SOME REMARKS ON TOPOLOGICAL LATTICES

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Abstract: In this paper we give a lattice-theoretic approach to compactification of topological spaces. We use Stone-Cech compactification for representing the remainder of the locally compact spaces.

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1. Introduction

Among the other results we investigate a lattice theoretic approach for the remainder $\beta X \setminus X$ of a topological space $X$. The problem of compactification of certain topological spaces is considered by many authors in recent years [1,3]. We use an algebraic systematic way using lattice theoretic techniques.

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2. Preliminaries

Knowledge and some of the basic concepts of topology (such as separation axioms) is assumed.

A subset $S$ of a topological space $X$ is called clopen if $S$ is both open and closed with respect to the topology on $X$.

A compactification $\alpha X$ of a space $X$ is a compact Hausdorff space that contains $X$ as a dense subset. A space has a compactification iff it is Tychonoff. From now on, unless otherwise noted, spaces will be assumed to be Tychonoff. $\alpha X \setminus X$ denotes the remainder (i.e. the compactification $\alpha X$ minus the original space $X$) of the compactification $\alpha X$.

The one-point compactification of $X$, sometimes denoted $\omega X$, is the compactification of $X$ whose remainder consists of a single point. Locally compact, non-compact spaces are precisely those that have one-point compactification.

The maximum compactification of a space $X$ is called the Stone-Čech compactification of $X$ and is denoted $\beta X$. The Stone-Čech compactification of $X$ can be characterized as the unique compactification of $X$ such that every bounded continuous function on $X$ can be extended to a continuous function on $\beta X$. If $X$ is compact and connected, it is called continuum.

2.1. Definitions, Lattice Theoretic

A partial order $\leq$ is a binary relation that is reflexive, transitive, and antisymmetric. A partially ordered set $(P, \leq)$ is a set $P$ paired with a partial order $\leq$. We usually just write $P$ instead of $(P, \leq)$.

Given a partially ordered set $P$ and elements $a, b \in P$, the supremum of $a$ and $b$ (denoted $a \lor b$) is the least element that is greater than or equal to both $a$ and $b$. The infimum of $a$ and $b$ (denoted $a \land b$) is the greatest element that is less than or equal to both $a$ and $b$. If $L$ is a partially ordered set (Poset) where $a \lor b$ and $a \land b$ exist for all $a, b \in L$, then we call $L$ a lattice.

A lattice is called distributive if

$$a \land (b \lor c) = (a \land b) \lor (a \land c) \text{ and } a \lor (b \land c) = (a \lor b) \land (a \lor c)$$

for all $a, b \in L$.

A bounded lattice is a lattice with both greatest and a least element (usually denoted 1 and 0, respectively).

A filter $p$ on a lattice $L$ is a subset of $L$ such that

(i). $0 \notin p$

(ii). $a, b \in p$ implies that $a \land b \in p$. 


(iii). $a \in p$ and $b \in L$ with $b \geq a$ implies that $b \in p$.

An ultrafilter is a maximal filter (with respect to set inclusion).

Let $a \in L$. Then $a' \in L$ is said to be the pseudocomplement of $a$ if $a \land a' = 0$ and $b \land a = 0 \Rightarrow b \leq a'$, $\forall b \in L$.

A lattice is called pseudocomplemented if every $a \in L$ has a pseudocomplement.

Let $a \in L$. Then $a^c \in L$ is said to be the complement of $a$ if $a \land a^c = 0$ and $b \land a = 0 \Rightarrow b \leq a^c$, $\forall b \in L$.

A lattice is called Boolean if every element of $L$ has a complement is called a Boolean algebra.

If $B$ is a Boolean algebra, then the stone space of $B$, denoted $St(B)$, is the
space formed by the set of all ultrafilters on $B$. We also use this notation to
describe the collection of ultrafilters of a lattice (i.e. if $L$ is a lattice, $St(L)$ is
the space of all ultrafilters on $L$).

2.2. General Lattices

Suppose that $L$ is a bounded distributive lattice. A filter on $L$ is said to be
prime if $a \lor b \in L$ implies that $a \in L$ or $b \in L$. With the help of the following
lemma, we show that every ultrafilter is prime.

**Lemma 2.2.1.** Let $p$ be an ultrafilter of a lattice $L$. Suppose $b \in L$ and
$\forall a \in p$, $a \land b \neq 0$. Then $b \in p$.

