ON THE GONALITY SEQUENCE AND THE BIRATIONAL GONALITY SEQUENCE OF SMOOTH CURVES

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Abstract: Let $C$ be a smooth curve of genus $g$. For any integer $r \geq 2$ the $r$-gonality of $C$ is a minimal degree of a line bundle $L$ such that $h^0(C, L) = r + 1$. If we assume that the associated map $C \to \mathbb{P}^r$ is birational onto its image, then we get the $r$-birational gonality $s_r(C)$. For $g = 29, 30$ we prove the existence of $C$ with $d_2(C) = 10$ and $d_3(C) = 16$. For many genera we prove the existence of $C$ with $s_4(C)/4 > s_3(C)/3$ and $s_6(C)/6 > s_5(C)$ (both inequalities for the same curve).

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1. Introduction

Let $C$ be a smooth and projective curve defined over an algebraically closed base field with characteristic zero. For each integer $r \geq 1$ the $r$-gonality $d_r(C)$ of $C$ is the minimal degree of a line bundle $L$ on $C$ such that $h^0(C, L) \geq r + 1$ (see [15]). For many (but not all) curves the slope inequality $d_{r+1}(C)/(r + 1) \leq d_r(C)/r$ holds for all $r \geq 2$ (it is trivially true for $r = 1$) (see [15]). It is usually easier to study the following invariant $s_r(C)$ (see [9]). For each integer $r \geq 2$ let $s_r(C)$ denote the minimal degree of a line bundle $L$ such that $h^0(C, L) \geq r + 1$ and the associated rational map $C \dasharrow \mathbb{P}^r$ is birational onto its image (see [9], §2). It is easy to check that each line bundle $L$ evincing $s_r(C)$ is spanned and
\[ h^0(C, L) = r + 1. \] Obviously \( s_r(C) \geq d_r(C) \). Equality holds if \( d_r(C) < r \cdot d_1(C) \) and \( C \) has no non-trivial morphisms onto a smooth curve of positive genus. The sequence \( \{ s_r(C) \}_{r \geq 2} \) is called the birational gonality sequence of \( C \). It was introduced in [9] and it seems to be a natural invariant of \( C \): ancient algebraic geometers often looked at birational model of an object, say the curve \( C \), in a fixed projective space. Fix an integer \( r \geq 2 \). It is often technically easier to compute the integer \( s_r(C) \) than the integer \( d_r(C) \). Quite often \( s_r(C) = d_r(C) \) (see Remark 1). Fix an integer \( r \geq 3 \). Another natural numerical invariant of \( C \) is the minimal integer, \( \beta_r(C) \), of a very ample line bundle \( L \) on \( C \) such that \( h^0(C, L) = r + 1 \). We have nothing to say here for the sequence \( \beta_r(C) \).

In Section 2 we prove the following result.

**Theorem 1.** For each \( g \in \{29, 30\} \) there is a smooth curve \( C \) of genus \( g \) with \( d_2(C) = 10 \) and \( d_3(C) = 16 \).

In Section 3 we construct several examples of smooth curves \( C \) for which two slopes inequalities for \( s_r \) fails (in our case we have \( s_4(C)/4 > s_3(C)/3 \) and \( s_6(C)/6 > s_5(C)/5 \)). We stress that both inequalities hold for the same curve \( C \), i.e. in the birational gonality sequence of \( C \) two non-consecutive slope inequalities fail. The examples give the following result.

**Theorem 2.** Fix integers \( a, x \) such that \( a \geq 5 \) and \( x \leq a^2/9 \). Let \( F_0 \subset \mathbb{P}^3 \) be a smooth quadric surface. Fix a general \( S \subset F_0 \) such that \( z(S) = x \) and a general \( Y \in |\mathcal{I}_2S(ah + bf)| \). Then \( Y \) is integral, nodal and \( \text{Sing}(Y) = S \). Let \( C \) be the normalization of \( Y \). The smooth curve \( C \) has genus \( (a - 1)^2 - x \), \( s_2(C) = 2a - 1 - \min\{1, x\} \), \( s_3(C) = 2a, s_5(C) \leq 3a \) and \( s_7(C) \leq 4a \).

(a) Fix an integer \( z \leq 3a - 2 - \min\{1, x\} \). If \( x < -1 + (a - 1)^2 - (z - 1)(z + 3)/8 \), then \( z + 1 \leq s_4(C) \leq 3a - 1 - \min\{1, x\} \).

(b) If \( x \leq a^2/9 - 2a + 1 \), then \( s_4(C)/4 > s_3(C)/3 \).

(c) Assume \( a \geq 108 \). Fix an integer \( z \leq 4a - 2 - \min\{1, x\} \). If \( x < (a - 1)^2 - (z - 1)(z + 15)/14 \), then \( z + 1 \leq s_4(C) \leq 3a - 1 - \min\{1, x\} \).

(d) If \( a \geq 108 \) and \( x < (a - 1)^2 - (18a - 5)(18a + 75)/350 \), then \( s_6(C)/6 > s_5(C)/5 \).

Notice that \( (a - 1)^2 - (18a - 5)(18a + 75)/350 \sim (13/175)a^2 \) when \( a \gg 0 \). From the proof of Theorem 2 we get the following result.

**Corollary 1.** There is an integer \( g_0 \) such that for every integer \( g \geq g_0 \) there is a smooth curve \( C \) of genus \( g \) with \( s_4(C)/4 > s_3(C)/3 \) and \( s_6(C)/6 > s_5(C)/5 \).

**Question 1.** (a) Are there a smooth curve \( C \) and an integer \( r \geq 3 \) such that \( s_{r+1}(C)/s_r(C) \geq (r + 2)/r \)? (or with \( s_{r+1}(C)/s_r(C) > (r + 2)/r \)?
(b) Are there a smooth curve C and an integer \( r \geq 3 \) such that \( s_{r+2}(C)/(r+2) > s_{r+1}(C)/(r+1) > s_r(C)/r? \)

**Remark 1.** Fix a smooth curve \( C \) of genus \( \geq 3 \) and an integer \( r \geq 2 \). Let \( L \in \text{Pic}^d(C) \), \( d := s_r(C) \), be a line bundle evincing \( d_r(C) \). Let \( \phi : C \to \mathbb{P}^r \) the morphism associated to \(|L|\). Let \( F \) be the normalization of \( \phi(C) \). If \( s_r(C) \neq d_r(C) \), then \( \deg(\phi) \geq 2 \), i.e. \( F \) has genus \( q < g \). Assume that there is no non-constant map \( C \to E \) with \( E \) a smooth curve of genus \( > 0 \). We get \( q = 0 \). Since \( \phi \) is induced by a complete linear system, we get that \( \phi(C) \) is a rational normal curve. Hence \( d_r(C) = r \cdot d_1(C) \) and any complete linear system evincing \( d_r(C) \) is of the form \(|R^{\otimes r}|\) with \( R \) evincing the gonality \( d_1(C) \) of \( C \).

