THE MAXIMALITY OF $p$-FACTORABLE 
MULTI-LINEAR OPERATORS

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Abstract: This work demonstrate the equivalence of the following definitions “let $1 \leq p \leq \infty$, be, $\Phi \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is called $p$–factorable, if exist a measure space $(\Omega, \Sigma, \mu)$ and operators $A \in \mathcal{L}(L_p(\mu), Y^{**})$ and $\Psi \in \mathcal{L}(X_1, \ldots, X_n; L_p(\mu))$ such that $K_Y \circ \Phi = A \circ \Psi$.

The collection of the $p$-factorable multi-linear operators of $X_1, \ldots, X_n$ to $Y$ will be denoted for $\mathcal{L}_{p-\text{fact}}(X_1, \ldots, X_n; Y)$. Also $\tilde{\gamma}_p(\Phi) = \inf \|\Psi\|\|A\|$, where the infimum is taken over all possible factorzation of $\Phi$ is a norm over $\mathcal{L}_{p-\text{fact}}(X_1, \ldots, X_n; Y)$” and “let $1 \leq p \leq \infty$.

A operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called $p$-factorable relative to $(q_1, \ldots, q_n)$ if belongs to the normed ideal:

\[
[\mathcal{L}_{p-\text{fact}}, \tilde{\gamma}_p] = [\mathcal{L}_{ap}, \|\cdot\|]^{-1} \circ [\mathcal{N}_{(\infty;p,q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)] \circ ([\mathcal{L}_{ap}, \|\cdot\|]^{-1} ; \cdots ; [\mathcal{L}_{ap}, \|\cdot\|]^{-1}),
\]

such that $\frac{1}{p} + \frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1$, with norm

\[
\tilde{\gamma}_p(\Phi) = \sup N_{(\infty;p,q_1,\ldots,q_n)}(B\Phi(T_1, \ldots, T_n))
\]

where the supremum is taken on all the $T = (T_1, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $\|B\| \leq 1, \|T_i\| \leq 1$, $i = 1, \ldots, n$.

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1. Introduction

In the paper from Cerna (see [7]) titled “Some properties of multi-linear operators of Nuclear Type”, we give the motivations for which we are studying the properties of the multi-linear operators of nuclear type, in this work we obtain two important results. The first result establishes that the definition: An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is $p$-factorable ($1 \leq p < \infty$) if there are operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_p(u)), A \in \mathcal{L}(L_p(u), F^{**}), K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = A \circ \Psi$. Here $(\Omega, u)$ is a suitable measure space. Also we have that $\tilde{\gamma}_p(\Phi) = \inf \|A\| \|\Psi\|$, where the infimum is taken over all possible factorizations. The second result establishes that the definition: An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is $\infty$-factorable if one of the following statements is true:

1. There exist an Hausdorff compact space $K$ and operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; C(K))$ and $Y \in \mathcal{L}(C(K), F^{**}), K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = Y \circ \Psi$, also we have that $\tilde{\gamma}_\infty(\Phi) = \inf \|Y\| \|\Psi\|$, where the infimum is taken over all possible factorizations.

2. There exist a measure space $(\Omega, \mu)$ and operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_\infty(\Omega, \mu)), Y \in \mathcal{L}(L_\infty(\Omega, \mu); F^{**})$ and $K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = Y \circ \Psi$. Also we have that $\tilde{\gamma}_\infty(\Phi) = \inf \|\Psi\| \|Y\|$, where the infimum is taken over all possible factorization.