**Proof.** Let $q = \{c \in L : \exists a \in p, c \geq a \land b\}$

It is easy to show that $q$ is a fitter. Let $a \in p$. Since $a \geq a \land b$, then $a \in q$. So
$p \subseteq q$. But $p$ is an ultrafilter, so $p = q$. since for any $a \in p$, $b \geq b \land a$, then
$b \in q$. But $p = q \Rightarrow b \in p$. □

**Proposition 2.2.2.** Suppose that $p$ is a filter of a lattice $L$: then $p$ is maximal
$\Rightarrow p$ is prime.

**Proof.** Let $p$ be a maximal filter of $L$. Let $x, y \in L$ such that $x \lor y \in p$. Then
$x \lor y \in p \Rightarrow (x \lor y) \land a \neq 0, \forall a \in p \Rightarrow (x \land a) \lor (y \land a) \neq 0, \forall a \in p$ suppose that
$\exists a, b \in p$ such that $x \land a = 0$ and $y \land b = 0$. Then $(x \land (a \land b)) \lor (y \land (a \land b)) = 0,
which contradict above since $a \land b \in p$. So either $(x \land a \neq 0, \forall a \in p)$ or
$(y \land a \neq 0, \forall a \in p)$ so by lemma 1.3.1., either $x \in p$ or $y \in p$. □

Now let’s define the topology on $St(L)$ by considering a base for the closed
sets.
**Notation.** Let \([f] = \{p \in \text{St}(L) : f \in p\}\), for each \(f \in L\), and let \(B(\text{St}(L)) = \{[f] : f \in L\}\).

Obviously \(B(\text{St}(L))\) is a base for the closed sets of some topology on \(\text{St}(L)\), which denoted \(\tau_B\).

The following are fundamental properties of these basic closed sets.

**Lemma 2.2.3.** (i) \([a] \cup [b] = [a \lor b], \forall a, b \in L\)
(ii) \([a] \cap [b] = [a \land b], \forall a, b \in L\).

**Proof.** The proof is obvious.

**Theorem 2.2.4.** A subset \(E\) of a lattice \(L\) has the finite intersection property \(\Rightarrow E\) is contained in an ultrafilter of \(L\).

**Proof.** Assume \(E \subseteq L\) has the f.i.p. So, if \(p_0\) is the filter generated by \(E\), then \(p_0\) is proper. Let \(P = \{q : \exists p \in [f_{\alpha_1}] \cap \cdots \cap [f_{\alpha_n}] \Rightarrow f_{\alpha_1}, \cdots, f_{\alpha_n} \in p \Rightarrow f_{\alpha_1} \land \cdots \land f_{\alpha_n} \neq 0\} \subseteq p\) since \(0 \notin p\).

(ii). Let \(g_1, g_2 \in S\). Then \(g_1 = f_{\alpha_1} \land \cdots \land f_{\alpha_n}\) and \(g_2 = f_{\beta_1} \land \cdots \land f_{\beta_m}\). So \(g_1 \land g_2 = f_{\alpha_1} \land \cdots \land f_{\alpha_n} \land f_{\beta_1} \land \cdots \land f_{\beta_m} \in S\). So, by (i) and (ii), \(S\) is a base for some filter \(p\), and thus by corollary 1.3.5. \(\exists\) an ultrafilter \(p'\) such that \(p' \supseteq p\). By construction, \(f_{\alpha} \in p', \forall \alpha \in \Delta\). Hence \(p' \in [f_{\alpha}], \forall \alpha \in \Delta\). So \(\cap C \neq 0\). Therefore \(\text{St}(L)\) is compact. □

**Corollary 2.2.5.** Any filter \(p\) of a lattice \(L\) is contained in an ultrafilter.

With this corollary, we prove that \(\text{St}(L)\) is compact.

**Proposition 2.2.6.** \(\text{St}(L)\) is compact.

**Proof.** Let \(C\) be a collection of basic closed sets of \(\text{St}(L)\) with the finite intersection property, denote \(C = \{[f_{\alpha}] : \alpha \in \Delta\}\). Let \(S\) be the collection of all finite infimums of the \(f_{\alpha}\)'s, \(\alpha \in \Delta\). (we will show that \(S\) is a filter base.)

