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THE MAXIMALITY OF p-FACTORABLE MULTI-LINEAR OPERATORS

B.M. Cerna

Departamento de Ciencias, Escuela de Matemáticas
Universidad Nacional "Santiago Antunez de Mayolo"
Ciudad Universitária de Shancayán
Avenida Centenario 100, Huaraz, PERU

Abstract: This work demostrate the equivalence of the following definitions "let $1 \leq p \leq \infty$, be, $\Phi \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is called $p-$factorable, if exist a measure space (Ω, Σ, μ) and operators $A \in \mathcal{L}(L_p(\mu), Y^{**})$ and $\Psi \in \mathcal{L}(X_1, \ldots, X_n; L_p(\mu))$ such that $K_Y \circ \Phi = A \circ \Psi$.

The collection of the p-factorable multi-linear operators of $X_1, \ldots, X_n$ to Y will be denoted for $\mathcal{L}_{p\text{-fact}}(X_1, \ldots, X_n, Y)$. Also $\widehat{\gamma}_p(\Phi) = \inf \|\Psi\| \, \|A\|$, where the infimun is taken over all possible factorzation of Φ is a norm over $\mathcal{L}_{p\text{-fact}}(X_1, \ldots, X_n; Y)$" and "let $1 \leq p \leq \infty$.

A operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called p-factorable relative to $(q_1, \ldots, q_n)$ if belongs to the normed ideal:

$$[\mathcal{L}_{p\text{-fact}}, \tilde{\gamma}_p] = [\mathcal{L}_{ap}, \|.\|]^{-1} \circ \left[\mathcal{N}_{(\infty:p,q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)\right] \circ ([\mathcal{L}_{ap}, \|.\|]^{-1}; \cdots; [\mathcal{L}_{ap}, \|.\|]^{-1}),$$

such that $\frac{1}{p'} + \frac{1}{q'_1} + \cdots + \frac{1}{q'_n} = 1$, with norm

$$\tilde{\gamma}_p(\Phi) = \sup N_{(\infty;p,q_1,\ldots,q_n)}(B\Phi(T_1, \ldots T_n))$$

where the supremun is taken on all the $T = (T_1, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $\|B\| \leq 1, \|T_i\| \leq 1$, $i = 1, \ldots, n$".

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1. Introduction

In the paper from Cerna (see [7]) titled *"Some properties of multi-linear operators of Nuclear Type"*, we give the motivations for which we are studying the properties of the multi-linear operators of nuclear type, in this work we obtain two important results. The first result establishes that the definition: An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is p-factorable $(1 \leq p < \infty)$ if there are operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_p(u))$, $A \in \mathcal{L}(L_p(u), F^{**})$, $K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = A \circ \Psi$. Here (Ω, u) is a suitable measure space. Also we have that $\widehat{\gamma}_p(\Phi) = \inf \|A\| \, \|\Psi\|$, where the infimun is taken over all possible factorizations. The second result establishes that the definition: An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is ∞-factorable if one of the following statements is true:

1. There exist an Hausdorff compact space K and operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; C(K))$ and $Y \in \mathcal{L}(C(K), F^{**})$, $K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = Y \circ \Psi$, also we have that
$$\widehat{\gamma}_\infty(\Phi) = \inf \|Y\| \, \|\Psi\| \, ,$$
where the infimun is taken over all possible factorizations.

2. There exist a measure space (Ω, μ) and operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_\infty(\Omega, \mu))$, $Y \in \mathcal{L}(L_\infty(\Omega, \mu); F^{**})$ and $K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = Y \circ \Psi$. Also we have that
$$\widehat{\gamma}_\infty(\Phi) = \inf \|\Psi\| \|Y\|,$$
where the infimun is taken over all possible factorization.