The two mentioned definitions are equivalent to the following definitions. An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called $p$-factorable relative to $(q_1, \ldots, q_n)$ respectively if it belongs to the normed ideal

$$[\mathcal{L}_{p\text{-fact}}, \tilde{\gamma}_p] = [\mathcal{L}_{ap}, ||.||]^{-1} \circ [N_{(\infty,p,q_1,\ldots,q_n)}(E_1, \ldots, E_n; F), N_{(\infty,p,q_1,\ldots,q_n)}]$$

$$\cdots \circ ([\mathcal{L}_{ap}, ||.||]^{-1}, \ldots, [\mathcal{L}_{ap}, ||.||]^{-1})$$

such that $\frac{1}{p} + \frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1$, with a norm given by

$$\tilde{\gamma}(\Phi) = \sup N_{(\infty;p,q_1,q_2,\ldots,q_n)}(B \Phi(T_1, T_2, \ldots, T_n)),$$

where the supremum is taken of all the $T = (T_1, T_2, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $||B|| \leq 1, ||T_i|| \leq 1, i = 1, \ldots, n$ where $X_i, F_0$ are arbitrary Banach Spaces.
We introduce the notations in the present work, for Banach Spaces $E_1, \ldots, E_n$ and $F$ over the field $\mathbb{K}(\mathbb{R} \text{ or } \mathbb{C})$, we denote $\mathcal{L}(E_1, \ldots, E_n; F)$ to the Banach Space of all multi-linear and continuous applications of $E_1 \times \cdots \times E_n$ over $F$ with a natural norm given by
\[
\|T\| = \sup_{x_i \in B_{E_i}}\|T(x_1, \ldots, x_n)\|,
\]
where $B_{E_i}$ denote the unitary ball of $E_i$, centered in $0$. $E^*_k$ denote the dual topological of $E_k$, $k = 1, \ldots, n$;
\[\operatorname{Dim}(E) = \{\text{collection of all finite dimensional subspaces}\};\]
\[\operatorname{Cod}(F) = \{\text{collection of all finite codimensional subspaces}\};\]
\[\mathcal{L}_{ap}(E, F) = \text{Ideal of approximable operators of } E \text{ onto } F;\]
\[K_F \text{ is the isometric immersion of } F \text{ into } F^{**};\]
\[J_{M}^E = \{\text{Embedding map from } M \text{ into } E\};\]
\[Q_{N}^F = \{\text{Canonical map from } E \text{ onto } F/N\};\]
\[\mathcal{F}(E, F) = \text{Ideal of finite operators of finite range from } E \text{ onto } F;\]
\[p' : \text{dual exponent, } \frac{1}{p} + \frac{1}{p'} = 1.\]

2. Ultra Stability

In this section using A. Santana Soares definition, from Soares (see [3]), we extended the lemma 8.8.4 from Pietsch (see [2]) for the multi-linear case.

**Definition 2.1.** Given a set of indexes $I$, for each $i \in I$, we consider Banach Spaces $E^i_1, \ldots, E^i_n, F_i$ and $A_i \in \mathcal{L}(E^i_1, \ldots, E^i_n; F_i)$ with $\sup \|A_i\| < \infty$ and $\mathcal{U}$ ultrafilter on $I$. We will define then
\[
(A_i)_{\mathcal{U}} : (E^i_1)_{\mathcal{U}} \times \cdots \times (E^i_n)_{\mathcal{U}} \longrightarrow (F_i)_{\mathcal{U}}
\]
where
\[
(A_i)_{\mathcal{U}} ( (x^i_1)_{\mathcal{U}}, \ldots, (x^i_n)_{\mathcal{U}} ) = (A_i(x^i_1, \ldots, x^i_n))_{\mathcal{U}}.
\]

Let us suppose $\Phi \in \mathcal{L}(E_1, \ldots, E_n : F)$. Given $I$ the set of all indexes $i = (M^i_1, \ldots, M^i_n : N_i)$ whith $M^i_k \in \operatorname{Dim}(E_k)$ and $N_i \in \operatorname{Cod}(F)$, $k = 1, \ldots, n$. We choose an ultrafilter $\mathcal{U}$ containing all subsets
\[
\{i \in I/M^i_k \supseteq M^{i_0}_k, N_i \subseteq N_{i_0}\}
\]
where $i_0 = (M^{i_0}_1, \ldots, M^{i_0}_n; N_{i_0})$ is fixed.
Finally, put
\[
E^i_k = M^i_k, \ k = 1, \ldots, n, \ F_i = F/N_i \text{ and } \Phi_i = Q^F_{N_i} \Phi \left( J^{E_i}_{M^i_1}, \ldots, J^{E_i}_{M^i_n} \right),
\]
Lemma 2.1. There are operators