**2. Proof of Theorem 1**

**Notation 1.** For any smooth surface \( M \) and any finite set \( S \subset M \) let \( 2S \) denote the closed subscheme of \( M \) with \(|I_S|^2 \) as its ideal sheaf. The scheme \( 2S \) is a zero-dimensional scheme, \((2S)_{\text{red}} = S \) and \( \deg(2S) = 3 \cdot \sharp(S) \). In this section we take \( M = \mathbb{P}^2 \). In the next section we will take \( M = F_0 \) (a smooth quadric surface).

**Lemma 1.** Fix a general \( S \subset \mathbb{P}^2 \) such that \( \sharp(S) \leq 7 \). Fix any \( P \in \mathbb{P}^2 \setminus S \). Then \( h^1(I_{S \cup \{P\}}(3)) = 0 \).

**Proof.** Set \( S' := S \cup \{P\} \). First assume the existence of a line \( D \subset \mathbb{P}^2 \) such that \( \sharp(D \cap S') \geq 3 \). Since \( S \) is general, we have \( P \in D \) and \( \sharp(S \cap D) = 2 \). Since \( \sharp(S' \cap D) \leq 4 \), we have \( h^1(D, I_{S' \cap D}^2(3)) = 0 \). Since \( \sharp(S' \cap D) \leq 5 \) and \( S' \setminus S' \cap D \) is general, we have \( h^1(I_{S' \setminus S' \cap D}^2(2)) = 0 \). The exact sequence

\[
0 \to I_{S' \setminus S' \cap D}(2) \to I_{S'}(3) \to I_{S' \cap D}(3) \to 0
\]

gives \( h^1(I_{S'}(3)) = 0 \). Now assume \( \sharp(D \cap S') \leq 2 \) for every line \( D \). Fix a conic \( E \) such that \( \sharp(S' \cap E) \) is maximal. Since \( h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6 \), we have \( \sharp(S' \cap E) \geq 5 \). Hence \( E \) is a smooth conic and \( \sharp(S' \setminus S' \cap E) \leq 3 \). Since no 3 of the points of \( S' \) are collinear, we have \( h^1(I_{S' \setminus S' \cap E}(1)) = 0 \). Since \( S \) is general, \( E \) contains at most 5 points of \( S \). Hence \( \sharp(S' \cap E) \leq 6 \). Since \( E \) is a smooth conic, \( h^1(E, I_{S' \cap E}(3)) = 0 \). As above an obvious exact sequence gives \( h^1(I_{S' \cup \{P\}}(3)) = 0 \).

Fix an integer \( x \in \{6, 7\} \). Let \( S \subset \mathbb{P}^2 \) be a general set with \( \sharp(S) = x \). Let \( Y \subset \mathbb{P}^2 \) be a general element of \(|I_{2S}(10)|\). It is easy to check that \( h^1(I_{2S}(10)) = 0 \) and that a general \( Y \in |I_{2S}(10)| \) is integral, nodal and with \( \text{Sing}(Y) = S \) (see
[1], Theorem 3.2). Hence the normalization $C$ of $Y$ is a smooth and connected curve of genus $36 - x$. To prove Theorem 1 it is sufficient to prove that $d_2(C) = 10$ and $d_3(C) = 16$ for at least one $C$. We prove it when $Y$ is general. More precisely we assume that there is no order two automorphism $\sigma$ of $\mathbb{P}^2$ with $\sigma(Y) = Y$.

**Lemma 2.** Assume $Y$ integral, nodal and with $\text{Sing}(Y) = S$. Assume that there is no order two automorphism of $\mathbb{P}^2$ sending $Y$ into itself. Then $C$ has no order two automorphism.

**Proof.** The line bundle $\mathcal{O}_Y(1)$ is the only line bundle $R$ on $Y$ with $\deg(R) \leq d$ and $h^0(Y, R) \geq 3$ by Noether’s theorem for integral singular curves (see [13], Theorem 2.1). Hence $Y$ has no order 2 automorphism. The case $k = 1$ and $\delta = 1$ of [8], Theorem 2.3, gives $d_1(C) = 8$, that the only $g_8^1$’s on $C$ are the ones induced by $|\mathcal{I}_{P}(1)|$ for some $P \in S$. Every $g_8^1$ on $C$ is induced by a pencil of lines through some point of $Y \setminus S$ (part i) of [8], Theorem 2.4). Hence $u^*(\mathcal{O}_Y(1))$ is the only $g_{10}^2$ on $C$. Hence each automorphism of $C$ induces an automorphism of $Y$. Hence $C$ has no order two automorphism.

**Proof of Theorem 1.** Fix $x \in \{6, 7\}$ and a general $S \subset \mathbb{P}^2$ such that $\sharp(S) = x$. Fix any $Y$ as in Lemma 2. Let $u : C \to Y$ be the normalization map. The smooth curve $C$ has genus $36 - x$ and we will check that $d_2(C) = 10$ and $d_3(C) = 16$. As in Lemma 2 we see that $d_2(C) = 10$. Fix $S' \subset S$ such that $\sharp(S') = 2$. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, the linear system $|\mathcal{I}_{S'}(2)|$ gives $d_3(C) \leq 16$. Assume $z := d_3(C) \leq 15$ and take $L \in \text{Pic}^z(C)$ evincing $d_3(C)$. The line bundle $L$ is spanned (see [15], Lemma 3.1). Hence $|L|$ induces a morphism $v : C \to \mathbb{P}^4$.