The two mentioned definitions are equivalent to the following definitions. An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called p-factorable $(1 \leq p < \infty)$, ∞-factorable relative to $(q_1, \ldots, q_n)$ respectively if it belongs to the normed ideal

$$[\mathcal{L}_{p-\text{fact}}, \tilde{\gamma}_p] = \left[\mathcal{L}_{a_p}, \|.\|\right]^{-1} \circ \left[\mathcal{N}_{(\infty,p,q_1,\cdots,q_n)}(E_1, \cdots, E_n; F), \mathsf{N}_{(\infty,p,q_1,\cdots,q_n)}\right] \cdots \circ \left(\left[\mathcal{L}_{a_p}, \|.\|\right]^{-1}, \cdots, \left[\mathcal{L}_{a_p}, \|.\|\right]^{-1}\right)$$

such that $\frac{1}{p'} + \frac{1}{q_1'} + \cdots + \frac{1}{q_n'} = 1$, with a norm given by

$$\widetilde{\gamma}(\Phi) = \sup N_{(\infty;p,q_1,q_2,\ldots,q_n)} \left(B\Phi(T_1, T_2, \ldots, T_n)\right),$$

where the supremun is taken of all the $T = (T_1, T_2, \ldots, T_n)$ with $T_i \in \mathcal{L}_{a_p}(X_i, E_i)$ and $B \in \mathcal{L}_{a_p}(F, F_0)$ such that $\|B\| \leq 1, \|T_i\| \leq 1, i = 1, \ldots, n$ where X_i, F_0 are arbitrary Banach Spaces.

We introduce the notations in the present work, for Banach Spaces $E_1, \cdots, E_n$ and F over the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, we denote $\mathcal{L}(E_1, \ldots, E_n; F)$ to the Banach Space of all multi-linear and continuous applications of $E_1 \times \cdots \times E_n$ over F with a natural norm given by

$$\|T\| = \sup_{\substack{x_i \in B_{E_i} \\ i=1,\cdots,n}} \|T(x_1, \cdots, x_n)\|,$$

where B_{E_i} denote the unitary ball of E_i, centered in 0. E_k^* denote the dual topological of E_k, $k = 1, \ldots, n$;

$\mathrm{Dim}(E) = \{$collection of all finite dimensional subspaces$\}$;
$\mathrm{Cod}(F) = \{$collection of all finite codimensional subspaces$\}$;
$\mathcal{L}_{a_p}(E, F) =$ Ideal of approximable operators of E onto F;
K_F is the isometric immersion of F into F^{**};
$J_M^E = \{$Embedding map from M into $E\}$;
$\mathcal{Q}_N^F = \{$Canonical map from E onto $F/N\}$;
$\mathcal{F}(E, F) =$ Ideal of finite operators of finite range from E onto F;
p': dual exponent, $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Ultra Stability

In this section using A. Santana Soares definition, from Soares (see [3]), we extended the lema 8.8.4 from Pietsch (see [2]) for the multi-linear case.

Definition 2.1. Given a set of indexes I, for each $i \in I$, we consider Banach Spaces $E_1^i, \ldots, E_n^i$, F_i and $A_i \in \mathcal{L}(E_1^i, \ldots, E_n^i; F_i)$ with $\sup \|A_i\| < \infty$ and $\mathcal{U}$ ultrafilter on I. We will define then

$$(A_i)_{\mathcal{U}} : (E_1^i)_{\mathcal{U}} \times \cdots \times (E_n^i)_{\mathcal{U}} \longrightarrow (F_i)_{\mathcal{U}}$$

where

$$(A_i)_{\mathcal{U}} \left((x_1^i)_{\mathcal{U}}, \ldots, (x_n^i)_{\mathcal{U}}\right) = \left(A_i(x_1^i, \ldots, x_n^i)\right)_{\mathcal{U}}.$$

Let us suppose $\Phi \in \mathcal{L}(E_1, \ldots, E_n : F)$. Given I the set of all indexes $i = \left(M_1^i, \ldots, M_n^i : N_i\right)$ whit $M_k^i \in \mathrm{Dim}(E_k)$ and $N_i \in \mathrm{Cod}(F)$, $k = 1, \ldots, n$. We choose an ultrafilter $\mathcal{U}$ containing all subsets

$$\{i \in I / M_k^i \supseteq M_k^{i_0}, N_i \subseteq N_{i_0}\}$$

where $i_0 = (M_1^{i_0}, \ldots, M_n^{i_0}; N_{i_0})$ is fixed.
Finally , put

$$E_k^i = M_k^i,\ k = 1, \ldots, n,\ F_i = F/N_i \text{ and } \Phi_i = Q_{N_i}^F \Phi\left(J_{M_1^i}^{E_1}, \ldots, J_{M_n^i}^{E_n}\right),$$

we have the following

Lemma 2.1. *There are operators*

$$J \in \mathcal{L}\left(E_1 \times \cdots \times E_n, \left(E_1^i\right)_{\mathcal{U}} \times \cdots \times \left(E_n^i\right)_{\mathcal{U}}\right)$$

and $Q \in \mathcal{L}\left((F_i)_{\mathcal{U}}, F^{**}\right)$ *such that* $\|J\| \leq 1$, $\|Q\| \leq 1$, *and* $K_F \Phi = Q(\Phi_i)_{\mathcal{U}} J$.