\[ J \in \mathcal{L}(E_1 \times \cdots \times E_n, (E_1^i)_U \times \cdots \times (E_n^i)_U) \]

and \( Q \in \mathcal{L}((F_i)_U, F^{**}) \) such that \( \|J\| \leq 1, \|Q\| \leq 1 \), and \( K_F \Phi = Q(\Phi_i)_U J \).

Proof. The operator \( J \) is defined by

\[ J(x_1, \ldots, x_n) := (J_1(x_1), \ldots, J_n(x_n)) \]

where \( J_k(x_k) = (x_k^i)_u, k = 1, \ldots, n \) with

\[ x_k^i = \begin{cases} x_k & \text{if } x_k \in M^i_k, \\ 0 & \text{if } x_k \notin M^i_k. \end{cases} \]

It is clear that \( J \) is linear and continuous for any norm over \( E_1 \times \cdots \times E_n, (E_1^i)_U \times \cdots \times (E_n^i)_U \) and we obtain \( \|J\| \leq 1 \). Moreover, let

\[ Q((y_i^0)_U) := F' - \lim U K_F y_i \]

where \( (y_i) \in l_\infty(F_i, I) \) such that \( Q^F_{N_i}(y_i) = y_i^0 \).

First we check that the right-hand expression does not depend on the special choice of \( (y_i) \). For this purpose assume that

\[ (Q^F_{N_i}y_i)_U = 0 \text{ and } y'' := F' - \lim U K_F y_i. \]

Then \( |\langle K_F y_i, b \rangle| \leq \|Q^F_{N_i}(y_i)\| \|b\| \) whenever \( F_i \) is contained in the null space of \( b \in F' \). Therefore

\[ |\langle y'', b \rangle| = \left| \lim_U \langle K_F y_i, b \rangle \right| \leq \lim_U \|Q^F_{N_i}y_i\| \|b\| = 0 \]

Hence \( y'' = 0 \). This proves that \( Q \) is well-defined.

If the family \( (y_i) \) is chosen such that

\[ \sup_I \|y_i\| \leq (1 + \epsilon) \|(y_i^0)_U\| \]

then we get

\[ \|Q((y_i^0)_U)\| \leq \lim_U \|y_i\| \leq (1 + \epsilon) \|(y_i^0)_U\| \]
Therefore \( \|Q\| \leq 1 \). Finally, observe that
\[
Q(\Phi_i)_U \cdot J(x_1, \ldots, x_n) = Q(\Phi_i_\mu U, \forall(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n
\]
\[
= f' - \lim U K_F \Phi \left( J^{E_1}_{M_1^j}, \ldots, J^{E_n}_{M_n^j} \right) (x_1^i, \ldots, x_n^i)
\]
\[
= K_F \Phi(x_1, \ldots, x_n), \forall(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n.
\]

3. \( p \)-Factorable Multi-Linear Operators

In this section we will demonstrate the equivalence of the two definitions from Cerna (see [7]), see appendix Theorem 3.1.

**Theorem 3.1.** Let \( 1 \leq p < \infty \), be \( \Phi \in \mathcal{L}(E_1, \ldots, E_n; F) \) is \( p \)-factorable relative to \( (q_1, \ldots, q_n) \) if and only if there exist a commutative diagram such that \( K_F \circ \Phi = A \circ \Psi \), where \( K_F \in \mathcal{L}(F, F^{**}) \), \( \Psi \in \mathcal{L}(E_1, \ldots, E_n; L_p(u)) \), \( A \in (L_p(u), F^{**}) \). Also
\[
\hat{\gamma}_p(\Phi) = \hat{\gamma}_p(\Phi) = \inf \|A\| \|\Psi\|,
\]
where the infimum is taken over all possible factorizations.

