h^0(\mathcal{I}_B(2)) = 0$, we have $\sharp(B \cap D) \le 5$ for all conics, $E$ is irreducible. We have $h^1(\mathcal{I}_{S\setminus E}(4)) = 0$, because $S$ is general. If $h^1(E, \mathcal{I}_{E\cap(S\cup B),E}(7)) = 0$. Hence the exact sequence

$$0 \to \mathcal{I}_{S\setminus S\cap E}(4)) \to \mathcal{I}_{S\cup B}(7) \to \mathcal{I}_{E\cap(S\cup B),E}(7) \to 0$$

gives $h^1(E, \mathcal{I}_{E\cap(S\cup B),E}(7)) > 0$. Since $p_a(E) = 1$ and $\deg(\mathcal{O}_E(7)) = 21$, either $x + z = 22$ and $S \cup B \subset E$ or $\sharp(E \cap (S \cup B)) = 21$ and $E \cap (S \cup B)$ is the complete intersection of $E$ and a plane curve of degree 7. In all cases there is $S' \subseteq S$ such that $\sharp(S') = 6$ and $E \supset S' \cup B$. Since $S'$ is general, we have $h^0(\mathcal{I}_{S'}(3)) = 4$. Since $h^0(C, L) = 4$, we get that $L$ is induced (after deleting the base points) by $|\mathcal{I}_{S'}(2)|$. Lemma 1 gives that $\mathcal{I}_{S'}(3)$ is spanned. Hence $z = 30 - 12$, a contradiction.

(b4) Assume $h^0(\mathcal{I}_B(2)) > 0$. Fix a conic $F$ containing $B$. Since $h^1(\mathcal{I}_{S \setminus S \cap F}(5)) = 0$, as in step (b3) we get $h^1(F, \mathcal{I}_{F \cap (S \cup B), F}(7)) > 0$. Since $\sharp(D \cap B) \leq 2$ for every line $D$, $F$ is irreducible. Since $F \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_F(7)) = 14$, we get $\sharp((S \cup B) \cap F) \geq 16$, i.e. $\sharp(S \cap F) \geq 16 - z > 0$. Since $B \subset F$, $|L|$ is induced (after deleting base points) from a subspace of $|\mathcal{I}_{S \cap F}(2)|$. Since $S \cap F$ is general, we have $h^0(\mathcal{I}_{S \cup F}(2)) = 6 - \sharp(F \cap S)$. Hence $\sharp(F \cap S) \in \{1, 2\}$. First assume $\sharp(F \cap S) = 2$. Since $\mathcal{I}_{F \cap S}(2)$ is spanned, we get $z = 20 - 4$, a contradiction. Now assume $\sharp(F \cap S) = 1$. Since $h^1(\mathcal{I}_Z(2)) = 0$ for every zero-dimensional scheme of degree 3, we get every codimension 1 linear subspace of $H^0(\mathcal{I}_{F \cap S}(2))$ induces (after deleting base points) a linear system of degree $\geq 20 - 4$, a contradiction. □

## 3. The Invariants $s_r(C)$

For any integer $e \geq 0$ let $F_e$ denote the Hirzebruch surface with a ruling with minimal self-intersection $-e$ (see [12], pp. 379–381). We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}^2$ and we take as a basis of $\mathrm{Pic}(F_e)$ a fiber of a ruling of $F_e$ and a section of the same ruling with self-intersection $-e$. Hence $\mathcal{O}_{F_e}(h) \cdot \mathcal{O}_{F_e}(h) = -e$, $\mathcal{O}_{F_e}(h) \cdot \mathcal{O}_{F_e}(h) = -e$ and $\mathcal{O}_{F_e}(h) \cdot \mathcal{O}_{F_e}(h) = -e$. We have $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the complete linear system $|\mathcal{O}_{F_0}(h + f)|$ embeds $F_0$ into $\mathbb{P}^3$ as a smooth quadric surface.

Fix $A \in |\mathcal{O}_{F_0}(ah + af)|$, $a \geq 1$. Since $\omega_{F_0} \cong \mathcal{O}_{F_0}(-2h - 2f)$, the adjunction formula gives $\omega_A \cong \mathcal{O}_A((a-2)h + (a-2)f)$ and hence $p_a(A) = (a-1)^2$.

