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THE RATIOS BETWEEN THE NUMERICAL RADIUS AND THE SPECTRAL RADIUS OF A MATRIX AND THE SQUARE ROOT OF THE SPECTRAL NORM OF THE SQUARE OF THIS MATRIX

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Abstract: In this paper, we introduce some ratios between the numerical radius and the spectral radius of a matrix and the square root of the spectral norm of the square of this matrix, and we review the existing results for extreme cases.

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1. Introduction

Let $B(H)$ denote the $\mathbb{C}^*$-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle ., . \rangle$. For $A \in B(H)$. Let $w(A), r(A)$ and $\|A\|$denote the numerical radius, the spectral radius and the usual operator norm of A, respectively. It is well known that $w(.)$ defines a norm on $B(H)$, and that for every $A \in B(H)$,

$$r(A) \leq w(A) \leq \|A\|,$$

and that equality holds if A is normal.

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The best known result in the affirmative direction is the power inequality, which asserts that $A \in B(H)$, then

$$r(A^n) = (r(A))^n \text{ and } w(A^n) \leq (w(A))^n,$$

for every positive integer n.

Numerical radii estimate of companion matrices have been invoked by many Mathematicians. Also, we know that

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

In (2009)., we find that

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right) \leq \|A\|,$$

whenever, $A^2 \neq [0]_{n\times n}$.

Also, if $A^2 = [0]_{n\times n}$, then $w(A) = \frac{1}{2}\|A\|$

$$w(A) = \frac{1}{2}\|A\|$$

In this work, let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices $M_n(\mathbb{C})$.

Kui Du presents the ratios between the spectral norm, the numerical radius and the spectral radius. In this note, we consider the ratios

$$s_1(A) = \frac{\|A^2\|^{\frac{1}{2}}}{w(A)}.$$

and

$$\tau(A) = \frac{w(A)}{r(A)}.$$

It is well known that

$$0 \leq r(A) \leq w(A) \leq \|A^2\|^{\frac{1}{2}} \leq 2w(A).$$

Thus

$$0 \leq s_1(A) \leq 2,$$

and

$$0 \leq \tau(A) \leq \infty.$$

Here we employ the convention that $\tau(A) = \infty$ for $r(A) = 0$. Obviously

$$s_1(zA) = s_1(A) \text{ and } \tau(zA) = \tau(A) \text{ for all } z \neq 0.$$

It follows from $r(A^m) = [r(A)]^m$ and $w(A^m) \leq [w(A)]^m$ that

$$\tau(A^m) \leq [\tau(A)]^m.$$

Now, we list some known results as a background and reminder for the reader.

Definition 1.1 If $A \in M_n(\mathbb{C})$, then:

(i) The spectral norm (or the operator norm) is defined by

$$\|A\| = \max\left\{\|Ax\| : \|x\| = 1\right\} = \max\left\{\frac{\|Ax\|}{\|x\|} : \|x\| \neq 0\right\}.$$

(ii) The numerical radius of A is defined by

$$w(A) = \max\left\{|(Ax, x)| : x \in \mathbb{C}^n, \|x\| = 1\right\}.$$

(iii) The spectral radius of A is defined by

$$r(A) = \max\left\{|\lambda| : \lambda \text{is an eigenvalue of } A\right\}.$$

Theorem 1.1. *(i) If $A \in M_n(\mathbb{C})$, then:*

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

(ii) If $A \in M_n(\mathbb{C})$, then

$$0 \leq r(A) \leq w(A) \leq \|A\|,$$

(iii) If $A \in M_n(\mathbb{C})$, then t

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right) \leq \|A\|,$$

2. The Ratios Between the Numerical Radius and the Spectral Radius of a Matrix and the Square Root of the Spectral Norm of the Square of This Matrix

The extreme cases $\tau(A) = 1$, $s_1(A) = 1$ and s(A) = 2. In this subsection, we review the existing results for the extreme cases $\tau(A)$=1, $s_1(A)$)=1 and $s_1(A)$ = 2, respectively. We focus on the relation between $s_1(A)$ and $\tau(A)$.