(i). Let \(g \in S\). Then \(g = f_{\alpha_1} \land \cdots \land f_{\alpha_n}\) since \(C\) has the f.i.p. \(\exists p \in [f_{\alpha_1}] \cap \cdots \cap [f_{\alpha_n}] \Rightarrow f_{\alpha_1}, \cdots, f_{\alpha_n} \in p \Rightarrow f_{\alpha_1} \land \cdots \land f_{\alpha_n} \neq 0\) if \(0 \notin p\).

(ii). Let \(g_1, g_2 \in S\). Then \(g_1 = f_{\alpha_1} \land \cdots \land f_{\alpha_n}\) and \(g_2 = f_{\beta_1} \land \cdots \land f_{\beta_m}\). So \(g_1 \land g_2 = f_{\alpha_1} \land \cdots \land f_{\alpha_n} \land f_{\beta_1} \land \cdots \land f_{\beta_m} \in S\). So, by (i) and (ii), \(S\) is a base for some filter \(p\), and thus by corollary 1.3.5. \(\exists\) an ultrafilter \(p'\) such that \(p' \supseteq p\). By construction, \(f_{\alpha} \in p', \forall \alpha \in \Delta\). Hence \(p' \in [f_{\alpha}], \forall \alpha \in \Delta\). So \(\cap C \neq 0\). Therefore \(\text{St}(L)\) is compact. □

Next, we’ll consider the following lattice property, labeled \((\ast)\):

\((\ast)\) If \(p, q \in L\) such that \(p \land q = 0\), then \(\exists a, b \in L\) such that \(p \land a = 0, q \land b = 0,\) and \(a \lor b = 1\).

**Definition 2.2.7.** A lattice \(L\) is called normal if \(L\) has the \((\ast)\) property.

**Proposition 2.2.8.** If \(L\) is normal, then \(\text{St}(L)\) is \(T_2\).
Proof. Let \( p, q \in \text{St}(L) \) such that \( p \neq q \). Then \( \exists f \in p \) and \( g \in q \) s.t. \( f \land g = 0 \). So by \((*)\), \( \exists a, b \in L \) s.t. \( f \land a = 0, g \land b = 0 \), and \( a \lor b = 1 \) so \( p \not\in [a] \) and \( q \not\in [b] \). Also by Lemma 1.3.3 \([a] \lor [b] = [a \lor b] = [1] = \text{St}(L)\), so \( \text{St}(L) \) is \( T_2 \).

Definition 2.2.9. A lattice \( L \) is called external provided that \( 1 \) is the only element of \( L \) that is in every ultrafilter on \( L \).

Proposition 2.2.10. Assume that \( L \) is external. If \( \text{St}(L) \) is \( T_2 \), then \( L \) is normal.

Proof. Suppose \( a, b \in L \) s.t. \( a \land b = 0 \). Then \([a] \cap [b] = \phi\) (i.e. they have no ultrafilters in common). Since \( \text{St}(L) \) is \( T_2 \) and compact, \( \text{St}(L) \) is normal. Note that the base \( B = \{[a] : a \in L\} \) for the closed sets of \( \text{St}(L) \) is closed under finite intersections. This, combined with the fact that \( \text{St}(L) \) is normal and compact, implies the following:

\[
\exists [c], [d] \in B s.t. [a] \cap [c] = \phi, [b] \cap [d] = \phi
\]

and s.t. \([c] \cup [d] = \text{St}(L)\).

Suppose \( a \land c \neq 0 \). Then \( \exists \) an ultrafilter \( p \in \text{St}(L) \) s.t. \( a \land c \in p \). But then \( a, c \in p \) (since \( a, c \geq a \land c \)), which means that \( p \in [a] \cap [c] \), a contradiction. Hence \( a \land c = 0 \).

Similarly, \( h \land d = 0 \). Since \( \text{St}(L) = [c] \cup [d] = [c \lor d] \) and \( L \) assumed to be external, then \( c \lor d = 1 \). Hence \( L \) is normal. \( \square \)

Corollary 2.2.11. Let \( L \) be external. Then \( L \) is normal iff \( \text{St}(L) \) is normal.

We now consider the disjunction property and how it relates to our external property. First let’s define the disjunction property as it is given in [3].