**Lemma 3.** *Fix integers $a, x$ such that $a \geq 5$ and $0 \leq x \leq \lfloor a^2/3 \rfloor$. Fix a general $S \subset F_0$ such that $\sharp(S) = x$ and a general $Y \in |\mathcal{I}_{2S}(ah + af)|$. Then $Y$ is integral and nodal and* Sing$(Y) = S$. *Let $u : C \to Y$ be the normalization map. Then $C$ has genus $(a-1)^2 - x$.*

*Proof.* Since $S$ is general, $a - 1 \geq 4$, and $3x \leq h^0(F_0, \mathcal{O}_{F_0}((a-1)h + (a-1)f))$, we have $h^1(F_0, \mathcal{I}_{2S}((a-1)h + (a-1)f)) = 0$ (see [14], Proposition 5.2 and Theorem 7.2). Obviously $h^2(F_0, \mathcal{I}_{2S}((a-2)h + (a-2)f)) = h^2(F_0, \mathcal{O}_{F_0}((a-2)h + (a-2)f)) = 0$. Since $\mathcal{O}_{F_0}(h + f)$ is very ample, Castelnuovo-Mumford's lemma gives that $\mathcal{I}_{2S}(ah + af)$ is spanned. Since $Y$ is general, Bertini's theorem gives $S = \mathrm{Sing}(Y)$. Fix $P \in S$. Since $\mathcal{I}_{2S}(ah + af)$ is spanned, a general element of $|\mathcal{I}_{2S}(ah+af)|$ has an ordinary node at $P$. Since $S$ is finite, each point of $S$ is an ordinary node. Since $S$ is general in $F_0$, $a \ge 3$, $S = \mathrm{Sing}(Y)$ and $Y$ is nodal, the curve $Y$ is integral. The adjunction formula gives $p_a(C) = (a-1)^2 - x$. $\square$

**Notation 2.** From now on $C$ is the normalization of a general $Y \in |\mathcal{I}_{2S}(ah+af)|$ with $a, x, S$ as in Lemma 3. Hence $p_a(C) = (a-1)^2 - x$.

**Proposition 1.** *Set* $\beta := \lceil (2a-1)/3 \rceil$.

*(a) Assume* $0 < x \le (a-\beta+1)^2$. *Then* $s_2(C) = 2a-2$ *and each line bundle* $R$ *of degree* $2a-2$ *(resp.* $2a-1$*) inducing a morphism* $C \to \mathbb{P}^2$ *birational onto its image is obtained in the following way. See* $Y$ *as a space curve using the composition of the inclusion* $Y \hookrightarrow F_0$ *and the embedding* $F_0 \hookrightarrow \mathbb{P}^3$ *induced by* $|\mathcal{O}_{F_0}(h+f)|$. *Then* $R$ *is induced by the linear projection of* $Y$ *into* $\mathbb{P}^2$ *from a point of* $S$ *(resp. a point of* $Y \setminus S$*).*

*(b) Assume* $x = 0$. *Then* $s_2(C) = 2a-1$ *and each line bundle evincing* $s_2(C)$ *is induced by the linear projection from some* $P \in Y$.

*Proof.* If $x > 0$, then the linear projection from $x$ gives $d_2(C) \ge 2a-2$. It also gives $s_2(C) \ge 2a-2$ for the following reason. Fix $P \in S$ and take a general $P' \in Y$. Since the line $D'$ spanned by $P$ and $P'$ is not contained in the smooth quadric $F_0 \subset \mathbb{P}^3$, it intersects $Y$ only at $P$ and $P'$. Hence the linear projection from $P$ is birational onto its image. Similarly, the linear projection from a smooth point of $Y$ is birational onto its image.