**Proof.** Let \( \Phi \in \mathcal{L}_{p-fat}(E_1, \ldots, E_n; F) \) relative to \( (q_1, \ldots, q_n) \), then
\[
V \Phi T \in \mathcal{N}_{(\infty:p,q_1,\ldots,q_n)}(X_1, \ldots, X_n; F_0) \quad \forall V \in \mathcal{L}_{a_p}(F, F_0),
\]
\[
T = (T_1, \ldots, T_n), \quad T_i \in \mathcal{L}_{a_p}(X_i, E_i), \quad \text{such that } \frac{1}{p'} + \frac{1}{q_1'} + \cdots + \frac{1}{q_n'} = 1, \quad i = 1, \ldots, n.
\]
Here \( F_0 \) \( \text{and } X_i \) are arbitrary Banach Space.

In particular for \( X_1 = M_1^j, \ldots, X_n = M_n^j, V = Q^{F}_{N_j}, F_0 = F/N_j, T_k = J^{E_k}_{M_k^j} \),
where \( M_k^j \in \text{Dim}(E_k) \) and \( F/N_j \in \text{Cod}(F), k = 1, \ldots, n \) and \( j \in I \), where \( I \) is a set of indexes. It is clear that \( J^{E_k}_{M_k^j} \in \mathcal{L}_{a_p}(M_k^j, E_k) \).

We define
\[
\Phi_j = Q^{F}_{N_j} \circ \Phi \circ \left( J^{E_1}_{M_1^j}, \ldots, J^{E_n}_{M_n^j} \right) : M_1^j \times \cdots \times M_n^j \rightarrow F_j = F/N_j.
\]

For the Lemma 2.1, the operator \( K_F \Phi \) we can write as a ultraproduct \( Q(\Phi_i)_u J \), where \( \Phi_j = Q^{F}_{N_j} \circ \Phi \circ \left( J^{E_1}_{M_1^j}, \ldots, J^{E_n}_{M_n^j} \right) \) with \( j = (M_1^j, \ldots, M_n^j; N_j) \), \( M_k^j \in \text{Dim}(E_k) \) and \( F/N_j \in \text{Cod}(F) \).
By the Theorem 3.2 from Cerna (see [6]) there are a factorization of $\Phi_j$ with $\Phi_j = Y_j \circ \Psi_j$, $\Psi_j \in \mathcal{L}(M_1^j, \ldots, M_n^j, l_p)$ and $Y_j \in \mathcal{L}(l_p; F_j)$ such that $\|\Psi_j\| \leq 1$ and $\|Y_j\| \leq (1 + \epsilon)N_{(\infty,p,q_1,\ldots,q_n)}(\Phi_j)$.

For the Lemma 19.3.4 from Pietsch (see [2]) we have that the ultraproduct $(l_p)_{\mathcal{U}}$, it can be represented as a Banach Space $L_p(\Omega, \mu)$. Hence $K_F \Phi$ is factored through $L_p(\Omega, \mu)$.

Inversely, we will suppose that $K_F \Phi$ have the described factorization. Then by the lemma 1.55 from Pietsch (see [2]) we have

$$S \circ \Phi \circ (T_1, \ldots, T_n) = S^\pi \circ K_F \circ \Phi \circ (T_1, \ldots, T_n) = S^\pi \circ A \circ \Psi \circ (T_1, \ldots, T_n)$$

for all $S \in \mathcal{F}(F, F_0)$ and $T_i \in \mathcal{F}(X_i, E_i)$. By the Lemma 2.1 from Cerna (see [7]) of (1) we obtain

$$N_f,_{(\infty,p,q_1,\ldots,q_n)}(S \circ \Phi \circ (T_1, \ldots, T_n)) = N_f,_{(\infty,p,q_1,\ldots,q_n)}(S^\pi \circ A \circ \Psi \circ (T_1, \ldots, T_n))$$

$$\leq \|S^\pi \circ A\| N_f,_{(\infty,p,q_1,\ldots,q_n)}(\Psi \circ (T_1, \ldots, T_n))$$

$$\leq \|S\| \|A\| \|\Psi \circ (T_1, \ldots, T_n)\|$$

$$\leq \|S\| \|A\| \prod_{i=1}^n \|T_i\|$$

(2)

