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SOME QUATERNION GEOMETRIC INTUITION

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Abstract: The utility of complex analysis in providing a rich family of holomorphic functions has motivated the extension of the planar theory based on complex numbers to a 4-space study with functions of a quaternion variable. A function of a quaternion variable is a function with domain and range in the quaternion. Such functions naturally appear when one considers the projection of a quaternion onto its scalar part or onto its vector part, as well as the modulus and versor functions. A very useful function of quaternion variable is the function $f(v) = qvq^{-1}$, which rotates the vector part of a quaternion v by twice the angle of a quaternion q. In this paper, we present a brief survey of some interesting discrete properties of $f(v)$ as well as propose its utility in the description of pythagorean triples. Representing the quaternion in matrix form, along with a microscopic modification of our results in this paper, resolves an open problem by Mircea Crasmareanu [3] concerning a connection between the pythagorean triple preserving matrix and the algebra of quaternions.

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1. Introduction

It is well known that quaternions play fundamental roles in quantum physics and computer graphics. Nowadays, quaternions are used by computer program-

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mers to perform rotations in three space because of their obvious advantages over the “good old” Euler angles (see [1], [5]). In mathematics, quaternions appear as a noncommutative division algebra or skew field. Quaternions are best described as a non-commutative extension of the complex numbers. While complex numbers are obtained by adding the element i to the real numbers, where $i^2 = -1$, quaternions are obtained by adding further elements j and k, to the complex numbers, such that

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

After setting up the basic notation in the rest of this section, we present in Section 2, some well known properties of the quaternions in light of our discussion here. In Section 3, we demonstrate how quaternions give rise to rotations and study the useful functions induced by quaternion variables. In Section 4, we present some concepts not so widely considered in dealing with quaternions; for instance, we introduce a similarity relation ‘$\sim$’ induced by the function of quaternion variables.[1] Also in this section, we prepare a general geometric intuition for a more focused utility in generating pythagorean angles (see [6]) in Section 5. Using this as a springboard, we propose a new way of generating the pythagorean triples in Section 5. If we put the quaternion in matrix form, our results here produce an answer to an open problem by Mircea Crasmareanu [3] concerning a connection between the pythagorean triple preserving matrix and the algebra of quaternions.

Recall that as it is with complex numbers, addition of quaternions is accomplished by adding corresponding coefficients and by linearity, multiplication is completely determined by the following multiplication rules for the basis quaternions: $ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$. These equations follow from equation (1), which is the fundamental formula for quaternion multiplication. Under this multiplication, the basis quaternions together with their negatives, form the so called quaternion group $\mathbf{Q_8}$ of order 8. If we regard a quaternion as the sum of a scalar and a vector, namely $q = q_0 + q_1 i + q_2 j + q_3 k$, then the scalar part of the quaternion is q_0, while the remaining part $q_1 i + q_2 j + q_3 k$ is the vector part. So a vector may be conceived as a quaternion that has zero for the scalar part. We may also adopt the notation $q = 1 \cdot q_1 + i \cdot q_2 + j \cdot q_3 + k \cdot q_3 = [q_1[q_2, q_3, q_4]] = [q_r, q_v] = [q_1, q_2, q_3, q_4]$. In other words, a quaternion may be viewed as a four dimensional vector space $\mathbf{Q}$ over the reals $\mathbb{R}$, with ordered basis $1, i, j, k$. We write q_r for the real quaternion, that is if the vector part is zero, and write q_v for the a vector quaternion, that is if the real part is zero.

[1]with this, one can conclude that the quaternion functions that induce the same pythagorean angles considered in Section 5 are similar.

2. Properties

As has already been stated in Section 1, addition of quaternions is performed component-wise; that is given two quaternions p and q,

$$p + q = (p_1, p_2, p_3, p_4) + (q_1, q_2, q_3, q_4) = (p_1 + q_1, p_2 + q_2, p_3 + q_3, p_4 + q_4).$$

Obviously, addition of quaternions is associative and commutative, as one can easily check. Multiplication of quaternions is performed as follows:

$$p \cdot q = [p_r, p_v] \cdot [q_r, q_v] = [p_r q_r - p_v q_v, p_r q_v + q_r p_v + p_v \times q_v] \tag{2}$$

It is a folklore to check that quaternion multiplication is not commutative; however, it is distributive over addition, as well as associative. Note that in the case where p and q are vector quaternions, equation (2) reduces to $p \cdot q = [-p_v q_v, p_v \times q_v]$, in which case, the following two special cases emerge: (i) when p and q are parallel; we have $p \cdot q = [-p_v q_v, 0]$; and (ii) when p and q are perpendicular; we have $p \cdot q = [0, p_v \times q_v]$. The conjugate of $q = q_0 + q_1 i + q_2 j + q_3 k$, is denoted by q^* and defined as $q^* = q_0 - q_1 i - q_2 j - q_3 k = [q_r, -q_v]$. The absolute value (or modulus or norm) of q is denoted by $|q|$ and defined as $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ The inverse of q is computed as $q^{-1} = \frac{q^*}{|q|^2}$. When $|q| = 1$, then q is called a unit quaternion, and for unit quaternions one has that $q^{-1} = q^*$. In modern mathematics language, the quaternions are examples of a division algebra and by using the distance function $d(p, q) = |p - q|$, the quaternions form a metric space isometric to the usual euclidean metric on $\mathbb{R}^4$, with the arithmetic properties continuous. Moreover, because for every quaternion p and q we have that $|pq| = |p||q|$, the quaternion is a real Banach algebra.