Take an integer $z \le 2a-1$ and a spanned $L \in \mathrm{Pic}^z(C)$ which induces a morphism $v : C \to \mathbb{P}^2$ birational onto its image. Fix a general $A \in |L|$ and set $B := u(A)$. Since $L$ is spanned and $A$ is general, we have $B \cap S = \emptyset$. Since $h^i(F_0, \mathcal{O}_{F_0}) = 0$ for $i = 1, 2$, $\omega_{F_0} \cong \mathcal{O}_{F_0}(-2h-2f)$ and $Y$ is nodal with $S$ as its singular locus, we have $H^0(C, \omega_C) \cong H^0(F_0, \mathcal{I}_S((a-2)h+(a-2)f))$. Since $L$ is spanned, Riemann-Roch gives $h^0(C, \omega_C \otimes \mathcal{O}_C(A \setminus \{Q\})) = h^0(C, \omega_C \otimes \mathcal{O}_C(A))$ for every $Q \in A$. Hence $h^0(F_0, \mathcal{I}_{S\cup(B\setminus\{O\}}((a-2)h+(a-2)f)) = h^0(F_0, \mathcal{I}_{S\cup B}((a-2)h+(a-2)f))$ for every $O \in B$. Hence $h^1(F_0, \mathcal{I}_{S\cup B}((a-2)h+(a-2)f)) > 0$. Since $S$ is general and $\sharp(S) \le (a-2)(a-3)$, we have $h^1(F_0, \mathcal{I}_S((a-2)h+(a-3)f)) = h^1(F_0, \mathcal{I}_S((a-3)h+(a-2)f)) = 0$. Using adjunction we get that two $g^1_a$'s on $C$ induced by the two rulings of $F_0$ are complete.

Since the monodromy group of the generic hyperplane section of $v(C)$ is the full symmetric group $S_z$, for all integers $u \ge 0$ and $v \ge 0$ such that $(u+1)(v+1) \le z$, and any $T \in |\mathcal{O}_{F_0}(uh+bf)|$, either $\sharp(T \cap B) \le (u+1)(v+1)-1$ or $B \subset T$. For $(u,v) = (0,1)$ (resp. $(u,v) = (0,1)$) we get that for any $D \in |\mathcal{O}_{F_0}(h)|$ (resp. $|\mathcal{O}_{F_0}(f)|$) either $\sharp(D \cap B) \le 1$ or $B \subset D$. The latter case is impossible, because

$\sharp(D \cap Y) \le a$ and the two $g^1_a$'s on $C$ induced by the two rulings of $F_0$ are complete.

Hence $\sharp(D \cap B) \le 1$ for every $D \in (|\mathcal{O}_{F_0}(h)| \cup |\mathcal{O}_{F_0}(f)|)$. Now assume the existence of $T \in |\mathcal{O}_{F_0}(h+f)|)$ such that $\sharp(B \cap T) \ge 4$. Hence $B \subset T$. Thus $|L|$ is induced by a subseries of $|\mathcal{O}_{F_0}(h+f)|$ (after deleting the base points). Let $V \subset H^0(F_0, \mathcal{O}_{F_0}(h+f))$ be the codimension 1 linear subspace inducing $|L|$ after deleting the base points. Since $\mathcal{O}_{F_0}(h+f)$ is very ample, the scheme-theoretic base locus of $V$ in $F_0$ is either empty or a point, $P$. We get that the base locus of the $g^2_{2a}$ induced by $V$ has degree $0, 1, 2$ and it has degree 2 if and only if $P \in S$, while it has degree 1 if and only if $P \in Y \setminus S$. Hence Proposition 1 is proved in this case.