Proof. The operator J is defined by

$$J(x_1, \ldots, x_n) := (J_1(x_1), \ldots, J_n(x_n))$$

where $J_k(x_k) = \left(x_k^i\right)_u, k = 1, \ldots, n$
with

$$x_k^i = \begin{cases} x_k & ;\text{ if } x_k \in M_k^i, \\ 0 & ;\text{ if } x_k \notin M_k^i. \end{cases}$$

It is clear that J is linear and continuous for any norm over $E_1 \times \cdots \times E_n$, $\left(E_1^i\right)_{\mathcal{U}} \times \cdots \times \left(E_n^i\right)_{\mathcal{U}}$ and we obtain $\|J\| \leq 1$. Moreover, let

$$Q\left(\left(y_i^0\right)_{\mathcal{U}}\right) := F' - \lim_{\mathcal{U}} K_F y_i$$

where $(y_i) \in l_\infty(F_i, I)$ such that $Q_{N_i}^F(y_i) = y_i^0$.
First we check that the right-hand expression does not depended on the special choise of (y_i). For this purpose assume that

$$\left(Q_{N_i}^F y_i\right)_{\mathcal{U}} = 0 \text{ and } y'' := F' - \lim_{\mathcal{U}} K_F y_i.$$

Then $|\langle K_F y_i, b\rangle| \leq \|Q_{N_i}^F(y_i)\| \|b\|$ whenever F_i is contained in the null space of $b \in F'$. Therefore

$$|\langle y'', b\rangle| = \left|\lim_{\mathcal{U}} \langle K_F y_i, b\rangle\right| \leq \lim_{\mathcal{U}} \|Q_{N_i}^F y_i\| \|b\| = 0$$

Hence $y'' = 0$. This proves that Q is well-defined.

If the family (y_i) is chosen such that

$$\sup_I \|y_i\| \leq (1+\epsilon)\|(y_i^0)_{\mathcal{U}}\|$$

then we get

$$\|Q\left((y_i^0)_{\mathcal{U}}\right)\| \leq \lim_{\mathcal{U}} \|y_i\| \leq (1+\epsilon)\|(y_i^0)_{\mathcal{U}}\|$$

Therefore $\|Q\| \leq 1$. Finally, observe that

$$\begin{aligned} Q(\Phi_i)_{\mathcal{U}} J(x_1, \ldots, x_n) &= Q(\Phi_i x_i)_{\mathcal{U}}, \ \forall (x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n \\ &= F' - \lim_{\mathcal{U}} K_F \Phi \left(J^{E_1}_{M^i_1}, \ldots, J^{E_n}_{M^i_n} \right) (x^i_1, \ldots, x^i_n) \\ &= K_F \Phi(x_1, \ldots, x_n), \ \forall (x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n. \end{aligned}$$ □

3. p-Factorable Multi-Linear Operators

In this section we will demonstrate the equivalence of the two definitions from Cerna (see [7]), see appendix

Theorem 3.1. *Let $1 \leq p < \infty$, be $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is p-factorable relative to $(q_1, \ldots, q_n)$ if and only if there exist a commutative diagram such that $K_F \circ \Phi = A \circ \Psi$, where $K_F \in \mathcal{L}(F, F^{**}), \Psi \in \mathcal{L}\left(E_1, \ldots, E_n; L_p(u)\right), A \in (L_p(u), F^{**})$. Also*

$$\widetilde{\gamma}_p(\Phi) = \hat{\gamma}_p(\Phi) = \inf \|A\| \|\Psi\|,$$

where the infimun is taken aver all possible factorizations.