3. Geometric Rotations

As is well known, a unit quaternion q is used in rotating vectors $\overline{v}$ or objects in three space ([1], [9]). This is achieved by setting $q = \cos\theta + \overline{u}\sin\theta$, and with appropriate application, gives a rotation of $\overline{v}$ by an angle $2(\theta)$ about the three dimensional axis $\overline{u}$. The rotated vector is

$$v' := f(\overline{v}) := q\overline{v}q^*. \tag{3}$$

More precisely, we have:

Theorem 3.1. *If $q := \cos\theta + \overline{u}\sin\theta$, then the function in equation (3) is the result of a rotation of a vector $\overline{v}$ by an angle $2(\theta)$ about the three dimensional axis $\overline{u}$.*

Proof. To demonstrate this, we show that $f(\overline{v})$ is an orthonormal transformation without a reflection component. In other words, we show first: that $f(\overline{v})$ is a three dimensioanl vector; that is we show that $W(f(\overline{v})) = 0$, where $W(q) = \frac{q+q^*}{2} = q_r$. But a simple verification shows this is true. Next, we show that it is length preserving; this follows because a check shows that $|f(\overline{v})| = |\overline{v}|$. Finally, we show that $f(v)$ is a linear transformation; this also follows, since for any real scalar x and three dimensional vectors v and w, we have $f(x\overline{v} + \overline{w}) = xf(\overline{v}) + f(\overline{w})$.

The rotation axis is the three dimensional unit vector $\overline{u}$, since $f(\overline{u}) = \overline{u}$; and to show that the rotation angle is $2(\theta)$, suppose $\overline{u}, \overline{v}$, and $\overline{w}$ are a right-handed set of orthonomal vectors, where $\overline{v}$ is rotated by an angle θ to the vector $q\overline{u}q^*$. But by simple verification, we have that $\cos(\theta) = \overline{v} \cdot (q\overline{v}q^*) = \cos(2\theta)$. □

4. Some Geometric Intuition

As a general motivation towards subsequent discussion in Section 5, we start this section by giving an elementary bird's eye view of a few low dimensional plane curves. Recall that plane curves of degree 1 are called lines. They are defined by equations of the form $ax + by + c = 0$, where a, b are not both zero. Plane curves of degree 2 are called conics. These curves are of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ for some $a, b, c, d, e, f \in \mathbb{R}$. The conics include ellipses, some special case of circles, parabolas, hyperbolas, and some "degenerate" cases such as[2] $xy = 0$, $x^2 - 1 = 0$, or $x^2 = 0$. cubic curves of the form $ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + ky + l = 0$, where $a, b, c, d, e, fg, h, k, l \in \mathbb{R}$. Examples of cubic curves include elliptic curves; that is, curves defined by equations of the form $y^2 = f(x)$, where $f(x)$ is a square free polynomial of degree 3. Recall that by mere scaling of the coordinates and translating, any elliptic curve can be reformed to the form $y^2 = x^3 + Ax + B$, where $A, B \in \mathbb{R}$. Therefore a general curve of the form $y^2 = x^3 + Ax + B$ is an elliptic curve[3] if and only if $-(4A^3 + 27B^2) = 0$. Finding the connections between these curves and the quaternion algebra will be discussed in our forth

[2]most literatures usually exclude some or all of the last three examples from the definition of a conics.

[3]note that $x^3 + Ax + B$ is square free

coming paper. Now, recall that a pythagorean triple is an ordered triple (x, y, z) of positive integers satisfying the equation

$$x^2 + y^2 = z^2. \tag{4}$$

Therefore a pythagorean triple (x, y, z) may be regarded as a coordinate of a point $X \in \mathbb{Z}^3$ on a quadric $x^2 + y^2 - z^2 = 0$. This centuries old concept has been studied by both amateur and professional mathematicians since ancient times (see [8]). A pythagorean triple is said to be primitive if $gcd(x, y, z) = 1$. Therefore every pythagorean triple is a multiple of some primitive pythagorean triples. Euclids provided a formula for finding pythagorean triples from any two positive integers m and n, $m > n$, namely

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2. \tag{5}$$

It is an easy exercise to verify that x, y, z defined in (5) readily satisfy equation (4). We now propose a new method of generating the pythagorean triples in the context of the function of quaternion variables. Before we do this, recall that real numbers, vectors and complex numbers may be regarded as quaternions in the natural way. Furthermore, the pair $(m, n) \in \mathbb{Z}^2$ form a spinor description of the pythagorean triples.[4]

Definition 4.1. We say that two quaternions p and q are similar (written as $p \sim q$) if there is a nonzero quaternion w such that $p = wqw^{-1}$.