Now assume that there is no such a curve $T$. We saw that $\sharp(T \cap B) \le 3$ for all $T \in |\mathcal{O}_{F_0}(h+f)|$. Set $\alpha := \lfloor z/3 \rfloor$, $B_0 := B$ and $S_0 := S$. Fix $A_1 \in |\mathcal{O}_{F_0}(h+f)|$ such that $a_1 := \sharp(A_1 \cap B_0)$ is maximal and set $S_1 := S_0 \setminus S_0 \cap A_1$ and $B_1 := B_0 \setminus B_0 \cap A_1$. For each integer $i \ge 2$ define the curve $A_1 \in |\mathcal{O}_{F_0}(h+f)|$, the integer $a_i$ and the sets $B_i$ and $S_i$ in the following way. Fix $A_i \in |\mathcal{O}_{F_0}(h+f)|$ such that $a_i := \sharp(A_i \cap B_{i-1})$ is maximal and set $S_i := S_{i-1} \setminus S_{i-1} \cap A_i$ and $B_i := B_{i-1} \setminus B_{i-1} \cap A_i$. Since $h^0(F_0, \mathcal{O}_{F_0}(h+f)) = 4$ if $a_i \le 2$, then $B_{i-1} \subset A_i$ and $B_i = \emptyset$. Since we proved that $a_i \le 3$ for all $i$, we have $a_i = 3$ for all $i \le \alpha$, $a_{\alpha+1}) = z - 3 \cdot \lfloor z/3 \rfloor$ and $a_i = 0$ for all $i \ge \alpha + 2$. For each $i \ge 1$ we have an exact sequence

$$\begin{aligned} &0 \to \mathcal{I}_{S_i \cup B_i}((a-2-i)h + (a-2-i)f) \\ &\to \mathcal{I}_{S_{i-1} \cup B_{i-1}}((a-1-i)h + (a-1-i)f) \\ &\to \mathcal{I}_{A_i \cap (B_{i-1} \cup S_{i-1}}((a-1-i)h + (a-1-i)f) \to 0 \end{aligned} \tag{1}$$

We take $A_1, \dots, A_\alpha$ with the additional condition that $x_i := \sharp(S_{i-1} \cap A_i)$ is maximal among all $A_i$ with $\sharp(A_i \cap B_{i-1}) = 3$. Notice that $x_i \ge x_j$ for all $i \le j \le \alpha$. Since $S$ is general, we have $x_i \le 3$ for all $i$. Hence $a_i + x_i \le 6$ for all $i$. Since $\sharp(D \cap B) \le 1$ for every $D \in (|\mathcal{O}_{F_0}(h)| \cup |\mathcal{O}_{F_0}(f)|)$, each $A_i$, $i \le \alpha$, is irreducible, i.e. $A_i \cong \mathbb{P}^1$. We may also take $A_{\alpha+1}$ irreducible. We have $\alpha \le a-3$ and equality holds if and only if $a = 5$ and $z = 9$. Since $\alpha \le a-3$ we get $h^1(A_i, \mathcal{I}_{A_i \cap (B_{i-1} \cup S_{i-1}, A_i}((a-1-i)h + (a-1-i)f)) = 0$; if $x = 0$, we would get the same result even if $\alpha = a-2$. Hence applying (1) for $i = 1, \dots, \alpha$ we get $h^1(F_0, \mathcal{I}_{S_\alpha \cup B_\alpha}((a-2-\alpha)h + (b-2-\alpha)f)) > 0$. The set $S_\alpha$ is general in $F_0$ and $\sharp(S_\alpha) \le 2$. We have $\alpha \le \beta$ and equality holds if and only if $z = 2a-1$ and $2a - 1 \equiv 0 \pmod 3$. Hence $\alpha < \beta$ if $B_\alpha \ne \emptyset$. Hence applying $\beta - \alpha$ times (1) we get $h^1(F_0, \mathcal{I}_{S_\beta}((a-\beta, a-\beta)) > 0$. Since $S$ is general and $S_\beta \subseteq S$, we get $x > (a-\beta+1)^2$, a contradiction. In the case $x = 0$ we just use that $\sharp(B_\alpha) \le 2$ and $a - \alpha > 0$. □