A matrix A is said to be spectral if $w(A) = r(A)$, i.e. $\tau(A) = 1$, see (11). By using similar method in (11), we present the following theorems.

Theorem 2.1. *Let $A \in C^{n\times n}$ such that $\tau(A) = 1$.*

1) If $n \leq 2$, then $s_1(A) = 1$, and A is a normal matrix.

2) If $n > 2$, then A is unitarily similar to a triangle matrix of the form

$$\begin{bmatrix} \Lambda_k & 0 \\ 0 & B \end{bmatrix},$$

where $1 \leq k \leq n$,

$$\Lambda_k = \begin{bmatrix} \lambda_1 & & \cdots & & \\ & \lambda_2 & \cdots & & \\ & & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \cdots & & \lambda_k \end{bmatrix}. \tag{1}$$

$$B = \begin{bmatrix} \lambda_{k+1} & * & \cdots & * & * \\ & \lambda_{k+2} & \cdots & * & * \\ & & \ddots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & \cdots & & \lambda n \end{bmatrix}.$$

and

$$r(B) < r(A) = |\lambda_1| = \cdots = |\lambda_k|,,$$

$$w(B) < r(A) = |\lambda_1| = \cdots = |\lambda_k|,,$$

Furthermore, if

$$r(A) < \|B^2\|^{\frac{1}{2}},$$

$$1 < s_1(A) \leq 2.$$

otherwise,

$$s_1(A) = 1.$$

A matrix A is said to be radial if

$$w(A) = \|A^2\|^{\frac{1}{2}}$$

i.e.,

$$s_1(A) = 1.$$

Theorem 2.2. *Let $A \in C^{n\times n}$ such that $s_1(A) = 1$. Then $\tau(A) = 1$.*

1) If $n \leq 2$, then A is a normal matrix.

2) If $n > 2$, then A is unitarily similar to a block diagonal matrix of the form (1) such that

$$r(B) \leq r(A),$$

and

$$r(A) \geq \|B^2\|^{\frac{1}{2}}.$$

Remark 2.3. Note that $s_1(A) = 1$ does not imply that $\tau(A) = 1$.

See the following example.

Example 2.4. (A Scaled Jordan Block) Let

$$J_n^\alpha(\lambda) = \begin{bmatrix} \lambda & \alpha & \cdots & & \\ & \lambda & \alpha & & \\ & & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \alpha \\ & & \cdots & & \lambda \end{bmatrix}_{n\times n} = \lambda I + N. \tag{2}$$

be a matrix of order $n > 1$. Then the numerical range of $J_n^\alpha(\lambda)$ is a disk centered at λ with radius $|\alpha|\cos\frac{\pi}{n+1}$ (see [10, Theorem 2.1]). We have

$$r(J_n^\alpha(\lambda)) = |\lambda|, \quad w(N) = |\alpha|\cos\frac{\pi}{n+1}, \quad w(J_n^\alpha(\lambda)) = |\lambda| + |\alpha|\cos\frac{\pi}{n+1},$$

and

$$\|J_n^\alpha(\lambda)\| = |\lambda| + |\alpha|.$$

So,

$$s_1(J_n^\alpha(\lambda)) = \frac{\sqrt{|\lambda^2| + |2\alpha\lambda|}}{|\lambda| + |\alpha|\cos\frac{\pi}{n+1}} \leq \frac{\sqrt{1 + |\frac{2\alpha}{\lambda}|}}{1 + |\frac{\alpha}{\lambda}|\cos\frac{\pi}{n+1}}$$

and

$$\tau(J_n^\alpha(\lambda)) = 1 + |\frac{\alpha}{\lambda}| \cos\frac{\pi}{n+1}.$$

Thus, when $n \to \infty$ and $|\alpha|/|\lambda| \to \infty$, $s_1(J_n^\alpha(\lambda)) \to 1$. However $\tau(J_n^\alpha(\lambda)) \to \infty$.

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