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PRACTICAL STABILIZATION OF DISCRETE CONTROL SYSTEMS

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Abstract: In this paper the properties of optimal sets of initial conditions in the problem of practical stabilization of discrete control systems are considered. In linear case Minkowski function, inverse Minkowski function, and support function of these sets are obtained.

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Key Words: discrete systems, practical stability, stabilization

1. Introduction

The issues of control systems stabilization are intensively researched due to their application [3, 4, 8]. The practical stabilization criteria for dynamical systems based on the Lyapunov function and the conditions for feedback control was obtained in [2, 3]. This paper deals with the practical stability problem for discrete control system. Our research is based on the properties of practical stability maximal set and the solutions of discrete inclusions (see [1, 5, 6]).

We use the following notation: $\mathbb{R}^n$ n-dimensional Euclidean space; $\|\cdot\|$ Euclidian norm; $\langle\cdot,\cdot\rangle$ scalar product that generates Euclidean norm in $\mathbb{R}^n$; $intA$ the set of internal points of the set A; ∂A bound of the set A; $[0, N] = \{0, 1, ..., N\}$

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set of indices; $c(A,\psi) = \sup\limits_{a\in A} \langle a,\psi\rangle$, $\psi \in \mathbb{R}^n$ support function of the set $A \subset \mathbb{R}^n$; S the unit sphere; $K_r(a)$ closed ball with the radius r and center at $a \in \mathbb{R}^n$; $comp\,(\mathbb{R}^n)$ the set of all nonempty compact sets in $\mathbb{R}^n$; $conv\,(\mathbb{R}^n)$ the set of all nonempty convex compact sets in $\mathbb{R}^n$; $A^\sigma = A + \sigma K_1(0)$ σ-extension of the set $A \subset \mathbb{R}^n$; $A^{<\sigma>} = \{a : a + K_\sigma(0) \subset A\}$ σ-constriction of the set $A \subset \mathbb{R}^n$.

2. Strong Stability of Discrete Systems

Let us consider a control system

$$x(k+1) = f_k(x(k), u(k)),\ \ k \in [0, N], \tag{1}$$

where continuous function $f_k : D \to comp(D)$ is defined on a domain $D \subset \mathbb{R}^n$; $x = (x_1, x_2, ..., x_n)^*$ is the state vector, $u = (u_1, u_2, ..., u_m)^*$ is the control vector, $u(k) \in U(k)$, $U(k) \in conv(\mathbb{R}^m)$ are control constraints, $k \in [0, N]$, $0 \in intU(k)$. Denote by $\mathcal{U}$ a set of admissible controls, $x(k, x_0, u)$ a solution of the system (1) which corresponds to initial conditions $x(0) = x_0$ and control $u \in \mathcal{U}$.

Suppose $G_0 \in comp(\mathbb{R}^n)$ is a set of initial conditions, $\Phi(k) \in comp(\mathbb{R}^n)$ is a set of state constraints, $0 \in G_0$, $0 \in int\Phi(k)$, $k \in [0, N]$.

Definition 1. The $\{G_0, \Phi(k), 0, N\}$-practical stability problem of discrete system (1) is to find a control $u \in \mathcal{U}$ such that for any solution of the system (1) $x(k, x_0, u) \in \Phi(k)$ for any $k \in [0, N]$ and $x_0 \in G_0$.

Definition 2. We say that a control $u \in \mathcal{U}$ practically stabilizes the system (1) from $x_0 \in \Phi_0$ under the state constraints $\Phi(k)$, $k \in [0, N]$, if $x(k, x_0, u) \in \Phi(k)$ for any $k \in [0, N]$.

Definition 3. We say that a control $u \in \mathcal{U}$ boundary stabilizes the system (1) from $x_0 \in \Phi_0$ under the state constraints $\Phi(k)$, if $x(k, x_0, u) \in \Phi(k)$ for any $k \in [0, N]$. Furthermore, there exists $\hat{k} \in [0, N]$ such that $x(\hat{k}, x_0, u) \in \partial\Phi(\hat{k})$.