Proof. Let $\Phi \in \mathcal{L}_{p-\text{fat}}(E_1, \ldots, E_n : F)$ relative to $(q_1, \ldots, q_n)$, then

$$V \Phi T \in \mathcal{N}_{(\infty; p, q_1, \ldots, q_n)}(X_1, \ldots, X_n; F_0) \text{ for all } V \in \mathcal{L}_{a_p}(F, F_0),$$

$T = (T_1, \ldots, T_n)$, $T_i \in \mathcal{L}_{a_p}(X_i, E_i)$, such that $\frac{1}{p'} + \frac{1}{q'_1} + \cdots + \frac{1}{q'_n} = 1$, $i = 1, \ldots, n$. Here F_0 y X_i are arbitrary Banach Space.

In particular for $X_1 = M^j_1, \ldots, X_n = M^j_n$,$V = Q^F_{N_j}$, $F_0 = F/N_j$, $T_k = J^{E_k}_{M^j_k}$, where $M^j_k \in \text{Dim}(E_k)$ and $F/N_j \in \text{Cod}(F), k = 1, \ldots, n$ and $j \in I$, where I is a set of indexes. It is clear that $J^{E_k}_{M^j_k} \in \mathcal{L}_{a_p}\left(M^j_k, E_k\right)$.

We define

$$\Phi_j = Q^F_{N_j} \circ \Phi \circ \left(J^{E_1}_{M^j_1}, \ldots, J^{E_n}_{M^j_n} \right) : M^j_1 \times \cdots \times M^j_n \longrightarrow F_j = F/N_j$$

For the Lemma 2.1, the operator $K_F \Phi$ we can write as a ultraproduct $Q\left(\Phi_i\right)_u J$, where $\Phi_j = Q^F_{N_j} \circ \Phi \circ \left(J^{E_1}_{M^j_1}, \ldots, J^{E_n}_{M^j_n} \right)$ with $j = \left(M^j_1, \ldots, M^j_n; N_j\right)$, $M^j_k \in \text{Dim}(E_k)$ and $F/N_j \in \text{Cod}(F)$.

By the Theorem 3.2 from Cerna (see [6]) there are a factorization of Φ_j with $\Phi_j = Y_j \circ \Psi_j$, $\Psi_j \in \mathcal{L}\left(M_1^j, \ldots, M_n^j, l_p\right)$ and $Y_j \in \mathcal{L}(l_p; F_j)$ such that $\|\Psi_j\| \leq 1$ and $\|Y_j\| \leq (1+\epsilon) N_{(\infty,p,q_1,\ldots,q_n)}(\Phi_j)$.

For the Lemma 19.3.4 from Pietsch (see [2]) we have that the ultraproduct $(l_p)_{\mathcal{U}}$, it can be represented as a Banach Space $L_p(\Omega, \mu)$. Hence $K_F \Phi$ is factored through $L_p(\Omega, \mu)$.

Inversely, we will suppose that $K_F \Phi$ have the described factorzation. Then by the lema 1.55 from Pietsch (see [2]) we have

$$\begin{aligned} S \circ \Phi \circ (T_1, \ldots, T_n) &= S^\pi \circ K_F \circ \Phi \circ (T_1, \ldots, T_n) \\ &= S^\pi \circ A \circ \Psi \circ (T_1, \ldots, T_n) \end{aligned} \tag{1}$$

for all $S \in \mathcal{F}(F, F_0)$ and $T_i \in \mathcal{F}(X_i, E_i)$. By the Lemma 2.1 from Cerna (see [7]) of (1) we obtain

$$\begin{aligned} &N_{f,(\infty;p,q_1,\ldots,q_n)}(S \circ \Phi \circ (T_1, \ldots, T_n)) \\ &\qquad = N_{f,(\infty;p,q_1,\ldots,q_n)}(S^\pi \circ A \circ \Psi \circ (T_1, \ldots, T_n)) \\ &\qquad \leq \|S^\pi \circ A\| \mathsf{N}_{f,(\infty,p,q_1,\ldots,q_n)}(\Psi \circ (T_1, \ldots, T_n)) \\ &\qquad \leq \|S^\pi \circ A\| \, \|\Psi \circ (T_1, \ldots, T_n)\| \\ &\qquad \leq \|S\| \, \|A\| \, \|\Psi(T_1, \ldots, T_n)\| \\ &\qquad \leq \|S\| \, \|A\| \, \|\Psi\| \prod_{i=1}^{n} \|T_i\| \end{aligned} \tag{2}$$