Definition. 2.2.12. A lattice \( L \) has the disjunction property provided that whenever \( a, b \in L \) s.t. \( a \neq b \), \( \exists c \in L \setminus \{0\} \) s.t. one of \( a \land c \) and \( b \land c \) is 0 and the other is not 0.

Theorem 2.2.13. \( L \) has the disjunction property iff there is a 1-1 correspondence between the elements of \( L \) and the elements of \( B(\text{St}(L)) = \{[f] : f \in L\} \).

Proof. [3].

The following proposition can easily proved.

Proposition 2.2.14. Suppose that \( f : L \rightarrow L' \) is a lattice embedding and that \( p \) is a prime filter on \( L' \). Then \( f^{-1}(p) \) is a prime filter on \( L \).
Definition 2.2.15. Suppose \( f : L \to L' \) is a lattice embedding and that \( L \) is normal. Define \( f^D : \text{St}(L') \to \text{St}(L) \), the stone-dual map of \( f \), by \( f^D(p) = q_p \), where \( q_p \) is the unique ultrafilter in \( L \) containing \( f^{-1}(p) \).

If \( p \in \text{St}(L') \), then \( p \) is also prime. Also \( f^{-1}(p) \) is a prime filter on \( L \). Since \( L \) is normal, then \( f^{-1}(p) \) is contained in a unique ultrafilter on \( L \). Thus \( f^D \) is well-defined.

Furthermore,

Proposition 2.2.16. \( f^D \) is onto.

Proof. Use Corollary 1.3.5.

3. \( L(X) \)

This section will begin by defining a lattice based on the closed sets of some topological space \( X \). Ultimately the goal is to use this lattice, which we will call \( L(X) \), to characterize \( \beta X \setminus X \).

Notation 2.1. Suppose \( X \) is a topological space and \( F \) is a closed subset of \( X \). Then we define:

\[
[F] = \{G \subseteq X : G \text{ closed and } cl_X(F \Delta G) \text{ is compact}\}.
\]

Using the above defined equivalence class, we can construct a lattice:

Definition 2.2. For a topological space \( X \), let \( L(X) = \{[F] : F \text{ is closed in } X\} \), with the operations \( \land \) and \( \lor \) defined by \([F_1] \land [F_2] = [F_1 \cap F_2]\) and \([F_1] \lor [F_2] = [F_1 \lor F_2]\).

In the what follows we shall give a partial proof of the fact that \( L(X) \) is a lattice.

Proposition 2.3. The operations \( \land \) and \( \lor \) defined above are well defined.

Proof. We’ll just show this for \( \lor \) as the proof that \( \land \) is well defined in nearly identical. Suppose that \([F_1] \lor [F_2] = [F]\) and \([F_1] \lor [F_2] = [G]\) (i.e. \( cl_X(F \Delta (F_1 \cup F_2)) \) and \( cl_X(G \Delta (F_1 \cup F_2)) \) are compact). We’d like to show that \( cl_X(F \Delta G) \) is compact. We’ll start by observing that \( F \Delta G = (F \setminus G) \cup (G \setminus F) \subseteq (F \setminus (F_1 \cup F_2)) \cup (F_1 \cup F_2) \setminus F \cup (G \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus G) \). So \( cl_X((F \setminus G) \cup (G \setminus F)) \subseteq cl_X((F \setminus (F_1 \cup F_2)) \cup (F_1 \cup F_2) \setminus F) \cup (G \setminus (F_1 \cup F_2)) \cup ((F_1 \cup F_2) \setminus G) \). Therefore \( cl_X(F \Delta (F_1 \cup F_2)) \cup cl_X(G \Delta (F_1 \cup F_2)) = cl_X(F \Delta (F_1 \cup F_2)) \).
Since the union of two compact sets is compact, then \( Cl_X(F \Delta G) \) is a closed subset of a compact set, and therefore compact. So \([F] = [G]\).

**Proposition 2.4.** \( L(X) \) with the given operations is a bounded distributive lattice.

**Proof.** The non-trivial part is stated below:
\[ [F] < [G] \] is equivalent to \([F] \lor [G] = [G] \) (or \([F] \land [G] = [F] \)). The lattice bounded, as \([X] \) and \([\phi] \) are the 1 and 0 elements respectively. \(\square\)

From this point on, unless otherwise stated, we will assume that our topological spaces are locally compact. In what follows, we will show that for any subset of a compact set, and therefore compact. So \( F \)

**Notation 2.5.** Let \( A \) be a subset of \( X \). Then the notation \( A^* \) means \((Cl_{\beta X}(A))\setminus X\).