(a) In this step we prove that $v$ is birational onto its image. Assume that $v$ is not birational onto its image. Set $w := \deg(v)$. Hence $\deg(v(C)) = z/w \in \mathbb{N}$. Since $C$ has no order two automorphism, we have $w \geq 3$. Since $v(C)$ spans $\mathbb{P}^3$ we have $z/w \geq 3$. Let $\nu : J \to v(C)$ denote the normalization map. Since $C$ is smooth, $v$ induces a morphism $\phi : C \to J$ such that $v = \nu \circ \phi$. Since $h^0(C, L) = 4$, we get $h^0(J, v^*(\mathcal{O}_{v(C)}(1))) = 4$. We have $d_1(C) \leq w \cdot d_1(J)$. If $J$ has genus 0, then we get $d_1(C) \leq z/3$ and hence $z \geq 3(d - 2) > 2d - 4$, a contradiction. Hence $z/w \geq 4$. Thus it is sufficient to check the case $(z, w) = (15, 3)$. Assume $(z, w) = (15, 3)$ and hence $\deg(v(C)) = 5$. Hence $g(J) \leq 2$. Hence $d_1(J) \leq 2$. Hence $8 = d_1(C) \leq 4$, a contradiction.

(b) Fix a general $A \in |L|$ and set $B := u(A)$. Since $L$ has no base points and $A$ is general, we have $B \cap S = \emptyset$. Since $v$ is birational onto its image, the monodromy group of a general hyperplane section of $v(C)$ is the full symmetric
group. Hence for every $t \in \{1, 2, 3\}$ either $h^0(\mathcal{I}_B(t)) > 0$ or $h^0(\mathcal{I}_{B_1}(t)) = \max\{0, (t+2)/2 − \sharp(B_1)\}$ for all $B_1 \subset B$. Since $Y$ is a degree 10 nodal curve with $S$ as its singular locus, adjunction theory gives $H^0(C, \omega_C) \cong H^0(\mathcal{I}_S(7))$. Since $L$ is spanned and $A \in |L|$, Riemann-Roch gives $h^0(C, \omega_C \otimes \mathcal{O}_C(A \setminus \{Q\}) = h^0(C, \omega_C \otimes \mathcal{O}_C(A))$ for every $Q \in A$. Hence $h^0(\mathcal{I}_{S \cup (B \setminus \{O\})}(7)) = h^0(\mathcal{I}_{S \cup B}(7))$ for each $O \in B$. Hence $h^1(\mathcal{I}_{S \cup B}(7)) > 0$.

(b1) Assume $h^0(\mathcal{I}_B(4)) = 0$. Hence $h^0(\mathcal{I}_B(t)) = 0$ for all $t \leq 3$. Hence for every $t \in \{1, 2, 3, 4\}$ we have $\sharp(B \cap D_t) \leq (t+2)/2 − 1$ for every degree $t$ curve $D_t$. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(4)) = 15$, there is a degree 4 curve $W$ containing 14 points of $B$. Since $h^0(\mathcal{I}_B(4)) = 0$, we have $z = 15$ and $B \cap W$ is a unique point, $P$. Set $S' := S \setminus W$. First assume $h^1(W, \mathcal{I}_{W \cap (S \cup B)}(7)) = 0$. The exact sequence

$$0 \rightarrow \mathcal{I}_{S' \cup \{P\}}(3) \rightarrow \mathcal{I}_{S \cup B}(7) \rightarrow \mathcal{I}_{W \cap (S \cup B)}(7) \rightarrow 0$$

gives $h^1(\mathcal{I}_{S' \cup \{P\}}(3)) > 0$. Lemma 1 gives a contradiction.

Now assume $h^1(W, \mathcal{I}_{W \cap (S \cup B)}(7)) \neq 0$. Since for every $t \in \{1, 2, 3\}$ we have $\sharp(B \cap D_t) \leq (t+2)/2 − 1$ for every degree $t$ curve $D_t$, $W$ is irreducible. We have $p_a(W) = 3$ and $\deg(\mathcal{O}_W(7)) = 28$. Since $\deg(\omega_W) = 4$ and the torsion free sheaf $\mathcal{I}_{W \cap (S \cup B)}(7)$ satisfies $h^1(W, \mathcal{I}_{W \cap (S \cup B)}(7)) > 0$, we have $z + \sharp(W \cap S) \geq 24$, contradicting the inequality $x + z \leq 22$.

(b2) Assume $h^0(\mathcal{I}_B(4)) > 0$ and $h^0(\mathcal{I}_B(3)) = 0$. Let $G \subset \mathbb{P}^2$ be a quartic curve containing $B$. Since $S$ is general, we have $h^1(\mathcal{I}_{S \setminus (S \cap G)}(3)) = 0$. Hence the exact sequence

$$0 \rightarrow \mathcal{I}_{S \setminus (S \cap G)}(3) \rightarrow \mathcal{I}_{S \cup B}(7) \rightarrow \mathcal{I}_{G \cap (S \cup B), G}(7) \rightarrow 0$$

gives $h^1(G, \mathcal{I}_{G \cap (S \cup B), G}(7)) > 0$. Since $h^0(\mathcal{I}_B(t)) = 0$ for $t = 2, 3$, we have $\sharp(B \cap D) \leq 5$ for all conics $D$ and $\sharp(B \cap D') \leq 9$ for all cubics $D'$. Hence $G$ is irreducible. We have $p_a(G) = 3$ and $\deg(\mathcal{O}_G(7)) = 28$. Since $\deg(\omega_G) = 4$ and the torsion free sheaf $\mathcal{I}_{G \cap (S \cup B), G}$ satisfies $h^1(G, \mathcal{I}_{G \cap (S \cup B), G}(7)) > 0$, we have $z + \sharp(G \cap S) \geq 24$, contradicting the inequality $x + z \leq 22$.

(b3) Assume $h^0(\mathcal{I}_B(3)) > 0$ and $h^0(\mathcal{I}_B(2)) = 0$. Let $E \subset \mathbb{P}^2$ be a cubic curve containing $B$. Since $h^0(\mathcal{I}_B(2)) = 0$, we have $\sharp(B \cap D) \leq 5$ for all conics, $E$ is irreducible. We have $h^1(\mathcal{I}_{S \setminus E}(4)) = 0$, because $S$ is general. If $h^1(E, \mathcal{I}_{E \cap (S \cup B), E}(7)) = 0$. Hence the exact sequence

$$0 \rightarrow \mathcal{I}_{S \setminus (S \cap E)}(4) \rightarrow \mathcal{I}_{S \cup B}(7) \rightarrow \mathcal{I}_{E \cap (S \cup B), E}(7) \rightarrow 0$$

gives $h^1(E, \mathcal{I}_{E \cap (S \cup B), E}(7)) > 0$. Since $p_a(E) = 1$ and $\deg(\mathcal{O}_E(7)) = 21$, either $x + z = 22$ and $S \cup B \subset E$ or $\sharp(E \cap (S \cup B)) = 21$ and $E \cap (S \cup B)$ is the
complete intersection of $E$ and a plane curve of degree 7. In all cases there is $S' \subseteq S$ such that $\sharp(S') = 6$ and $E \supset S' \cup B$. Since $S'$ is general, we have $h^0(I_{S'}(3)) = 4$. Since $h^0(C, L) = 4$, we get that $L$ is induced (after deleting the base points) by $|I_{S'}(2)|$. Lemma 1 gives that $I_{S'}(3)$ is spanned. Hence $z = 30 - 12$, a contradiction.

