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# COMPUTING PERMUTATION ENTROPY OF INTERVAL MAPS

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**Abstract:** The aim of this paper is to compute topological and Shannon permutation entropies in an effective way to be able of making massive computations. We point out some problems coming from the implementation and state some open questions.

**AMS Subject Classification:** 26A18, 37B40, 37B10, 37E05
**Key Words:** permutation entropy, one–dimensional dynamics, computational methods

## 1. Introduction, Main Definitions and Statement of Problems

Recently, permutations have been studied in the setting of one dimensional time series. These studies reveal that permutations can be a useful tool to decide

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whether a time series is generated by an i.i.d. random variable (see [16] or [9]), or to determine structural changes (see [10] or [12]). Additionally, a relationship between time series generated by one dimensional dynamical systems and the dynamical systems itself has been also established (see [4]). The key for the above mentioned relationship is given by entropy, both topological (see [1]) and metric (see [21, Chapter 4]). Let us introduce some basic definitions.

Let $I = [0,1]$ and let $f : I \to I$ be a continuous map. The topological entropy of $f$ is a non–negative number, $h(f)$, which can be taken as a measure of the dynamical complexity of the map $f$. For instance, if $h(f) > 0$, then the map $f$ is chaotic in the sense of Li and Yorke (see [13] or [2, Chapter 4]) and Devaney chaotic maps have topological entropy greater than $\frac{1}{2}\log 2$ (see [14]). Let us recall the definition [1].

Let $\alpha = \{A_1, ..., A_k\}$ be a finite open cover of $I$ (clearly we may assume that $A_i$ are intervals $i = 1, ..., k$) and let $n \in \mathbb{N}$. Denote by $\mathcal{N}(\bigvee_{i=0}^{n-1} f^{-i}\alpha)$ the smallest cardinality of a subcover chosen from the open cover $\bigvee_{i=0}^{n-1} f^{-i}\alpha = \{\bigcap_{i=0}^{n-1} f^{-i}(A_{j_i}) : A_{j_i} \in \alpha\}$. Then the topological entropy of $f$ is the non–negative number

$$h(f) = \sup_{\alpha} \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right).$$

Let $\pi$ be a permutation of length $n$, that is an element of the symmetric group of order $n!$, $S_n$. We say that $\pi$ is admissible (also $f$–admissible) if

$$P_\pi = \{x \in I : f^{\pi(0)}(x) < f^{\pi(1)}(x) < \cdots f^{\pi(n-1)}(x)\} \neq \emptyset.$$

Let $P_n^*$ be the family of all nonempty $P_\pi$ for $\pi \in S_n$ an admissible permutation. For a set $A$ denote by $\#A$ its cardinality. Define

$$h^*(f,n) = \frac{1}{n} \log \#P_n^*,$$

and the topological permutation entropy is given by

$$h^*(f) = \lim_{n\to\infty} h^*(f,n).$$

A map $f : I \to I$ is said to be piecewise monotone if there are points $0 = x_0 < x_1 < ... < x_n = 1$ such that $f|_{(x_{i-1},x_i)}$ is monotone and continuous. The following remarkable result shows the close relationship between topological and permutation entropy.

**Theorem 1.** *[4] If $I$ is a compact interval and $f : I \to I$ a piecewise continuous monotone function. Denote by $h(f)$ the topological entropy of $f$. Then*

$$h^*(f) = h(f).$$

However, topological entropy is a conjugacy invariant which is defined for arbitrary continuous interval maps. So, Misiurewicz (see [17]) constructed an example of a continuous function $f : I \to I$ with zero topological entropy and such that $h^*(f) \geq \log 2$. The key for constructing this example is to consider a map of type $2^\infty$ in the Sharkovsky's order, but having periodic points (of period $2^n$ for some integer $n$) with the maximal number of admissible permutations for periodic orbits.