**Lemma 2.6.** (i) \((F \cup G)^* = F^* \cup G^* \) (ii) \((F \cap G)^* = f^* \cap G^* \).

**Proof.** We prove only union case. The intersection will follow similarly.

There are two parts:

\((\supseteq)\) Let \( x \in F^* \cup G^* \). Then \( x \in f^* \) or \( x \in G^* \). Let \( U \) be an open neighborhood (in \( \beta X \)) of \( x \). Then \( U \) intersects \( F \) or \( U \) intersects \( G \). Hence \( U \cap (F \cup G) \neq \phi \). So \( x \in Cl_{\beta X}(F \cup G) \) and \( x \notin X \). So \( x \in (F \cup G)^* \).

\((\subseteq)\) Let \( x \in \beta X \setminus X \) s.t. \( x \notin (F^* \cup G^*) \) (i.e. \( x \notin F^* \) and \( x \notin G^* \)). So \( \exists \) an open (in \( \beta X \)) neighborhood \( V_1 \) of \( x \) s.t. \( V_1 \cap F = \phi \) and \( \exists \) an open (in \( \beta X \)) neighborhood \( V_2 \) of \( x \) s.t. \( V_2 \cap G = \phi \). Let \( U = V_1 \setminus V_2 \). Then \( U \) is an open neighborhood (in \( \beta X \)) of \( x \) s.t. \( U \cap (F \cup G) = \phi \).

Therefore \( x \notin (F \cup G)^* \). \(\square\)

**Proposition 2.7.** \( F = \{ F^* : F \text{ closed in } X \} \) is a base for the closed sets of \( \beta X \setminus X \).

**Proof.** Let \( C \) be a closed set in \( \beta X \setminus X \) and let \( x \in (\beta X \setminus X) \setminus C \). Since \( X \) is locally compact, \( C \) is closed in \( \beta X \). Let \( V \) and \( W \) be open sets in \( \beta X \) s.t. \( x \in V \cap W \), \( Cl_{\beta X}(V) \subseteq W \), and \( W \cap C = \phi \). Note that \( V' = V \cap X \) is open in \( X \).

Let \( F = X \setminus V' \). Clearly \( F \) is closed in \( X \). Since \( \beta X \setminus X = X^* = (V' \cup F)^* = (V')^* \cup F^* \) and \( (V')^* \cap C = \phi \) (because \( (V')^* \subseteq (Cl_{\beta X}(V') \setminus B) \subseteq W \)), then \( C \subseteq F^* \). Additionally, \( \beta X \setminus V \) is a closed set in \( \beta X \) containing \( F \Rightarrow Cl_{\beta X}(F) \subseteq \beta x \setminus V \Rightarrow F^* \subseteq \beta X \setminus V \).
Since $x \in V$, then $x \not\in F^*$. Therefore $\mathcal{F}$ is a base for the closed sets of $\beta X \setminus X$. \hfill \qed

**Notation 2.8.** Let $\mathcal{F}_p = \{F^* : [F] \in p\}$.

**Proposition 2.9.** For any $p \in St(L(X))$, $| \cap \mathcal{F}_p | = 1$.

**Proof.** We'll present the proof in two parts:

1. For any $p \in St(L(X))$, $\cap \mathcal{F}_p \neq \emptyset$:
   Let $p \in St(L(X))$ suppose $F_1^*, F_2^*, \ldots, F_n^* \in \mathcal{F}_p$. Then $[F_1], [F_2] \cap \cdots \cap [F_n] \in p$. Hence $F_1^* \cap F_2^* \cap \cdots \cap F_n^* = (F_1 \cap F_2 \cap \cdots \cap F_n)^* \in \mathcal{F}_p$. Therefore $\mathcal{F}_p$ is a family of closed sets with the finite intersection property. Since $X$ is locally compact, $\beta X \setminus X$ is closed ([2]) and thus compact, so $\cap \mathcal{F}_p \neq \emptyset$.