(b4) Assume $h^0(I_B(2)) > 0$. Fix a conic $F$ containing $B$. Since $h^1(I_{S \setminus S \cap F}(5)) = 0$, as in step (b3) we get $h^1(F, I_{F \cap (S \cap F)}(7)) > 0$. Since $\sharp(D \cap B) \leq 2$ for every line $D$, $F$ is irreducible. Since $F \cong \mathbb{P}^1$ and $\deg(O_F(7)) = 14$, we get $\sharp((S \cup B) \cap F) \geq 16$, i.e. $\sharp(S \cap F) \geq 16 - z > 0$. Since $B \subset F$, $|L|$ is induced (after deleting base points) from a subspace of $|I_{S \cap F}(2)|$. Since $S \cap F$ is general, we have $h^0(I_{S \cup F}(2)) = 6 - \sharp(F \cap S)$. Hence $\sharp(F \cap S) \in \{1, 2\}$. First assume $\sharp(F \cap S) = 2$. Since $I_{F \cap S}(2)$ is spanned, we get $z = 20 - 4$, a contradiction. Now assume $\sharp(F \cap S) = 1$. Since $h^1(I_Z(2)) = 0$ for every zero-dimensional scheme of degree 3, we get every codimension 1 linear subspace of $H^0(I_{F \cap S}(2))$ induces (after deleting base points) a linear system of degree $\geq 20 - 4$, a contradiction. \hfill \Box

3. The Invariants $s_r(C)$

For any integer $e \geq 0$ let $F_e$ denote the Hirzebruch surface with a ruling with minimal self-intersection $-e$ (see [12], pp. 379–381). We have $\text{Pic}(F_e) \cong \mathbb{Z}^2$ and we take as a basis of $\text{Pic}(F_e)$ a fiber of a ruling of $F_e$ and a section of the same ruling with self-intersection $-e$. Hence $O_{F_e}(h) \cdot O_{F_e}(h) = -e$, $O_{F_e}(h) \cdot O_{F_e}(h) = -e$ and $O_{F_e}(h) \cdot O_{F_e}(h) = -e$. We have $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the complete linear system $|O_{F_0}(h + f)|$ embeds $F_0$ into $\mathbb{P}^3$ as a smooth quadric surface.

Fix $A \in |O_{F_0}(ah + af)|, a \geq 1$. Since $\omega_{F_0} \cong O_{F_0}(-2h - 2f)$, the adjunction formula gives $\omega_A \cong O_A((a - 2)h + (a - 2)f)$ and hence $p_a(A) = (a - 1)^2$.

**Lemma 3.** Fix integers $a, x$ such that $a \geq 5$ and $0 \leq x \leq \lfloor a^2/3 \rfloor$. Fix a general $S \subset F_0$ such that $\sharp(S) = x$ and a general $Y \in |I_{2S}(ah + af)|$. Then $Y$ is integral and nodal and $\text{Sing}(Y) = S$. Let $u : C \to Y$ be the normalization map. Then $C$ has genus $(a - 1)^2 - x$.

**Proof.** Since $S$ is general, $a - 1 \geq 4$, and $3x \leq h^0(F_0, O_{F_0}((a - 1)h + (a - 1)f))$, we have $h^1(F_0, I_{2S}((a - 1)h + (a - 1)f)) = 0$ (see [14], Proposition 5.2 and Theorem 7.2). Obviously $h^2(F_0, I_{2S}((a - 2)h + (a - 2)f)) = h^2(F_0, O_{F_0}((a - 2)h + (a - 2)f)) = 0$. Since $O_{F_0}(h + f)$ is very ample, Castelnuovo-Mumford’s lemma gives that $I_{2S}(ah + af)$ is spanned. Since $Y$ is general, Bertini’s theorem gives $S = \text{Sing}(Y)$. Fix $P \in S$. Since $I_{2S}(ah + af)$ is spanned, a general element
of $|\mathcal{I}_{2S}(ah+af)|$ has an ordinary node at $P$. Since $S$ is finite, each point of $S$ is an ordinary node. Since $S$ is general in $F_0$, $a \geq 3$, $S = \text{Sing}(Y)$ and $Y$ is nodal, the curve $Y$ is integral. The adjunction formula gives $p_a(C) = (a-1)^2 - x$. \quad \Box

**Notation 2.** From now on $C$ is the normalization of a general $Y \in |\mathcal{I}_{2S}(ah+af)|$ with $a, x, S$ as in Lemma 3. Hence $p_a(C) = (a-1)^2 - x$.

**Proposition 1.** Set $\beta := \lfloor (2a - 1)/3 \rfloor$.

(a) Assume $0 < x \leq (a - \beta + 1)^2$. Then $s_2(C) = 2a - 2$ and each line bundle $R$ of degree $2a - 2$ (resp. $2a - 1$) inducing a morphism $C \to \mathbb{P}^2$ birational onto its image is obtained in the following way. See $Y$ as a space curve using the composition of the inclusion $Y \hookrightarrow F_0$ and the embedding $F_0 \hookrightarrow \mathbb{P}^3$ induced by $|\mathcal{O}_{F_0}(h + f)|$. Then $R$ is induced by the linear projection of $Y$ into $\mathbb{P}^2$ from a point of $S$ (resp. a point of $Y \setminus S$).

(b) Assume $x = 0$. Then $s_2(C) = 2a - 1$ and each line bundle evincing $s_2(C)$ is induced by the linear projection from some $P \in Y$.