As (2) is true for all $S \in \mathcal{F}(F, F_0)$ and $T_i \in \mathcal{F}(X_i, E_i)$ using the theorem (1) see appendix, we have $\Phi \in \mathcal{L}_{p\text{-fact}}(E_1, \ldots, E_n; F)$ and

$$\widetilde{\gamma}_p(\Phi) = \mathsf{N}^{\max}_{(\infty,p;q_1,\ldots,q_n)}(\Phi) \leq \|A\| \, \|\Psi\|$$

Hence is clear that

$$\widetilde{\gamma}_p(\Phi) = \inf \|A\| \, \|\Psi\| = \hat{\gamma}_p(\Phi)$$

where the infimun is taken over all possible factorizations. □

Theorem 3.2. *An operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is ∞-factorable if and only if there exist $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_p(u))$, $A \in \mathcal{L}(L_p(u), F)$, $K_F \in \mathcal{L}(F, F^{**})$ such that following statements are true:*

1. *There exist a compact Hausdorff Space and operators $\Psi \in \mathcal{L}(E_1, \ldots, E_n; C(K))$ and $Y \in \mathcal{L}(C(K), F^{**})$, $K_F \in \mathcal{L}(F, F^{**})$ such that $K_F \circ \Phi = Y \circ \Psi$, also we have that*

$$\widetilde{\gamma}_\infty(\Phi) = \hat{\gamma}_p(\Phi) = \inf \|Y\| \|\Psi\|,$$

where the infimun is taken over all possible factorzations.

2. *There exist a measure Space* (Ω, μ) *and operators* $\Psi \in \mathcal{L}(E_1, \ldots, E_n; L_\infty(\Omega, \mu))$, $Y \in \mathcal{L}(L_\infty(\Omega, \mu); F^{**})$ *and* $K_F \in \mathcal{L}(F, F^{**})$ *such that* $K_F \circ \Phi = Y \circ \Psi$, *also*
$$\widetilde{\gamma}_\infty(\Phi) = \hat{\gamma}_p(\Phi) = \inf \|\Psi\|\|Y\|,$$
where the infimun is taken over all possible factorizations.

Proof. The proof of the theorem for $p = \infty$ can be achieved following the same steps with some changes in the theorem (3.1) next to use the theorem 19.3.9 from Pietsch (see [2]). □

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A. Appendix

Definition A.1. Let $1 \leq p \leq \infty$, be, $\Phi \in \mathcal{L}(X_1, \ldots, X_n; Y)$ is called $p-$factorable, if exist a measure space (Ω, Σ, μ) and operators $A \in \mathcal{L}(L_p(\mu), Y^{**})$ and $\Psi \in \mathcal{L}(X_1, \ldots, X_n; L_p(\mu))$ such that $K_Y \circ \Phi = A \circ \Psi$.

The collection of the p-factorable multi-linear operators of $X_1, \ldots, X_n$ to Y will be denoted for $\mathcal{L}_{p\text{-fact}}(X_1, \ldots, X_n, Y)$. Also $\widehat{\gamma}_p(\Phi) = \inf \|\Psi\| \, \|A\|$, where the infimun is taken over all possible factorzation of Φ is a norm over

$$\mathcal{L}_{p\text{-fact}}(X_1, \ldots, X_n; Y).$$

Definition A.2. Let $1 \leq p \leq \infty$. A operator $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ is called p-factorable relative to $(q_1, \ldots, q_n)$ if belongs to the normed ideal:

$$\begin{aligned}&[\mathcal{L}_{p\text{-fact}}, \tilde{\gamma}_p] \\ &= [\mathcal{L}_{ap}, \|.\|]^{-1} \circ \big[\mathcal{N}_{(\infty:p,q_1,\ldots,q_n)}(E_1, \ldots, E_n; F)\big] \circ ([\mathcal{L}_{ap}, \|.\|]^{-1}; \cdots; [\mathcal{L}_{ap}, \|.\|]^{-1}),\end{aligned}$$

such that $\frac{1}{p'} + \frac{1}{q_1'} + \cdots + \frac{1}{q_n'} = 1$, with norm

$$\tilde{\gamma}_p(\Phi) = \sup N_{(\infty;p,q_1,\ldots,q_n)}(B\Phi(T_1, \ldots T_n)),$$

where the supremun is taken on all the $T = (T_1, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}\,(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $\|B\| \leq 1, \|T_i\| \leq 1, i = 1, \ldots, n$

Observation A.1. The definition given in (A.2) should be understand of the following $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ belongs to $[\mathcal{L}_{p\text{-fact}}, \tilde{\gamma}_p]$ if $B\Phi(T_1, \ldots, T_n) \in \mathcal{N}_{(\infty:p,q_1,\ldots,q_n)}(X_1, \ldots, X_n; F_0)$ for all $T_i \in \mathcal{L}_{ap}(X_i, E_i), i = 1, \ldots, n$ and for all $B \in \mathcal{L}_{ap}(F, F_0)$.