**Remark 2.** Take the set-up of Notation 2. Fix an odd integer $r \geq 3$. Since $h^0(F_0, \mathcal{O}_{F_0}(h + ((r-1)/2)f)) = r+1$, $\mathcal{O}_{F_0}(h+((r-1)/2)f)$ is very ample and $\mathcal{O}_{F_0}(h+((r-1)/2)f) \cdot \mathcal{O}_{F_0}(ah+af) = a(r-1)$, we have $s_r(C) \leq a(r-1)$. Let $\phi_r : F_0 \to \mathbb{P}^r$ denote the embedding of $F_0$ induced by $|\mathcal{O}_{F_0}(h + ((r-1)/2)f)|$. Fix any $P \in \phi_r(F_0)$ and call $\mathbb{D}$ the union of the lines of $\phi_r(F_0)$ containing $P$ (one line if $r \geq 5$, two lines if $r = 3$). Since $\phi_r(F_0)$ is cut out by quadrics, for any line $D \subset \mathbb{P}^r$ either $D \subset \phi_r(F_0)$ or $\deg(D \cap \phi_r(F_0)) \leq 2$. Hence the linear projection from $P$ induces an isomorphism of $F_0 \setminus \mathbb{D}$ into a degree $r-2$ surface of $\mathbb{P}^{r-1}$. Hence taking the linear projection from some point of $\phi_r(S)$ (if $S \neq \emptyset$) or of $\phi_r(Y)$ (if $S = \emptyset$) we get $s_{r-1}(C) \leq a(r-1) - 1 - \min\{1, x\}$.

**Lemma 4.** *Assume $3x \leq a^2$ and that $Y$ is general in $|\mathcal{I}_{2S}(ah+af)|$. Let $\rho : C \to \mathbb{P}^1$ and $\rho' : C \to \mathbb{P}^1$ denote the degree $a$ morphisms induced by $|\mathcal{O}_{F_0}(h)|$ and $|\mathcal{O}_{F_0}(h)|$. Then neither $\rho$ nor $\rho'$ factors as $C \stackrel{f_1}{\to} C' \stackrel{f_2}{\to} \mathbb{P}^1$ with $\deg(f_1) \geq 2$ and $\deg(f_2) \geq 2$.*

*Proof.* It is sufficient to find $O, O' \in \mathbb{P}^1$ such that $\sharp(\rho^{-1}(O)) = \sharp(\rho'^{-1}(O')) = a-1$ (indeed, if $O$ (resp. $O'$) exists, then the corresponding fiber of $\rho$ (resp. $\rho'$) has exactly one ramification point at which the ramification has order 2). Fix general $P_1, P_2 \in F_0$. Take $D_1 \in |\mathcal{O}_{F_0}(h)|$ containing $P_1$ and $D_2 \in |\mathcal{O}_{F_0}(f)|$ containing $P_2$. Take $S_i \subset D_i \setminus \{P_i\}$ such that $\sharp(S_i) = a-2$. We have $P_2 \notin D_1$ and $P_2 \notin D_2$. Let $Z_i \subset D_i$ the degree 2 divisor of $D_i$ with $P_i$ has its support. By [14], $h^1(F_0, \mathcal{I}_{2S}((a-1)h + ((a-1)f)) = 0$. Since $\deg(Z_i \cup S_i) = a$, we get $h^1(F_0, \mathcal{I}_{2S \cup Z_1 \cup S_i \cup Z_2 \cup S_2}(ah+bf)) = 0$. Hence $Z_1 \cup S_1 \cup Z_2 \cup S_2$ gives $2a$ independent conditions to $|\mathcal{I}_{2S}(ah+af)|$. Since $\dim(F_0) = 2$ and $\dim(D_i) = 1$, varying $P_1$ and $P_2$ we get the existence of $O$ and $O'$ for a general $Y \in |\mathcal{I}_{2S}(ah+bf)|$. □

**Lemma 5.** *Assume $x \leq (a - \lceil (2a-1)/3 \rceil + 1)^2$ and that $Y$ is general in $|\mathcal{I}_{2S}(ah+af)|$. Then $d_1(C) = a$.*

*Proof.* Let $\rho : C \to \mathbb{P}^1$ be a degree $a$ morphism induced by a ruling of $F_0$. The morphism $\rho$ gives $d_1(C) \leq a$. Assume $z := d_1(C) < a$ and take a degree $z$ morphism $u' : C \to \mathbb{P}^1$. The morphism $\phi = (\rho, u') : C \to \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is birational onto its image by Lemma 4. Hence $\phi(C) \in |\mathcal{O}_{F_0}(ah + zf)|$. Taking a linear projection from a general $O \in \phi(C)$ we get that $C$ has infinitely many morphisms $C \to \mathbb{P}^2$ birational onto its image and with image of degree $\leq a + z - 1 \leq 2a - 2$, contradicting Proposition 1. □

*Proof of Theorem 2.* Proposition 1 gives $s_2(C) = 2a - 1 - \min\{1, x\}$. Remark 2 gives $s_r(C) \leq a(r-1)/2$ for every odd integer $r$ and $s_r(C) \leq ar/2 - 1 - \min\{1, x\}$ for every even integer $r$.