Here, we consider a more restrictive definition of topological permutation entropy by considering some techniques for time series analysis (see e.g. [19] or [3]). Let $(x_n)_{n=0}^\infty$ be a sequence of real numbers. Let $\mathcal{A}_n \subset \mathcal{S}_n$ be the subset of permutations with the property that for any $\pi \in \mathcal{A}_n$ there is $k \in \mathbb{N}$ such that $x_{k+\pi(1)} < x_{k+\pi(2)} < ... < x_{k+\pi(n)}$ and define the permutation entropy of the sequence as

$$h^*((x_n)_{n=0}^\infty) = \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{A}_n.$$

Note that we can see easily that $h^*((x_n)_{n=0}^\infty) = 0$ if the sequence is periodic or has some monotonicity properties because for any $n \in \mathbb{N}$ the number of elements of $\mathcal{A}_n$ is bounded. It is just a simple observation that $\mathcal{P}_\pi \neq \emptyset$ provided there is $x \in I$ such that its orbit $\mathrm{Orb}(x, f) = (f^n(x))_{n=0}^\infty$ contains $\pi$ as a permutation. Hence, in [8] it is proved that if $f : I \to I$ be continuous and piecewise monotone, then, for any $y \in I$,

$$h^*(\mathrm{Orb}(y, f)) \leq h(f) = \sup_{x\in I} h^*(\mathrm{Orb}(x, f)).$$

However, in practice one could not reach the value $h(f)$ by following the above formula. For instance, one can consider an interval map with positive topological entropy such that almost every orbit is attracted by a periodic orbit (see e.g. [20]). Hence, by using a computer we will get that $h^*(f)$ would be zero and the value of $h(f)$ would not be approximated by this method. This example points out a typical "contradiction" from which topological chaos is not physically observable.

Additionally, the computation of $h^*(\mathrm{Orb}(x, f))$ for a single $x \in I$ depends on two parameters. On one hand we have a sample of the orbit, that is, we have to cut $\mathrm{Orb}(x, f)$ because computers cannot work with an infinite orbit. One might think that the larger number of points from $\mathrm{Orb}(x, f)$ is, the better the estimation we have, but the largest values of the orbit could be affected of round off effects. On the other hand, the permutation length $m$ has to increase, which increases the computation time as well. So, in [8], the question of how to make the above computation more effective was raised. The main aim of this paper is to improve in practice the computations made in [8] for an slightly different definition of permutation entropy as follows.

For an arbitrary continuous interval map $f$ we define

$$h^{\#}(f) = \inf_{X} \sup_{x \in X} h^*(\mathrm{Orb}(x,f)),$$

where the infimum is taken on the subsets $X \subseteq I$ with Lebesgue measure 1. Additionally, for $\pi \in \mathcal{A}_n$ let $p_T(\pi)$ be the relative frequency of $\pi$ in the finite sample orbit $\mathrm{Orb}(x,f,T) = \{x_n\}_{n=0}^{T}$ and define

$$h_S^*(\mathrm{Orb}(x,f,T)) = \limsup_{n\to\infty} \frac{1}{n}\left(-\sum_{\pi} p_T(\pi)\log p_T(\pi)\right),$$

$$h_S^*(\mathrm{Orb}(x,f)) = \limsup_{T\to\infty} h_S(\mathrm{Orb}(x,f,T)),$$

and

$$h_S^{\#}(f) = \inf_{X} \sup_{x \in X} h_S^*(\mathrm{Orb}(x,f)),$$

where again, the infimum is taken over all the measurable subsets of full measure. We are going to investigate whether the relationship between $h_S^{\#}(f)$ and $h^{\#}(f)$. One might expect that they agree for piecewise monotone maps. The motivation for the last statement is given by the variational entropy for topological entropy and the fact that piecewise monotone transitive continuous interval maps have invariant measure with maximal entropy (see [5]). Recall that an invariant measure $\mu$ is a probability measure on the Borel sets of $I$ such that $\mu(A) = \mu(f^{-1}A)$ for any measurable set $A$. The variational principle states that

$$h(f) = \sup_{\mu} h_\mu(f),$$

where $h_\mu(f)$ denotes the metric entropy of $f$ (see e.g. [21, Chapter 4]) for definition). An invariant measure $\mu$ is said to be a maximal entropy measure provided $h(f) = h_\mu(f)$.