As (2) is true for all $S \in \mathcal{F}(F, F_0)$ and $T_i \in \mathcal{F}(X_i, E_i)$ using the theorem (1) see appendix, we have $\Phi \in \mathcal{L}_{p-fact} (E_1, \ldots, E_n; F)$ and

$$\gamma_p(\Phi) = \max_{(\infty,p,q_1,\ldots,q_n)}(\Phi) \leq \|A\| \|\Psi\|$$

Hence is clear that

$$\gamma_p(\Phi) = \inf \|A\| \|\Psi\| = \gamma_p(\Phi)$$

where the infimum is taken over all possible factorizations. \hfill \Box

**Theorem 3.2.** An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is $\infty$-factorable if and only if there exist $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_p(u))$, $A \in \mathcal{L}(L_p(u), F)$, $K_F \in \mathcal{L}(F, F^{**})$ such that following statements are true:

1. There exist a compact Hausdorff Space and operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; C(K))$ and $Y \in \mathcal{L}(C(K), F^{**})$, $K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = Y \circ \Psi$.

Also we have that

$$\gamma_{\infty}(\Phi) = \gamma_p(\Phi) = \inf \|Y\| \|\Psi\|,$$
where the infimum is taken over all possible factorizations.

2. There exist a measure space \((\Omega, \mu)\) and operators \(\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_\infty(\Omega, \mu))\), \(Y \in \mathcal{L}(L_\infty(\Omega, \mu); F^{**})\) and \(K_F \in \mathcal{L}(F, F^{**})\) such that \(K_F \circ \Phi = Y \circ \Psi\), also

\[
\tilde{\gamma}_\infty(\Phi) = \hat{\gamma}_p(\Phi) = \inf \|\Psi\| \|Y\|,
\]

where the infimum is taken over all possible factorizations.

Proof. The proof of the theorem for \(p = \infty\) can be achieved following the same steps with some changes in the theorem (3.1) next to use the theorem 19.3.9 from Pietsch (see [2]).

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References


A. Appendix

**Definition A.1.** Let $1 \leq p \leq \infty$, be, $\Phi \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is called $p-$factorable, if exist a measure space $(\Omega, \Sigma, \mu)$ and operators $A \in \mathcal{L}(L_p(\mu), Y^{**})$ and $\Psi \in \mathcal{L}(X_1, \ldots, X_n; L_p(\mu))$ such that $K_Y \circ \Phi = A \circ \Psi$.

The collection of the $p$-factorable multi-linear operators of $X_1, \ldots, X_n$ to $Y$ will be denoted for $\mathcal{L}_{p\text{-fact}}(X_1, \ldots, X_n; Y)$. Also $\tilde{\gamma}_p(\Phi) = \inf \|\Psi\| \|A\|$, where the infimum is taken over all possible factorization of $\Phi$ is a norm over $\mathcal{L}_{p\text{-fact}}(X_1, \ldots, X_n; Y)$.

**Definition A.2.** Let $1 \leq p \leq \infty$. A operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called $p$-factorable relative to $(q_1, \ldots, q_n)$ if belongs to the normed ideal:

$$[\mathcal{L}_{p\text{-fact}}, \tilde{\gamma}_p] = [\mathcal{L}_{ap}, \|\|]\circ [N_{(\infty:p,q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)]\circ [\mathcal{L}_{ap}, \|\|]\circ \cdots \circ [\mathcal{L}_{ap}, \|\|],$$

such that $\frac{1}{p} + \frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1$, with norm

$$\tilde{\gamma}_p(\Phi) = \sup N_{(\infty:p,q_1,\ldots,q_n)}(B\Phi(T_1, \ldots, T_n)),$$

where the supremum is taken on all the $T = (T_1, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $\|B\| \leq 1, \|T_i\| \leq 1, i = 1, \ldots, n$