Definition 4. The system (1) is called strongly $\{G_0, \Phi(k), 0, N\}$-stabilized, if all admissible controls $u \in \mathcal{U}$ practically stabilize the system (1) from any point $x_0 \in G_0$ under the state constraints $\Phi(k)$, $k \in [0, N]$.

Denote by

$$X(k, X_0) = \bigcup_{x_0\in X_0} \bigcup_{u\in\mathcal{U}} x(k, x_0, u),\ \ k \in [0, N]$$

the attainability set of (1) corresponding to $x(0) \in X_0$. Then the following discrete inclusion corresponds to the control system (1)

$$x(k+1) \in f_k(x(k), U(k)),\ \ k \in [0, N-1]. \tag{2}$$

The attainability set of (2) is equal to the attainability set of (1). It satisfies the equation

$$X(k+1) = f_k(X(k), U(k)),\ k \in [0, N-1].$$

Denote by G_* maximal set of initial conditions that satisfies definition 4. From theorem 1 [6] it follows that G_* is a compact set.

Theorem 5. *The point $x_0 \in \partial G_*$ if and only if there exists an admissible control $u \in \mathfrak{U}$ which boundary stabilizes the system (1) from $x_0 \in G_*$ in state constraints $\Phi(k)$, $k \in [0, N]$.*

Proof. "If" case. Suppose $x_0 \in \partial G_*$. Then from theorem 2 [6] it follows that there exists $\hat{k} \in [0, N]$ and the solution $x(k, x_0, \hat{u})$, $\hat{u} \in \mathfrak{U}$ of the inclusion (2) such that $x(\hat{k}, x_0, \hat{u}) \in \partial\Phi(\hat{k})$. Since the solution $x(k, x_0, \hat{u})$ of the system (2) is the solution of the system (1), we proved "if" case.

"Only if" case. Assume that there exists $\hat{u} \in U$ such that $x(\hat{k}, x_0, \hat{u}) \in \partial\Phi(\hat{k})$, $\hat{k} \in [0, N]$. Since the solution $x(k, x_0, \hat{u})$ is the solution of the system (2), then using theorem 1 [6] we have $x_0 \in \partial G_*$. This completes the proof of the theorem. □

Let us consider a linear control system

$$x(k+1) = A_k x(k) + u(k),\ k \in [0, N-1], \tag{3}$$

where $x \in \mathbb{R}^n$, A_k is nonsingular $n \times n$-matrix, $u(k) \in U(k)$, $U(k) \in conv(\mathbb{R}^n)$, $k \in [0, N-1]$, $\Phi(k) \in conv(\mathbb{R}^n)$ is the set of state constraints, $k \in [0, N]$, $0 \in intG_*$.

The following discrete inclusion corresponds to the system (3)

$$x(k+1) \in A_k x(k) + U(k),\ X(0) = X_0, \tag{4}$$

The attainability sets of (3) and (4) are equal and satisfy the equation

$$X(k+1) = A_k X(k) + U(k),\ X(0) = X_0,$$

where $X_0 \in conv(\mathbb{R}^n)$. Let us write the attainability set as follows

$$X\,(k, X_0) = \Theta(k) X_0 + \Omega(k),$$

where

$$\Theta(k) = A_{k-1} \cdots A_1 A_0,$$

$$\Omega(k) = U(k-1) + A_{k-1}U(k-2) + \ldots + A_{k-1}A_{k-2}\ldots A_1 U(0). \tag{5}$$

Since $X(k, x_0) \subseteq \Phi(k)$, $x_0 \in G_*$, then from (5) we have

$$\Theta(k)x_0 + \Omega(k) \subseteq \Phi(k),$$

for any $k = 0, 1..., N$.