Next, we are going to show the implementation for computing the above mentioned quantities $h^{\#}(f)$ and $h_S^{\#}(f)$.

## 2. Computing Results

For testing our computations we choose four one parameter families of continuous maps. Two of them are well–known: the tent and logistic families. Both families are unimodal, with two pieces of monotonicity, as well as the third family, the Puu's model which was stated in an microeconomic model. For unimodal maps it is possible to compute topological entropy with prescribed accuracy (see [7]). However, for the last family, also related with an economic model called Kopel map (see [15]), the above mentioned computations are not possible and hence, our approach is useful to estimate the model complexity.

We followed the same methodology for all the families. We consider truncated orbit whose length changes from 500000 to 5000000 points, that is, we start evaluating orbits with 500000 and increase its length by adding 500000 new points till get 5000000 points. For all the above samples, we compute the number of admissible permutations for permutation length ranging the integers 15, 16,...,25.

Simulations have been programmed in matlab in a supercomputer called Hipatia at Universidad Politécnica de Cartagena. Hipatia is a cluster for high computing performance with Linux RedHat OS of 40 Intel Xeon processors (152 cores in total), 300 Gbytes of RAM and 8.7 terabytes of shared storage. The runing times for each model are summarized in the following table:

| **Model** | **Time** |
|---|---|
| Tent | 128:03:04 |
| Logistic | 64:14:15 |
| Puu | 56:40:15 |
| Kopel | 182:37:52 |

The results are given in what follows.

**Tent family**. The tent family is given by $f(x) = ax$ if $x \in [0, 1/2]$ and $f(x) = -ax + a$ if $x \in [1/2, 1]$, $a \in [1, 2]$. It is well known that its entropy is $\log a$ (see e.g. [2, Chapter 4]). For the tent family the results are summarized in Figures 1 to 7. From Figures 1 and 2 we see how the data size is quite important because not all the permutations appear at the beginning of orbits. On the other hand, the convergence seems to be more slow when the topological entropy is close to zero.

As we can see, permutation entropies increase with the parameter value.

**Logistic family**. We consider the logistic family $f(x) = ax(1 - x)$, where $x \in [0, 1]$ and the parameter value ranges the set $a \in [3.5, 4]$. The results are summarized in Figures 8 to 11. The difference with the tent family is

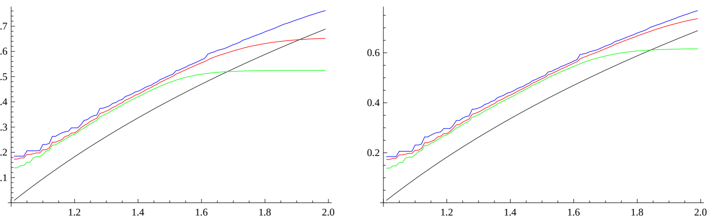


Figure 1: *Computation of* $h^{\#}(f)$*. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(blue),* $20$ *(red) and* $25$ *(green) and the exact value of topological entropy (black). The parameter value* $a$ *ranges the interval* $[1.01, 1.99]$ *with variation of* $0.01$*, and recall that the topological entropy is* $\log a$*. On the right, the same computation is done for* $T = 5000000$*.*