Lemma 4.2. *[2] Let $q = q_0 + q_1 i + q_2 j + q_3 k$, and $p = p_0 + p_1 i + p_2 j + p_3 k$ be quaternions. Then $q \sim p$ if $q_r = p_r$ and $|q_v| = |p_v|$.*

Rational points on a circle $\mathbf{C}$ are closely related to Eulid's parametrization of pythagorean triples because if x, y, and z are any integers satisfying (4) then $(\frac{x}{z}, \frac{y}{z})$ will be a rational point on $\mathbf{C}$. Conversely, if (a, b) is a rational point on $\mathbf{C}$, then by choosing a common denominator for a and b, we can write $a = \frac{x}{z}$ and $b = \frac{y}{z}$ for some integers x, y, z with $z \neq 0$, and the relation $a^2 + b^2 = 1$ implies $x^2 + y^2 = z^2$. Moreover, if a and b are nonzero, then x, y, z will all be nonzero, and $(|x|, |y|, |z|)$ will be a pythagorean triple. We now find a description using the quaternion rotation function described before.

5. Application to Pythagorean Triples

We begin this section by first getting a needed terminology out of the way.

[4]equation(5) may also be conceived as the square of an integer complex number

Definition 5.1. We say that an angle θ is pythagorean if $\cos\theta$ and $\sin\theta$ are rational.

Without loss of generality, we shall consider θ to sit on the range $0 \leq \theta \leq \frac{\pi}{2}$; in which case one has that θ is a pythagorean angle if and only if $\theta' = \frac{\pi}{2} - \theta$ is a pythagorean angle. Pythagorean angles are closely related to pythagorean triples in the sense that if $v = (\frac{x}{z}, \frac{y}{z}, 1) = (\sin\theta, \cos\theta, 1)$ for some θ (where $0 \leq \theta \leq \frac{\pi}{4}$) and[5] $q = \cos\theta + \overline{u}\sin\theta$, where $\overline{u} = (0, 0, 1)$, then it is easy to verify that the function[6] $f(v)$ only changes the first two components of v leaving the third component fixed. This suggests that there is no harm in working with only the first two components; that is considering the projection $\mathbb{Z}^3 \to \mathbb{Z}^2$.

Proposition 5.2. *Let θ be an angle. Then θ is a pythagorean angle if and only if there is a vector v in $\mathbb{Z}^2 - \{(0, 0\}$ such that $f(v) = qvq^*$ is in $\mathbb{Z}^2$.*

Proof. Suppose θ is a pythagorean angle. Then there exists an integer k such that $(k\cos\theta, k\sin\theta) \in \mathbb{Z}^2 - \{(0, 0)\}$. Obviously

$$f([k\cos(\theta), k(\sin\theta)]) = [\cos(\theta) + \overline{u}\sin(\theta)][k\cos(\theta), k\sin(\theta)][\cos(\theta) - \overline{u}\sin(\theta)] \in \mathbb{Z}^2.$$

Conversely, suppose $v = (x, y) \in \mathbb{Z}^2 - \{(0, 0)\}$ is such that $f(v) \in \mathbb{Z}^2$. Let $(x', y') = f(v)$. Then $\cos\theta = \frac{xx'+yy'}{x^2+y^2}$ and $\sin\theta = \frac{xy'-yx'}{x^2+y^2}$, as reqiured. □

Two pythagorean angles θ and θ' such that $\theta + \theta' = \frac{\pi}{2}$ can be generated by a pair of positive integers (x, y) such that $y < x$, where the application of equation (3) with θ transforms (x, y) into (y, x), and equation (3) applied to θ' transforms (y, x) into $(-y, x)$. It follows therefore that all the pythagorean angles generated as demonstrated above can be generated by pairs[7] (x, y) such that $gcd(x, y) = 1$ and $x - y$ is odd.[8] We state this in the next result.

Proposition 5.3. *An angle $\theta(0 \leq \theta \leq \frac{\pi}{2})$ is pythagorean if and only if there exists a vector (p, q) in $\mathbb{Z}^2$ such that $gcd(x, y) = 1, x - y$ is odd and either $f(x, y) = (y, x)$ or $f(y, x) = (-y, x)$.*

Proof. Recall that any given triple $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + y^2 = z^2$ and $gcd(x, y, z) = 1$, can be placed in one to one correspondence with pairs $(x, y) \in \mathbb{Z}^2$ such that $gcd(x, y) = 1, x > y$ and either x and y is even. Obviously, one necessarily has that $x > y, gcd(x, y) = 1$ and $x - y$ odd. This proposition is

[5] see theorem 3.1

[6] of equation (3)

[7] note that for each positive integer k, the angles generated by the pair $(kx, , ky)$ are the same as those generated by (x, y).

[8] otherwise the angles generated by the pairs (x, y) are the same as the angles generated by $(\frac{x+y}{2}, \frac{x-y}{2})$.

nothing else than an application of this result for (x, y, z) such that $\cos\theta = \frac{x}{z}$, and $\sin\theta = \frac{y}{z}$, where z is minimal. $\square$

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