Now we get

$$G_* = \bigcap_{\xi \in S} \{x : \langle x, \xi \rangle \leq \zeta(\xi)\},$$

where $\xi = \Theta^*(k)\psi$ and

$$\zeta(\xi) = \min_{k \in [0,N]} \left(c(\Phi(k), (\Theta^*(k))^{-1}\xi) - c(\Omega(k), (\Theta(k)^*)^{-1}\xi) \right).$$

Therefore we obtain the support function of G_* as [7]

$$c(G_*, \xi) = \overline{co}\zeta(\xi).$$

Further, we can find Minkowski functional of G_* from

$$m_*(x) = \max_{k \in [0,N]} \max_{\psi \in S} \frac{\langle \Theta(k)x_0, \psi \rangle}{c(\Phi(k), \psi) - c(\Omega(k), \psi)}.$$

Deformation function of G_* satisfies

$$d_*(\ell) = \min_{k \in [0,N], \psi \in Z(k,\ell)} \frac{c(\Phi(k), \psi) - c(\Omega(k), \psi)}{\langle \Theta(k)\ell, \psi \rangle},$$

where $Z(k, \ell) = \{\psi \in S : \langle \Theta(k)\ell, \psi \rangle > 0\}$.

Thus we obtain that any control $u \in \mathfrak{U}$ is the solution of $\{G_*, \Phi(k), 0, N\}$-practical stability problem of discrete system (1).

3. Weakly Stabilized Control Systems

Let us consider the control system (1). Suppose $I_0 \in comp(\mathbb{R}^n)$ is the set of the initial conditions, $\Phi(k) \in comp(\mathbb{R}^n)$ is the set of the phase constraints, $0 \in I_0$, $0 \in int\Phi(k)$, $k \in [0, N]$.

Definition 6. The system (1) is called weakly $\{I_0, \Phi(k), 0, N\}$-stabilized, if all admissible controls $u \in \mathfrak{U}$ practically stabilize the system (1) from any point $x_0 \in G_0$ under the state constraints $\Phi(k)$, $k \in [0, N]$.

Denote by $I_* \subseteq \Phi(t_0)$ the maximal set of initial conditions that satisfies definition 5. From the theorem on the maximum set compactness it follows that I_* is a compact set.

Theorem 7. *The point $x_0 \in \partial I_*$ if and only if there exists an admissible control $u \in \mathfrak{U}$ which boundary stabilizes the system (1) from $x_0 \in I_*$ under the state constraints $\Phi(k)$, $k \in [0, N]$.*

Proof. Suppose $x_0 \in \partial I_*$. Then from theorem on boundary point of maximal set it follows that for any solution $x(k, x_0, u)$ of the inclusion (2) such that $x(k, x_0, u) \in \Phi(k)$, $k \in [0, N]$ there exists $\hat{k} \in [0, N]$ such that

$$x(\hat{k}, x_0, u) \in \partial\Phi(\hat{k}).$$

Since the solution $x(k, x_0, \hat{u})$ is the solution of the system (1). This finishes the proof of the theorem. □

Let us consider the linear control system (3), where $0 \in intI_*$. The support function of I_* may be written as

$$c(I_*, \eta) = \overline{co} \min_{k \in [0,N]} c(\Phi(k) + (-1)\Omega(k), \Theta^*(k))^{-1}\eta), \ \eta \in \mathbb{R}^n.$$

We can find Minkowski functional of I_* from

$$m_*(x_0) = \max_{\psi \in S} \max_{k \in [0,N]} \frac{\langle \Theta(k)x_0, \psi \rangle}{c(\Phi(k) + (-1)\Omega(k), \psi)}.$$

Deformation function of I_* is equal to

$$d_*(x_0) = \min_{\psi \in Z} \min_{k \in [0,N]} \frac{c(\Phi(k) + (-1)\Omega(k), \psi)}{\langle \Theta(k)x_0, \psi \rangle},$$

where $Z(k, \ell) = \{\psi \in S : \langle \Theta(k)\ell, \psi \rangle > 0\}$.

Theorem 8. *Suppose that $x_0 \in I_*$. If there exists a control $u \in \mathfrak{U}$ which boundary stabilizes the system (3) from $x_0 \in I_*$ under the state constraints $\Phi(k)$, $k \in [0, N]$, then $x_0 \in \partial I_*$.*

Proof. Assume that $x_0 \in I_*$ and there exists a control $u \in \mathfrak{U}$ which boundary stabilizes the system (3) from $x_0 \in I_*$ under the state constraints $\Phi(k)$, $k \in [0, N]$. The solution $x(k, x_0, u) \in \Phi(k)$ is the solution of the system (4). Then for all solutions $x(k, x_0, u)$ of the inclusion (4) such that $x(k, x_0, u) \in \Phi(k), k \in [0, N]$ there exists $\hat{k} \in [0, N]$ such that $x(\hat{k}, x_0, u) \in \partial\Phi(\hat{k})$. This completes the proof. □

4. Practical Stabilization of Linear Control System

Consider the control system (3). Let us $x_0 \in I_*$. According to definition 5 there exists $u \in \mathcal{U}$ such that $x(k, x_0, u) \in \Phi(k)$ for all $k \in [0, N]$.