**Observation A.1.** The definition given in (A.2) should be understand of the following $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ belongs to $[\mathcal{L}_{p\text{-fact}}, \tilde{\gamma}_p]$ if $B\Phi(T_1, \ldots, T_n) \in N_{(\infty:p,q_1,\ldots,q_n)}(X_1, \ldots, X_n; F_0)$ for all $T_i \in \mathcal{L}_{ap}(X_i, E_i), i = 1, \ldots, n$ and for all $B \in \mathcal{L}_{ap}(F, F_0)$.

**Theorem A.1.** Let $[A, A]$ an ideal $s$-normed. Then $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ belongs to $A^{\text{max}}$ if and only if there exist a constant $\sigma \geq 0$ such that:

$$A(B\Phi T) \leq \sigma \|B\| \prod_{i=1}^n \|T_i\|, \ \forall T = (T_1, \ldots, T_n)$$

where $T_i \in \mathcal{F}(E_i, F_i), B \in \mathcal{F}(F, F_0)$ y $X_i, F_0$ are arbitrary Banach spaces.

Here

$$A^{\text{max}}(\Phi) = \inf \sigma$$

**Proof.** Let $\Phi \in A^{\text{max}}(E_1, \ldots, E_n; F)$ be, then $A^{\text{max}}(\Phi) = \sup A(B\Phi T)$, where the supremum is taken on all the $T = (T_1, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $\|T_i\| \leq 1, i = 1, \ldots, n$ and $\|B\| \leq 1$. 


Hence

\[ A(B\Phi T) \leq A^{\text{max}}(\Phi) \prod_{i=1}^{n} \|T_i\| \|B\|, \]

\[ \forall T_i \in \mathcal{F}(X_i, E_i), \ i = 1, \ldots, n \text{ and } B \in \mathcal{F}(F, F_0). \]

Conversely, let \( \Phi \in \mathcal{L}(E_1, \ldots, E_n; F) \) satisfy the above condition.

Let \( T = (T_1, \ldots, T_n) \) be with \( T_i \in \mathcal{L}_{\text{ap}}(X_i, E_i), i = 1, \ldots, n \) and \( B \in \mathcal{L}_{\text{ap}}(F, F_0) \), then there are \( T^k_i \in \mathcal{F}(X_i, E_i) \) and \( B_k \in \mathcal{F}(F, F_0) \) with

\[ T_i = \lim_{k \to \infty} T^k_i, \lim_{k \to \infty} (T^k_1, \ldots, T^k_n) = (T_1, \ldots, T_n) \text{ and } B = \lim_{k \to \infty} B_k. \]

It is followed from:

\[ A(B_k \Phi(T^k_1, \ldots, T^k_n) - B_m \Phi(T^m_1, \ldots, T^m_n)) \leq A((B_k - B_m)\Phi(T^k_1, \ldots, T^k_n) + B_m \Phi(T^k_1 - T^m_1, \ldots, T^k_n - T^m_n)) \]

\[ \leq \sigma \|B_k - B_m\| \prod_{i=1}^{n} \|T^k_i\| + \sigma \|B_m\| \prod_{i=1}^{n} \|T^k_i - T^m_i\| \]

that \( B_k \Phi T^k \) is an \( A \) Cauchy sequence. Since \( B\Phi T = \lim B_k \Phi(T^k_1, \ldots, T^k_n) \) is the only possible limit, we have that \( B\Phi T \in \mathcal{A}(X_1, \ldots, X_n; F_0) \). Moreover

\[ A(B\Phi T) \leq \sigma \|B\| \prod_{i=1}^{n} \|T_i\| \]

Hence

\[ \Phi \in \mathcal{A}^{\text{max}}(E_1, \ldots, E_n; F) \] and \( A^{\text{max}}(\Phi) \leq \sigma. \]