**Proof.** If $x > 0$, then the linear projection from $x$ gives $d_2(C) \geq 2a - 2$. It also gives $s_2(C) \geq 2a - 2$ for the following reason. Fix $P \in S$ and take a general $P' \in Y$. Since the line $D'$ spanned by $P$ and $P'$ is not contained in the smooth quadric $F_0 \subset \mathbb{P}^3$, it intersects $Y$ only at $P$ and $P'$. Hence the linear projection from $P$ is birational onto its image. Similarly, the linear projection from a smooth point of $Y$ is birational onto its image.

Take an integer $z \leq 2a - 1$ and a spanned $L \in \text{Pic}^z(C)$ which induces a morphism $v : C \to \mathbb{P}^2$ birational onto its image. Fix a general $A \in |L|$ and set $B := u(A)$. Since $L$ is spanned and $A$ is general, we have $B \cap S = \emptyset$. Since $h^i(F_0, \mathcal{O}_{F_0}) = 0$ for $i = 1, 2$, $\omega_C \cong \mathcal{O}_{F_0}(-2h - 2f)$ and $Y$ is nodal with $S$ as its singular locus, we have $H^0(C, \omega_C) \cong H^0(F_0, \mathcal{I}_S((a - 2)h + (a - 2)f))$. Since $L$ is spanned, Riemann-Roch gives $h^0(C, \omega_C \otimes \mathcal{O}_C(A \setminus \{Q\})) = h^0(C, \omega_C \otimes \mathcal{O}_C(A))$ for every $Q \in A$. Hence $h^0(F_0, \mathcal{I}_{S \cup B \setminus \{O\}}((a - 2)h + (a - 2)f)) = h^0(F_0, \mathcal{I}_{S \cup B}((a - 2)h + (a - 2)f))$ for every $O \in B$. Hence $h^1(F_0, \mathcal{I}_{S \cup B}((a - 2)h + (a - 2)f)) > 0$. Since $S$ is general and $\sharp(S) \leq (a - 2)(a - 3)$, we have $h^1(F_0, \mathcal{I}_S((a - 2)h + (a - 3)f)) = h^1(F_0, \mathcal{I}_S((a - 3)h + (a - 2)f)) = 0$. Using adjunction we get that two $g_a$’s on $C$ induced by the two rulings of $F_0$ are complete.

Since the monodromy group of the generic hyperplane section of $v(C)$ is the full symmetric group $S_2$, for all integers $u \geq 0$ and $v \geq 0$ such that $(u + 1)(v + 1) \leq z$, and any $T \in |\mathcal{O}_{F_0}(uh+bf)|$, either $\sharp(T \cap B) \leq (u+1)(v+1)-1$ or $B \subset T$. For $(u, v) = (0, 1)$ (resp. $(u, v) = (0, 1)$) we get that for any $D \in |\mathcal{O}_{F_0}(h)|$ (resp. $|\mathcal{O}_{F_0}(f)|$) either $\sharp(D \cap B) \leq 1$ or $B \subset D$. The latter case is impossible, because
\[ \sharp(D \cap Y) \leq a \] and the two \( g_a^1 \)'s on \( C \) induced by the two rulings of \( F_0 \) are complete.

Hence \( \sharp(D \cap B) \leq 1 \) for every \( D \in (|O_{F_0}(h)| \cup |O_{F_0}(f)|) \). Now assume the existence of \( T \in |O_{F_0}(h+f)| \) such that \( \sharp(B \cap T) \geq 4 \). Hence \( B \subset T \). Thus \( |L| \) is induced by a subseries of \( |O_{F_0}(h+f)| \) (after deleting the base points). Let \( V \subset H^0(F_0, O_{F_0}(h+f)) \) be the codimension 1 linear subspace inducing \( |L| \) after deleting the base points. Since \( O_{F_0}(h+f) \) is very ample, the scheme-theoretic base locus of \( V \) in \( F_0 \) is either empty or a point, \( P \). We get that the base locus of the \( g^2_{2a} \) induced by \( V \) has degree 0, 1, 2 and it has degree 2 if and only if \( P \in S \), while it has degree 1 if and only if \( P \in Y \setminus S \). Hence Proposition 1 is proved in this case.

Now assume that there is no such a curve \( T \). We saw that \( \sharp(T \cap B) \leq 3 \) for all \( T \in |O_{F_0}(h+f)|. \) Set \( \alpha := [z/3] \), \( B_0 := B \) and \( S_0 := S \). Fix \( A_1 \in |O_{F_0}(h+f)| \) such that \( a_1 := \sharp(A_1 \cap B_0) \) is maximal and set \( S_1 := S_0 \setminus S_0 \cap A_1 \) and \( B_1 := B_0 \setminus B_0 \cap A_1 \). For each integer \( i \geq 2 \) define the curve \( A_i \in |O_{F_0}(h+f)| \), the integer \( a_i \) and the sets \( B_i \) and \( S_i \) in the following way. Fix \( A_i \in |O_{F_0}(h+f)| \) such that \( a_i := \sharp(A_i \cap B_{i-1}) \) is maximal and set \( S_i := S_{i-1} \setminus S_{i-1} \cap A_i \) and \( B_i := B_{i-1} \setminus B_{i-1} \cap A_i \). Since \( h^0(F_0, O_{F_0}(h+f)) = 4 \) if \( a_i \leq 2 \), then \( B_{i-1} \subset A_i \) and \( B_i = \emptyset \). Since we proved that \( a_i \leq 3 \) for all \( i \), we have \( a_i = 3 \) for all \( i \leq \alpha \), \( a_{\alpha+1} = z - 3 \cdot [z/3] \) and \( a_i = 0 \) for all \( i \geq \alpha + 2 \). For each \( i \geq 1 \) we have an exact sequence

\[ 0 \to \mathcal{I}_{S_1 \cup B_1}((a - 2 - i)h + (a - 2 - i)f) \to \mathcal{I}_{S_{i-1} \cup B_{i-1}}((a - 1 - i)h + (a - 1 - i)f) \to \mathcal{I}_{A_i \cap (B_{i-1} \cup S_{i-1}]}((a - 1 - i)h + (a - 1 - i)f) \to 0 \]  

(1)