2. For any $p \in St(L(X))$, $\cap \mathcal{F}_p$ contains only one point:
   Let $p \in St(L(X))$ suppose $\exists r, s \in \cap \mathcal{F}_p$ with $r \neq s$. Then $\exists$ some basic closed set $G^*$ (with $G$ closed in $X$) with $r \in G^*$ and $s \not\in G^*$. Then $[G] \not\in p$ because $s \not\in G^*$. Since $p$ is maximal, $\exists [F] \in p$ s.t. $[\phi] = [F] \land [G] = [F \cap G]$. Hence $F \cap G$ is compact so $\phi = (F \cap G)^* = F^* \cap G^* \supseteq \{r\}$, a contradiction. \hfill \qed

We are now ready to define our homeomorphism candidate.

**Definition 2.10.** Let $h : St(L(X)) \to \beta X \setminus X$ be defined by $h(p) = r_p$, where $r_p$ is the unique point in $\cap \mathcal{F}_p$.

By proposition 3.9, it is clear that $h$ is well defined. That it is a homeomorphism comes from the result that follow.

**Proposition 2.11.** The function $h : St(L(X)) \to \beta X \setminus X$ as defined above is 1-1.

**Proof.** Suppose $p \neq q$, and that $h(p) = r_p, h(q) = r_q$ (show that $r_p \neq r_q$). Suppose that $r_q \in F^*, \forall [F] \in p$. Let $[G] \in q$. Then $r_q \in F^* \cap G^* = (F \cap G)^*, \forall [F] \in p$.

This implies that $[\phi] \neq [F \cap G] = [F] \land [G], \forall [F] \in p$. Since $P$ is maximal, $[G] \in$, Hence $q \subseteq p$. Since $q$ is maximal $p = q$ a contradiction. Therefore $\exists [F] \in p$ s.t. $r_q \not\in F^*$. So $r_q \neq h(p) = r_p$. So $h$ is 1-1. \hfill \qed

**Proposition 2.12.** $h$ is onto.

**Proof.** Let $x \in \beta X \setminus X$. Let $\mathcal{G} = \{G^* : G$ is closed in $X$ and $x \in G^*\}$.

Let $p = \{[F] : F^* \in G\}$.

Show that $p \in St(L(X))$:

1. $p$ is a filter:
   (i) Since $x \not\in \phi = \phi^*$, then $\phi^* \not\in \mathcal{G}$ and $[\phi] \not\in p$. 

(ii) Suppose \([F], [G] \in p\). So \(x \in F^* \cap G^* = (F \cap G)^*\) since \(F \cap G\) is closed, \((F \cap G)^* \in \mathcal{G}\). Thus \([F \cap G] \in p\). But \([F \cap G] = [F] \land [G]\).

(iii) Suppose \([F] \in p\) and \([G] \in L(X)\) such that \([F] < [G]\). So \([F] = [F] \land [G] = [F \cap G]\). So \(\text{Cl}_X(F \Delta (F \cap G))\) is compact \(\Rightarrow \text{Cl}_X((F \setminus (F \cap G)) \cup (\Delta(F \cap G) \setminus F))\) is compact \(\Rightarrow \text{Cl}_X((F \setminus G) \cup \phi)\) is compact \(\Rightarrow \text{cl}(F \setminus G)\) is compact \(\Rightarrow (F \setminus G)^* = \phi\). Thus \(F^* = ((F \setminus G) \cup (F \cap G))^* = (F \setminus G)^* \cup (F \cap G)^* = (F \cap G)^* = F^* \cap G^*\). Therefore \(F^* \subseteq G^* \Rightarrow x \in G^* \Rightarrow [G] \in p\).

2. \(p\) is maximal:

Suppose \([G] \land [F] \neq \phi, \forall [F] \in p\) (show \([G] \in p\)). Since \([\phi] \neq [G] \land [F] = [G \land F]\), then \(\forall [F] \in p, G \cap F\) is not compact. Hence \(\phi = (G \cap F)^* = G^* \cap F^*, \forall [F] \in p\). Suppose that \(x \notin G\). Then \(\exists J\) a basic closed set of the form \(J^*\) (with \(J\) closed in \(X\)) s.t. \(x \in J^*\) and \(J^* \cap G^* = \phi\) (because \(\beta X \setminus X\) is regular). But \(x \in J^* \Rightarrow [J] \in p\), which contradict that \(G^* \cap F^* \neq \phi, \forall [F] \in p\). Hence \(x \in G^*\), i.e. \([G] \in p\). Thus \(p\) is maximal.