From support function properties it follows that

$$\langle A_k x(k), \psi\rangle + \langle u(k), \psi\rangle \le c(\Phi(k+1), \psi), \ \psi \in S.$$

So we have

$$c(\Phi(k+1), \psi) - \langle A_k x(k), \psi\rangle - \langle u(k), \psi\rangle \ge 0, \ \psi \in S. \tag{6}$$

Therefore

$$\min_{\psi \in S}\{c(\Phi(k+1), \psi) - \langle A_k x(k), \psi\rangle - \langle u(k), \psi\rangle\} \ge 0.$$

Let us find the control $u \in \mathcal{U}$ in a form

$$u(k) = \alpha(k)\psi(k),$$

where $\alpha(k)$ be an unknown scalar, $\psi(k)$ be an unknown n-dimensional vector. Substituting $u(k) = \alpha(k)\psi(k)$ in (6) we obtain

$$c(\Phi(k+1), \psi) - \langle A_k x(k), \psi(k)\rangle - \langle \alpha(k)\psi(k), \psi(k)\rangle \ge 0.$$

Thus

$$\alpha(k) \le c(\Phi(k+1), \psi) - \langle A_k x(k), \psi(k)\rangle, \tag{7}$$

where $\psi(k)$ can be find from the condition

$$\psi(k) = arg \min_{\psi \in S}\{c(\Phi(k+1), \psi) - \langle A_k x(k), \psi\rangle\}.$$

Let us choose $\alpha(k)$ such that $\alpha(k)\psi(k) \in U(k)$. Since $U(k) \in conv(\mathbb{R}^n)$, $0 \in U(k)$ $k \in [0, N]$, there exists $r(k)$ such that

$$r(k) = \max_{r}\{K_r(0) : K_r(0) \subset U(k)\}.$$

Denote

$$P_k = (\Phi(k+1), \psi) - \langle A_k x(k), \psi(k)\rangle.$$

Then $\alpha(k) \in [0, \min\{P_k, r(k)\}]$.

Finally we have the sequence of stabilizing controls $u(k) = \alpha(k)\psi(k)$, where

$$\alpha(k) \in [0, \min\{P_k, r(k)\}],$$

$$\psi(k) = arg\min_{\psi\in S}\{c(\Phi(k+1),\psi) - \langle A_k\cdots A_1A_0x(0),\psi\rangle -$$

$$-\langle A_k\cdots A_2A_1\alpha(1)\psi(1) + \ldots + A_k\alpha(k-1)\psi(k-1),\psi\rangle\},$$

$$P_k = c(\Phi(k+1),\psi) - \langle A_k\cdots A_0x(0),\psi(k)\rangle -$$

$$-\langle A_k\cdots A_1\alpha(0)\psi(0),\psi(k)\rangle - \ldots - \langle A_k\alpha(k-1)\psi(k-1),\psi(k)\rangle.$$

$$x(k+1) = A_k\cdots A_1A_0x(0)+$$

$$+A_k\cdots A_2A_1\alpha(1)\psi(1) + \ldots + A_k\alpha(k-1)\psi(k-1) + \alpha(k)\psi(k).$$

Substituting $u(k) = \alpha(k)\psi(k)$ in (3), we get a system

$$x(k+1) = A_kx(k) + \alpha(k)\psi(k)\, k\in[0,N-1],\, x(0) = x_0,$$

where $x_0 \in I_*$. Solution of this problem belongs to the state constraints $\Phi(k)$, $k\in[0,N-1]$.

5. Conclusion

This paper proves the statement on compactness and conditions to the boundary of the optimal practical stability set. We obtain support function, Minkowski function and deformation function of the practical stability optimal set in strong and weak cases.

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