I) We first prove parts (a) and (b) of Theorem 2. Fix an integer $z \le 3a-2-\min\{1,x\}$ such that there is a morphism $v : C \to \mathbb{P}^4$ birational onto its image, with $\deg(v(C)) = z$ and $v(C)$ spanning $\mathbb{P}^4$. We have $p_a(v(C)) \ge p_a(C) = (a-1)^2 - x$. Set $m_1 := \lfloor (z-1)/4 \rfloor$, $\epsilon_1 = z-1-4m_1$, $\mu_1 = 1$ if $\epsilon_1 = 3$ and $\mu_1 := 0$ if $\epsilon_1 \ne 3$. Set $\pi_1(4,z) := 2m_1(m_1-1) + m_1(\epsilon_1+1) + \mu_1$. Assume $x < (a-1)^2 - (z-1)^2/8$. Since $\pi_1(4,z) \le 2m_1(m_1+1)+1$ and $m_1 \le (z-1)/4$, we have $p_a(C) > \pi_1(4,z)$. If $z \le 8a/3$, then to get a contradiction it is sufficient to assume $x < (a-1)^2 - (8a-3)^2/72$. The maximal value of $x$ allowed here is of order $a^2/9$ and we used a similar restriction to apply Proposition 1.

(i) In this step we assume that $T$ is a cone. The minimal desingularization of $T$ is isomorphic to $F_3$ and the composition of the desingularization map with the inclusion $T \hookrightarrow \mathbb{P}^4$ is induced by the linear system $|\mathcal{O}_{F_3}(h+3f)|$. Let $E$ be the strict transform of $v(C)$ in $F_3$. Let $c, y$ be the integers such that $E \in |\mathcal{O}_{F_3}(ch+yf)|$. We have $y \ge 3c$ and $z = \mathcal{O}_{F_3}(ch+yf) \cdot \mathcal{O}_{F_3}(h+3f) = y$. Since $d_1(C) = a$ (Lemma 5), we have $c \ge a$. Hence $z \ge 3a$, a contradiction.

(ii) In this step we assume $T \cong F_1$. Take $c, y$ such that $v(C) \in |\mathcal{O}_{F_1}(ch+yf)|$. We have $z = \mathcal{O}_{F_1}(ch+yf) \cdot \mathcal{O}_{F_1}(h+2f) = y+c$. Since $d_1(C) = a$, we have $c \ge a$. Since $s_2(C) \ge 2a-1-\min\{1,x\}$ (Proposition 1) and $|\mathcal{O}_{F_1}(h+f)|$ induces a birational morphism $F_1 \to \mathbb{P}^2$, we have $\mathcal{O}_{F_1}(ch+yf) \cdot \mathcal{O}_{F_1}(h+f) \ge 2a-1-\min\{1,x\}$, i.e. $y \ge 2a-1-\min\{1,x\}$. Hence $z \ge 3a-1-\min\{1,x\}$, a contradiction.