Therefore by 1. and 2. \(p \in \text{St}(L(X))\) and \(h(p) = x\), so \(h\) is onto.

\[\square\]

**Notation 2.13.** In the following proof \([[[F]]]\) will be used for the set of ultrafilters on \(L(X)\) that contain \([F]\) (i.e. \([[[F]]] = \{p \in \text{St}(L(X)) : [F] \in p\}\)). As discussed before, the \([[[F]]]\)'s form a base for closed sets of \(\text{St}(L(X))\).

**Proposition 2.14.** \(h\) is continuous.

**Proof.** Let \(F^*\) be a basic closed set in \(\beta X \setminus X\). The following will show that \(h^{-1}(F^*) = [[[F]]]: (\subseteq)\) let \(p \in h^{-1}(F^*)\) (so \(h(p) \in F^*\)) and suppose that \([F] \notin p\). Then \(\exists [G] \in p\) s.t. \([\phi] = [F] \land [G] = [F \cap G]\) (since \(p\) is maximal). So \(\text{Cl}_X(F \cap G)\) is compact. Hence \(\phi = (F \cap G)^* = F^* \cap G^*\), which contradicts that \(h(p) \in F^* \cap G^*\). Hence \([F] \in p\), so \(p \in [[[F]]]\).

\((\supseteq)\) let \(p \in [[[F]]]\). Then \([F] \in P \Rightarrow h(p) \in F^*\) (by definition of \(h\)) \(\Rightarrow p \in h^{-1}(F^*)\). Therefore, \(h^{-1}(F^*) = [[[F]]\) and since \([[F]]\) is closed in \(\text{St}(L(X))\), \(h\) is continuous.

Now since any bijective continuous function from a compact space to a Hausdorff space is a homeomorphism we have the following result.

**Main Theorem 3.16.** \(\text{St}(L(X)) \cong \beta X \setminus X\).

4. Example \(L(H)\)

Let \(H\) be the halfline \([0, \infty)\) with the usual topology. We now present a couple of facts concerning the associated lattice \(L(H)\).

**Proposition 4.1.** \(L(H)\) is normal.
Proof. This proof is a little long and interested reader may ask this proof from the authors. □

Proposition 4.2. $L(H)$ is not pseudocomplemented.

Proof. Look at the closed set $C \subseteq H$ given by $C = [1, 2] \cup [3, 4] \cup [5, 6] \cup \cdots = \bigcup [2i - 1, 2i], \forall i$. Clearly $C$ is not compact, hence $[C] \neq [\phi]$ suppose that $[F]$ is the pseudocomplement of $[C]$. Then $C \cap F$ is compact since $C \cap F \subseteq H$, then $C \cap F$ is bounded, call the upper bound $M(\in H)$. Let $G = F \cup [a, a + 1] \cup S$, where $a$ is some odd integer greater than $M$ and $S$ is defined in the following way:

For each $[2i - 1, 2i] \subseteq C$ with $i > (a + 1)/2$, we know that $2i \not\in F$. So look at the sequence $((2i)_n)$ defined by $(2i)_n = 2i + (1/2n)$. Since $(2i)_n \to 2i$, then $\exists$ an $m_{2i} \in \mathbb{N}$ s.t. $(2i)_{m_{2i}} \not\in F$. Let $S = \{(2i)_{m_{2i}} : i > (a + 1)/2\}$. Notice that for any $n \in \mathbb{N}$, $2i < 2i + (1/2n) < 2i + 1$, so $S \cap C = \emptyset$. Since $S$ is closed, unbounded and $S \cap F = \emptyset$, then $\text{Cl}_H(G\setminus F) = G\setminus F$ is not compact. This implies that $[G] \neq [F]$ since $G \supseteq F$ then $[G] > [F]$. But by construction $G \cap C = (F \cup [a, a + 1] \cup S) \cup [a, a + 1]$ which is compact. Thus $[G] \cap [C] = [\phi]$, which contradicts that $[F]$ is pseudocomplement of $[C]$. Therefore $C$ has no pseudocomplement, and thus $L(H)$ is not pseudocomplemented. □

References