Theorem A.1. *Let $[\mathcal{A}, A]$ an ideal s-normed. Then $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ belongs to $\mathcal{A}^{\max}$ if and only if there exist a constant $\sigma \geq 0$ such that:*

$$A\,(B\Phi T) \leq \sigma \,\|B\| \prod_{i=1}^{n} \|T_i\|\,, \ \forall T = (T_1, \ldots, T_n)$$

where $T_i \in \mathcal{F}\,(E_i, F_i)\,, B \in \mathcal{F}\,(F, F_0)$ y X_i, F_0 are arbitrary Banach spaces.

Here

$$A^{\max}(\Phi) = \inf \sigma$$

Proof. Let $\Phi \in \mathcal{A}^{\max}(E_1, \ldots, E_n; F)$ be, then $A^{\max}(\Phi) = \sup A(B\Phi T)$, where the supremun is taken on all the $T = (T_1, \ldots, T_n)$ with $T_i \in \mathcal{L}_{ap}(X_i, E_i)$ and $B \in \mathcal{L}_{ap}(F, F_0)$ such that $\|T_i\| \leq 1, i = 1, \ldots, n$ and $\|B\| \leq 1$.

Hence

$$A(B\Phi T) \leq A^{\max}(\Phi) \prod_{i=1}^{n} \|T_i\| \|B\|,$$
$$\forall T_i \in \mathcal{F}(X_i, E_i),\ i = 1, \ldots, n \ \text{ and } \ B \in \mathcal{F}(F, F_0).$$

Conversely, let $\Phi \in \mathcal{L}(E_1, \ldots, E_n; F)$ satisfy the above condition.

Let $T = (T_1, \ldots, T_n)$ be with $T_i \in \mathcal{L}_{ap}(X_i, E_i), i = 1, \ldots, n$ and $B \in \mathcal{L}_{ap}(F, F_0)$, then there are $T_i^k \in \mathcal{F}(X_i, E_i)$ and $B_k \in \mathcal{F}(F, F_0)$ with

$$T_i = \lim_{k\to\infty} T_i^k, \lim_{k\to\infty} (T_1^k, \ldots, T_n^k) = (T_1, \ldots, T_n) \text{ and } B = \lim_{k\to\infty} B_k.$$

It is followed from:

$$\begin{aligned} A(B_k\Phi(T_1^k, \ldots, T_n^k) - B_m\Phi(T_1^m, \ldots, T_n^m))^s = & A((B_k - B_m)\Phi(T_1^k, \ldots, T_n^k) \\ & + B_m\Phi(T_1^k - T_1^m, \ldots, T_n^k - T_n^m))^s \\ \leq & A((B_k - B_m)\Phi(T_1^k, \ldots, T_n^k))^s + \\ & + A(B_m\Phi(T_1^k - T_1^m, \ldots, T_n^k - T_n^m))^s \\ \leq & \sigma^s \|B_k - B_m\|^s \prod_{i=1}^{n} \|T_i^k\|^s + \\ & + \sigma^s \|B_m\|^s \prod_{i=1}^{n} \|T_i^k - T_i^m\|^s \end{aligned}$$

that $B_k\Phi T_k$ is an A Cauchy sequence. Since $B\Phi T = \lim B_k\Phi(T_1^k, \ldots, T_n^k)$ is the only possible limit, we have that $B\Phi T \in \mathcal{A}(X_1, \ldots, X_n; F_0)$. Moreover

$$A(B\Phi T) \leq \sigma \|B\| \prod_{i=1}^{n} \|T_i\|$$

Hence

$$\Phi \in \mathcal{A}^{\max}(E_1, \ldots, E_n; F) \text{ and } A^{\max}(\Phi) \leq \sigma. \qquad \square$$