II) Now we prove parts (c) and (d) of Theorem 2. We assume $a \ge 108$. Fix an integer $z \le 4a-2-\min\{1,x\}$ such that there is a spanned $L \in \mathrm{Pic}^z(C)$ with $h^0(C,L) = 7$ and whose associated morphism $v : C \to \mathbb{P}^6$ is birational onto its image. We have $z = \deg(v(C))$ and $p_a(v(C)) \ge p_a(C) = (a-1)^2 - x$. Set $m_2 := \lfloor (z-1)/7 \rfloor$, $\epsilon_2 = z-1-7m_1$, $\mu_2 = \max\{0, \lfloor (4+\epsilon_2)/2 \rfloor\}$ and $\pi_2(6,z) := 7m_2(m_2-1)/2 + m_2(\epsilon_2+2) + \mu_2$. Assume $x < (a-1)^2 - (z-1)^2/12$. Since $\pi_2(6,z) \le 2 + m_2(7m_2+16)/2$, we have $p_a(C) > \pi_2(6,z)$ if $x < (a-1)^2 - (z-1)(z+15)/14$. Hence [11], Theorem 3.22, gives that $v(C)$ is contained in a surface $T$ such that $\deg(T) \in \{5,6\}$. Since $s_5(C) \le 3a$, we have $6 \cdot s_5(C) \le 18a/5$. If we only need to exclude that $z \le 18a/5$, it is sufficient to assume $x < (a-1)^2 - (18a-5)(18a+75)/350$. Notice that $(a-1)^2 - (18a-5)(18a+75)/350 \sim (13/175)a^2$ when $a \gg 0$.

First assume $\deg(T) = 5$. By the classification of minimal degree surfaces either $T$ is a cone over a rational normal curve of $\mathbb{P}^5$ or $T \cong F_e$, $e \in \{1,3\}$, embedded by the complete linear system $|\mathcal{O}_{F_1}(h+(e+1)f)|$.

(i) In this step we assume that $T$ is a cone. It is well-known that the minimal desingularization of $T$ is isomorphic to $F_5$ and that the composition of the desingularization map with the inclusion $T \hookrightarrow \mathbb{P}^6$ is induced by the linear system $|\mathcal{O}_{F_5}(h+5f)|$. Let $E$ be the strict transform of $v(C)$ in $F_5$. Let $c, y$ be the integers such that $E \in |\mathcal{O}_{F_5}(ch+yf)|$. We have $y \geq 5c$ and $z = \mathcal{O}_{F_5}(ch+yf) \cdot \mathcal{O}_{F_3}(h+3f) = y$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Hence $z \geq 5a$, a contradiction.

(ii) In this step we assume $T \cong F_3$. Take $c, y$ such that $v(C) \in |\mathcal{O}_{F_3}(ch+yf)|$. We have $z = \mathcal{O}_{F_3}(ch+yf) \cdot \mathcal{O}_{F_3}(h+4f) = y+c$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Since $|\mathcal{O}_{F_3}(h+3f)|$ induces a birational morphism $F_1 \to \mathbb{P}^2$, we have $\mathcal{O}_{F_3}(ch+yf) \cdot \mathcal{O}_{F_3}(h+3f) \geq s_4(C)$, i.e. $y \geq s_4(C)$. Hence part (a) gives $z > a + 8a/3 > 18a/5$, a contradiction.

(iii) In this step we assume $T \cong F_1$. Take $c, y$ such that $v(C) \in |\mathcal{O}_{F_1}(ch+yf)|$. We have $z = \mathcal{O}_{F_1}(ch+yf) \cdot \mathcal{O}_{F_1}(h+3f) = y+2c$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Since $s_2(C) \geq 2a-1-\min\{1,x\}$(Proposition 1) and $|\mathcal{O}_{F_1}(h+f)|$ induces a birational morphism $F_1 \to \mathbb{P}^2$, we have $\mathcal{O}_{F_1}(ch+yf) \cdot \mathcal{O}_{F_1}(h+f) \geq 2a-1-\min\{1,x\}$, i.e. $y \geq 2a-1-\min\{1,x\}$. Since $c \geq a$, we get $z \geq 4a-1-\min\{1,x\}$, a contradiction.

From now on we assume $\deg(T) = 6$. We have $\deg(v(C)) > \deg(T)$ and the morphism $v : C \to \mathbb{P}^6$ is induced by a complete linear system. Hence $T$ is not an external linear projection of a degree 6 surface of $\mathbb{P}^7$. Hence $T$ is a normal del Pezzo surface (introduction of [2], [3], [5], [6], [7], Corollary 6.5 and Theorem 9.16). Hence a general hyperplane section $H \cap T$ of $T$ is a smooth degree 5 curve. Hence $p_a(T \cap H) \leq 1$ (and indeed $p_a(T \cap H) = 1$). If $T$ is not a cone, then $T$ is Gorenstein and $\mathcal{O}_T(-1) \cong \omega_T$ (see [7], theorem 9.16). If $T$ is not a cone, then each singular point of $T$ is a rational hypersurface singularity (see [4], p. 116) and hence $T$ is the canonical model of a weak del Pezzo surface in the sense of [10].