We take \( A_1, \ldots, A_\alpha \) with the additional condition that \( x_i := \sharp(S_{i-1} \cap A_i) \) is maximal among all \( A_i \) with \( \sharp(A_i \cap B_{i-1}) = 3 \). Notice that \( x_i \geq x_j \) for all \( i \leq j \leq \alpha \). Since \( S \) is general, we have \( x_i \leq 3 \) for all \( i \). Hence \( a_i + x_i \leq 6 \) for all \( i \). Since \( \sharp(D \cap B) \leq 1 \) for every \( D \in (|O_{F_0}(h)| \cup |O_{F_0}(f)|) \), each \( A_i \), \( i \leq \alpha \), is irreducible, i.e. \( A_i \cong \mathbb{P}^1 \). We may also take \( A_{\alpha+1} \) irreducible. We have \( \alpha \leq a - 3 \) and equality holds if and only if \( a = 5 \) and \( z = 9 \). Since \( \alpha \leq a - 3 \) we get \( h^1(A_i, \mathcal{I}_{A_i \cap (B_{i-1} \cup S_{i-1} \cup A_{i-1})(a - 1 - i)h + (a - 1 - i)f)} = 0 \); if \( x = 0 \), we would get the same result even if \( \alpha = a - 2 \). Hence applying (1) for \( i = 1, \ldots, \alpha \) we get \( h^1(F_0, \mathcal{I}_{S_\alpha \cup B_\alpha}((a - 2 - \alpha)h + (b - 2 - \alpha)f)) > 0 \). The set \( S_\alpha \) is general in \( F_0 \) and \( \sharp(S_\alpha) \leq 2 \). We have \( \alpha \leq \beta \) and equality holds if and only if \( z = 2a - 1 \) and \( 2a - 1 \equiv 0 \) (mod 3). Hence \( \alpha < \beta \) if \( B_\alpha \neq \emptyset \). Hence applying \( \beta - \alpha \) times (1) we get \( h^1(F_0, \mathcal{I}_{S_\beta}((a - \beta, a - \beta)) > 0 \). Since \( S \) is general and \( S_\beta \subseteq S \), we get \( x > (a - \beta + 1)^2 \), a contradiction. In the case \( x = 0 \) we just use that \( \sharp(B_\alpha) \leq 2 \) and \( a - \alpha > 0 \).
Remark 2. Take the set-up of Notation 2. Fix an odd integer \( r \geq 3 \). Since \( h^0(F_0, \mathcal{O}_{F_0}(h + ((r - 1)/2)f)) = r + 1 \), \( \mathcal{O}_{F_0}(h + ((r - 1)/2)f) \) is very ample and \( \mathcal{O}_{F_0}(h + ((r - 1)/2)f) \cdot \mathcal{O}_{F_0}(ah + af) = a(r - 1) \), we have \( s_r(C) \leq a(r - 1) \). Let \( \phi_r : F_0 \to \mathbb{P}^r \) denote the embedding of \( F_0 \) induced by \( |\mathcal{O}_{F_0}(h + ((r - 1)/2)f)| \). Fix any \( P \in \phi_r(F_0) \) and call \( \mathcal{D} \) the union of the lines of \( \phi_r(F_0) \) containing \( P \) (one line if \( r \geq 5 \), two lines if \( r = 3 \)). Since \( \phi_r(F_0) \) is cut out by quadrics, for any line \( D \subset \mathbb{P}^r \) either \( D \subset \phi_r(F_0) \) or \( \deg(D \cap \phi_r(F_0)) \leq 2 \). Hence the linear projection from \( P \) induces an isomorphism of \( F_0 \setminus \mathcal{D} \) into a degree \( r - 2 \) surface of \( \mathbb{P}^{r-1} \). Hence taking the linear projection from some point of \( \phi_r(S) \) (if \( S \neq \emptyset \)) or of \( \phi_r(Y) \) (if \( S = \emptyset \)) we get \( s_{r-1}(C) \leq a(r - 1) - 1 - \min\{1, x\} \).

Lemma 4. Assume \( 3x \leq a^2 \) and that \( Y \) is general in \( |\mathcal{I}_{2S}(ah + af)| \). Let \( \rho : C \to \mathbb{P}^1 \) and \( \rho' : C \to \mathbb{P}^1 \) denote the degree \( a \) morphisms induced by \( |\mathcal{O}_{F_0}(h)| \) and \( |\mathcal{O}_{F_0}(h)| \). Then neither \( \rho \) nor \( \rho' \) factors as \( C \stackrel{f_1}{\to} C' \stackrel{f_2}{\to} \mathbb{P}^1 \) with \( \deg(f_1) \geq 2 \) and \( \deg(f_2) \geq 2 \).

Proof. It is sufficient to find \( O, O' \in \mathbb{P}^1 \) such that \( h(\rho^{-1}(O)) = h(\rho'^{-1}(O')) = a - 1 \) (indeed, if \( O \) (resp. \( O' \)) exists, then the corresponding fiber of \( \rho \) (resp. \( \rho' \)) has exactly one ramification point at which the ramification has order 2). Fix general \( P_1, P_2 \in F_0 \). Take \( D_1 \in |\mathcal{O}_{F_0}(h)| \) containing \( P_1 \) and \( D_2 \in |\mathcal{O}_{F_0}(f)| \) containing \( P_2 \). Take \( S_i \subset D_i \setminus \{P_i\} \) such that \( h(S_i) = a - 2 \). We have \( P_2 \notin D_1 \) and \( P_2 \notin D_2 \). Let \( Z_i \subset D_i \) the degree 2 divisor of \( D_i \) with \( P_i \) has its support. By [14], \( h^1(F_0, \mathcal{I}_{2S}((a - 1)h + ((a - 1)f)) = 0 \). Since \( \deg(Z_i \cup S_i) = a \), we get \( h^1(F_0, \mathcal{I}_{2S \cup Z_i \cup Z_2 \cup Z_3}(ah + af)) = 0 \). Hence \( Z_1 \cup Z_1 \cup Z_2 \cup Z_3 \) gives 2a independent conditions to \( |\mathcal{I}_{2S}(ah + af)| \). Since \( \dim(F_0) = 2 \) and \( \dim(D_i) = 1 \), varying \( P_1 \) and \( P_2 \) we get the existence of \( O \) and \( O' \) for a general \( Y \in |\mathcal{I}_{2S}(ah + af)| \).

Lemma 5. Assume \( x \leq (a - [(2a - 1)/3] + 1)^2 \) and that \( Y \) is general in \( |\mathcal{I}_{2S}(ah + af)| \). Then \( d_1(C) = a \).