(iv) Here we assume that $T$ is a cone over a linearly normal elliptic curve $E_1$ of $\mathbb{P}^5$. Call $O$ the vertex of $T$ and $\mu$ the multiplicity of $v(C)$ at $O$. We have $z = \mu + 6k$, where $k \in \mathbb{N}$ and $k$ is the degree of the covering $\tau : C \to E_1$ induced from the linear projection from $O$. Since $E_1$ has infinitely many $g^2_3$'s, the covering $\tau$ shows that $C$ has infinitely many $g^2_{3k}$. Proposition 1 gives $3k \geq 2a-1$ and hence $z \geq 6(2a-1)/3$, a contradiction.

(v) Here we assume that $T$ is a weak del Pezzo surface of degree 6 in the sense of [10]. Fix $P_1 \in \mathbb{P}^2$ and call $\Pi(1)$ the blowing up of $\mathbb{P}^2$ at $\mathbb{P}^1$. Fix $P_2 \in \Pi(1)$ and call $\Pi(2)$ the blowing-up of $\Pi(1)$ at $P_2$. Fix $P_3 \in \Pi(2)$ and call $\Pi$ the blowing up of $\Pi(2)$ at $P_3$. The anticanonical line bundle $\omega_\Pi^\vee$ is spanned and it induces a morphism $\phi : \Pi \to \mathbb{P}^6$ which is birational onto its image. There are $P_1, P_2, P_3$ with $T = \phi(\Pi)$ and we use these points $P_1, P_2, P_3$. Let $E$ denote the strict transform of $v(C)$ in $\Pi$. Notice that $C$ is the normalization of $E$. We take a basis $\ell$, $\ell_1$, $\ell_2$, $\ell_3$ of $\mathrm{Pic}(\Pi) \cong \mathbb{Z}^4$ with $\ell$ coming from $\mathcal{O}_{\mathbb{P}^2}(1)$, each $\ell_i$ effective (but not necessarily irreducible) $\ell_i^2 = -1$ for all $i$, $\ell^2 = 1$, $\ell \cdot \ell_i = 0$ for all $i$ and $\ell_i \cdot \ell_j = 0$ for all $i \neq j$. Write $E = u\ell - u_1\ell_1 - u_2\ell_2 - u_3\ell_3$ for some $u, u_i \in \mathbb{N}$. Since $\omega_\Pi \cong -3\ell + \ell_1 + \ell_2 + \ell_3$ and $\phi$ is induced by the anticanonical linear system, we have $z = 3u - u_1 - u_2 - u_3$. Since $\dim(|\ell|) = 2$ and the linear system $|\ell|$ is birationally very ample, we have $s_2(C) \geq u$. Hence $u \geq 2a - 4$ (Proposition 1). Since $h^0(\Pi, \mathcal{O}_\Pi(2\ell - \ell_1 - \ell_2 - \ell_3)) = 3$ and $|\mathcal{O}_\Pi(2\ell - \ell_1 - \ell_2 - \ell_3)|$ is birationally very ample, we have $s_2(C) \leq 2u - u_1 - u_2 - u_3$. Proposition 1 gives $2u - u_1 - u_2 - u_3 \geq 2a - 4$. Hence $z \geq 4a - 8$, a contradiction. □

*Proof of Corollary 1.* Set $g_{a,x} := (a-1)^2 - x$. Notice that $g_{a,0} - g_{a-1,0} = 2a - 3$. Hence every integer $g \geq 19$ is of the form $g_{a,x}$ for uniquely determined integers $a, x$ such that $a \geq 5$ and $0 \leq x \leq 2a - 4$. Since $(a-1)^2 - (18a - 5)(18a + 75)/350 \sim (13/175)a^2$ when $a \gg 0$, for $a \gg 0$ all integers $x \leq 2a - 4$ are allowed in parts (b) and (d) of Theorem 2. □

## Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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