Proof. Let \( \rho : C \to \mathbb{P}^1 \) be a degree \( a \) morphism induced by a ruling of \( F_0 \). The morphism \( \rho \) gives \( d_1(C) \leq a \). Assume \( z := d_1(C) = a \) and take a degree \( z \) morphism \( u' : C \to \mathbb{P}^1 \). The morphism \( \phi = (\rho, u') : C \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) is birational onto its image by Lemma 4. Hence \( \phi(C) \in |\mathcal{O}_{F_0}(ah + zf)| \). Taking a linear projection from a general \( O \in \phi(C) \) we get that \( C \) has infinitely many morphisms \( C \to \mathbb{P}^2 \) birational onto its image and with image of degree \( \leq a + z - 1 \leq 2a - 2 \), contradicting Proposition 1.

Proof of Theorem 2. Proposition 1 gives \( s_2(C) = 2a - 1 - \min\{1, x\} \). Remark 2 gives \( s_r(C) \leq a(r - 1)/2 \) for every odd integer \( r \) and \( s_r(C) \leq ar/2 - 1 - \min\{1, x\} \) for every even integer \( r \).
I) We first prove parts (a) and (b) of Theorem 2. Fix an integer $z \leq 3a - 2 - \min\{1, x\}$ such that there is a morphism $v : C \to \mathbb{P}^4$ birational onto its image, with $\deg(v(C)) = z$ and $v(C)$ spanning $\mathbb{P}^4$. We have $p_a(v(C)) \geq p_a(C) = (a - 1)^2 - x$. Set $m_1 := [(z - 1)/4]$, $\epsilon_1 = z - 1 - 4m_1$, $\mu_1 = 1$ if $\epsilon_1 = 3$ and $\mu_1 := 0$ if $\epsilon_1 \neq 3$. Set $\pi_1(4, z) := 2m_1(m_1 - 1) + m_1(\epsilon_1 + 1) + \mu_1$. Assume $x < (a - 1)^2 - (z - 1)^2/8$. Since $\pi_1(4, z) \leq 2m_1(m_1 + 1) + 1$ and $m_1 \leq (z - 1)/4$, we have $p_a(C) > \pi_1(4, z)$. If $z \leq 8a/3$, then to get a contradiction it is sufficient to assume $x < (a - 1)^2 - (8a - 3)^2/72$. The maximal value of $x$ allowed here is of order $a^2/9$ and we used a similar restriction to apply Proposition 1.

(i) In this step we assume that $T$ is a cone. The minimal desingularization of $T$ is isomorphic to $F_3$ and the composition of the desingularization map with the inclusion $T \hookrightarrow \mathbb{P}^4$ is induced by the linear system $|O_{F_3}(h + 3f)|$. Let $E$ be the strict transform of $v(C)$ in $F_3$. Let $c, y$ be the integers such that $E \in |O_{F_3}(ch + yf)|$. We have $y \geq 3c$ and $z = O_{F_3}(ch + yf) \cdot O_{F_3}(h + 3f) = y$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Hence $z \geq 3a$, a contradiction.

(ii) In this step we assume $T \cong F_1$. Take $c, y$ such that $v(C) \in |O_{F_1}(ch + yf)|$. We have $z = O_{F_1}(ch + yf) \cdot O_{F_1}(h + 2f) = y + c$. Since $d_1(C) = a$, we have $c \geq a$. Since $s_2(C) \geq 2a - 1 - \min\{1, x\}$ (Proposition 1) and $|O_{F_1}(h + f)|$ induces a birational morphism $F_1 \to \mathbb{P}^2$, we have $O_{F_1}(ch + yf) \cdot O_{F_1}(h + f) \geq 2a - 1 - \min\{1, x\}$, i.e. $y \geq 2a - 1 - \min\{1, x\}$. Hence $z \geq 3a - 1 - \min\{1, x\}$, a contradiction.

II) Now we prove parts (c) and (d) of Theorem 2. We assume $a \geq 108$. Fix an integer $z \leq 4a - 2 - \min\{1, x\}$ such that there is a spanned $L \in \text{Pic}^2(C)$ with $h^0(C, L) = 7$ and whose associated morphism $v : C \to \mathbb{P}^6$ is birational onto its image. We have $z = \deg(v(C))$ and $p_a(v(C)) \geq p_a(C) = (a - 1)^2 - x$. Set $m_2 := [(z - 1)/7]$, $\epsilon_2 = z - 1 - 7m_1$, $\mu_2 = \max\{0, [(4 + \epsilon_2)/2]\}$ and $\pi_2(6, z) := 7m_2(m_2 - 1)/2 + m_2(\epsilon_2 + 2) + \mu_2$. Assume $x < (a - 1)^2 - (z - 1)^2/12$. Since $\pi_2(6, z) \leq 2 + m_2(7m_2 + 16)/2$, we have $p_a(C) > \pi_2(6, z)$ if $x < (a - 1)^2 - (z - 1)(z + 15)/14$. Hence $[11]$, Theorem 3.22, gives that $v(C)$ is contained in a surface $T$ such that $\deg(T) \in \{5, 6\}$. Since $s_5(C) \leq 3a$, we have $6 \cdot s_5(C) \leq 18a/5$. If we only need to exclude that $z \leq 18a/5$, it is sufficient to assume $x < (a - 1)^2 - (18a - 5)(18a + 75)/350$. Notice that $(a - 1)^2 - (18a - 5)(18a + 75)/350 \sim (13/175)a^2$ when $a \gg 0$.

First assume $\deg(T) = 5$. By the classification of minimal degree surfaces either $T$ is a cone over a rational normal curve of $\mathbb{P}^5$ or $T \cong F_5$, $e \in \{1, 3\}$, embedded by the complete linear system $|O_{F_1}(h + (e + 1)f)|$.

(i) In this step we assume that $T$ is a cone. It is well-known that the minimal desingularization of $T$ is isomorphic to $F_5$ and that the composition
of the desingularization map with the inclusion $T \hookrightarrow \mathbb{P}^6$ is induced by the linear system $|\mathcal{O}_{F_5}(h + 5f)|$. Let $E$ be the strict transform of $v(C)$ in $F_5$. Let $c, y$ be the integers such that $E \in |\mathcal{O}_{F_5}(ch + yf)|$. We have $y \geq 5c$ and $z = \mathcal{O}_{F_5}(ch + yf) \cdot \mathcal{O}_{F_5}(h + 3f) = y$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Hence $z \geq 5a$, a contradiction.

(ii) In this step we assume $T \cong F_3$. Take $c, y$ such that $v(C) \in |\mathcal{O}_{F_3}(ch + yf)|$. We have $z = \mathcal{O}_{F_3}(ch + yf) \cdot \mathcal{O}_{F_3}(h + 4f) = y + c$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Since $|\mathcal{O}_{F_3}(h + 3f)|$ induces a birational morphism $F_1 \rightarrow \mathbb{P}^2$, we have $\mathcal{O}_{F_3}(ch + yf) \cdot \mathcal{O}_{F_3}(h + 3f) \geq s_4(C)$, i.e. $y \geq s_4(C)$. Hence part (a) gives $z > a + 8a/3 > 18a/5$, a contradiction.

(iii) In this step we assume $T \cong F_1$. Take $c, y$ such that $v(C) \in |\mathcal{O}_{F_1}(ch + yf)|$. We have $z = \mathcal{O}_{F_1}(ch + yf) \cdot \mathcal{O}_{F_1}(h + 3f) = y + 2c$. Since $d_1(C) = a$ (Lemma 5), we have $c \geq a$. Since $s_2(C) \geq 2a - 1 - \min\{1, x\}$ (Proposition 1) and $|\mathcal{O}_{F_1}(h + f)|$ induces a birational morphism $F_1 \rightarrow \mathbb{P}^2$, we have $\mathcal{O}_{F_1}(ch + yf) \cdot \mathcal{O}_{F_1}(h + f) \geq 2a - 1 - \min\{1, x\}$, i.e. $y \geq 2a - 1 - \min\{1, x\}$. Since $c \geq a$, we get $z \geq 4a - 1 - \min\{1, x\}$, a contradiction.

From now on we assume $\deg(T) = 6$. We have $\deg(v(C)) > \deg(T)$ and the morphism $v : C \rightarrow \mathbb{P}^6$ is induced by a complete linear system. Hence $T$ is not an external linear projection of a degree 6 surface of $\mathbb{P}^7$. Hence $T$ is a normal del Pezzo surface (introduction of [2], [3], [5], [6], [7], Corollary 6.5 and Theorem 9.16). Hence a general hyperplane section $H \cap T$ of $T$ is a smooth degree 5 curve. Hence $p_a(T \cap H) \leq 1$ (and indeed $p_a(T \cap H) = 1$). If $T$ is not a cone, then $T$ is Gorenstein and $\mathcal{O}_T(-1) \cong \omega_T$ (see [7], theorem 9.16). If $T$ is not a cone, then each singular point of $T$ is a rational hypersurface singularity (see [4], p. 116) and hence $T$ is the canonical model of a weak del Pezzo surface in the sense of [10].

(iv) Here we assume that $T$ is a cone over a linearly normal elliptic curve $E_1$ of $\mathbb{P}^5$. Call $O$ the vertex of $T$ and $\mu$ the multiplicity of $v(C)$ at $O$. We have $z = \mu + 6k$, where $k \in \mathbb{N}$ and $k$ is the degree of the covering $\tau : C \rightarrow E_1$ induced from the linear projection from $O$. Since $E_1$ has infinitely many $g^2$’s, the covering $\tau$ shows that $C$ has infinitely many $g^2$. Proposition 1 gives $3k \geq 2a - 1$ and hence $z \geq 6(2a - 1)/3$, a contradiction.

(v) Here we assume that $T$ is a weak del Pezzo surface of degree 6 in the sense of [10]. Fix $P_1 \in \mathbb{P}^2$ and call $\Pi(1)$ the blowing up of $\mathbb{P}^2$ at $P_1$. Fix $P_2 \in \Pi(1)$ and call $\Pi(2)$ the blowing-up of $\Pi(1)$ at $P_2$. Fix $P_3 \in \Pi(2)$ and call $\Pi$ the blowing up of $\Pi(2)$ at $P_3$. The anticanonical line bundle $\omega_{\Pi}^\vee$ is spanned and it induces a morphism $\phi : \Pi \rightarrow \mathbb{P}^6$ which is birational onto its image. There are $P_1, P_2, P_3$ with $T = \phi(\Pi)$ and we use these points $P_1, P_2, P_3$. Let $E$ denote
the strict transform of $v(C)$ in $\Pi$. Notice that $C$ is the normalization of $E$. We take a basis $\ell, \ell_1, \ell_2, \ell_3$ of $\text{Pic}(\Pi) \cong \mathbb{Z}^4$ with $\ell$ coming from $\mathcal{O}_{\mathbb{P}^2}(1)$, each $\ell_i$ effective (but not necessarily irreducible) $\ell_i^2 = -1$ for all $i$, $\ell^2 = 1$, $\ell_i \cdot \ell_i = 0$ for all $i$ and $\ell_i \cdot \ell_j = 0$ for all $i \neq j$. Write $E = u\ell - u_1\ell_1 - u_2\ell_2 - u_3\ell_3$ for some $u, u_i \in \mathbb{N}$. Since $\omega_{\Pi} \cong -3\ell + \ell_1 + \ell_2 + \ell_3$ and $\phi$ is induced by the anticanonical linear system, we have $z = 3u - u_1 - u_2 - u_3$. Since $\dim(|\ell|) = 2$ and the linear system $|\ell|$ is birationally very ample, we have $s_2(C) \geq u$. Hence $u \geq 2a - 4$ (Proposition 1). Since $h^0(\Pi, \mathcal{O}_\Pi(2\ell - \ell_1 - \ell_2 - \ell_3)) = 3$ and $|\mathcal{O}_\Pi(2\ell - \ell_1 - \ell_2 - \ell_3)|$ is birationally very ample, we have $s_2(C) \leq 2u - u_1 - u_2 - u_3$. Proposition 1 gives $2u - u_1 - u_2 - u_3 \geq 2a - 4$. Hence $z \geq 4a - 8$, a contradiction. \hfill \Box

Proof of Corollary 1. Set $g_{a,x} := (a - 1)^2 - x$. Notice that $g_{a,0} - g_{a-1,0} = 2a - 3$. Hence every integer $g \geq 19$ is of the form $g_{a,x}$ for uniquely determined integers $a, x$ such that $a \geq 5$ and $0 \leq x \leq 2a - 4$. Since $(a - 1)^2 - (18a - 5)(18a + 75)/350 \sim (13/175)a^2$ when $a \gg 0$, for $a \gg 0$ all integers $x \leq 2a - 4$ are allowed in parts (b) and (d) of Theorem 2. \hfill \Box

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