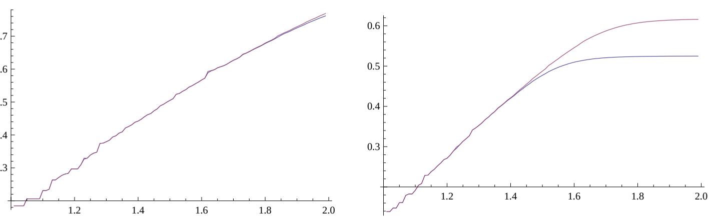


Figure 2: *Computation of* $h^{\#}(f)$*. On the left, we fix* $m = 15$ *and show the results for* $T = 500000$ *and* $5000000$*. The parameter value* $a$ *ranges the interval* $[1.01, 1.99]$ *with variation of* $0.01$*. On the right, the same computation is done for* $m = 25$*. We see that for large enough embedding dimension the size of the data is quite important.*

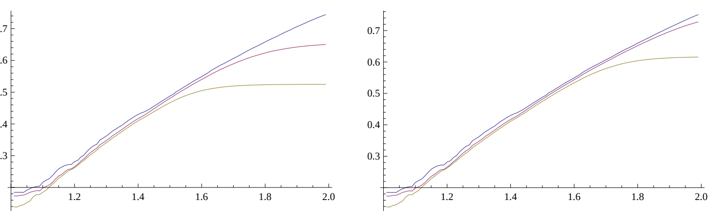


Figure 3: *Computation of* $h^{\#}_{\mathsf{S}}(f)$*. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(cyan),* $20$ *(magenta) and* $25$ *(yellow). The parameter value* $a$ *ranges the interval* $[1.01, 1.99]$ *with variation of* $0.01$*. On the right, the same computation is done for* $T = 5000000$*.*

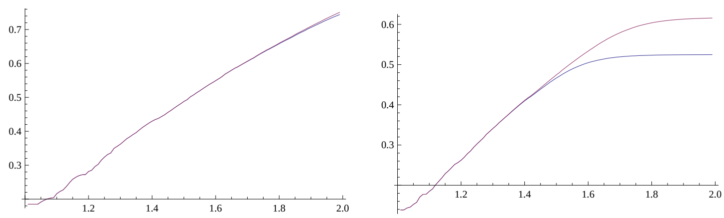


Figure 4: *Computation of* $h^{\#}_{\mathsf{S}}(f)$. *On the left, we fix* $m = 15$ *and show the results for* $T = 500000$ *and* $5000000$. *The parameter value* $a$ *ranges the interval* $[1.01, 1.99]$ *with variation of* $0.01$. *On the right, the same computation is done for* $m = 25$.

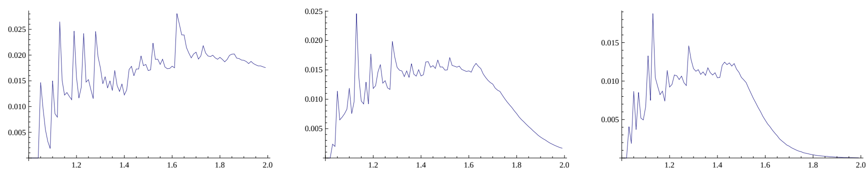


Figure 5: *Computation of* $h^{\#}(f)$ *minus* $h^{\#}_{\mathsf{S}}(f)$. *On the left, we fix* $m = 15$ *and show the results for data size* $T = 500000$. *The same computation is done for* $m = 20$ *(center) and* $25$ *(right).*

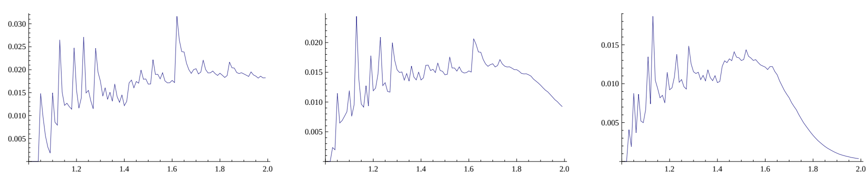


Figure 6: *Computation of* $h^{\#}(f)$ *minus* $h^{\#}_{\mathsf{S}}(f)$. *On the left, we fix* $m = 15$ *and show the results data size* $T = 5000000$. *The same computation is done for* $m = 20$ *(center) and* $25$ *(right).*

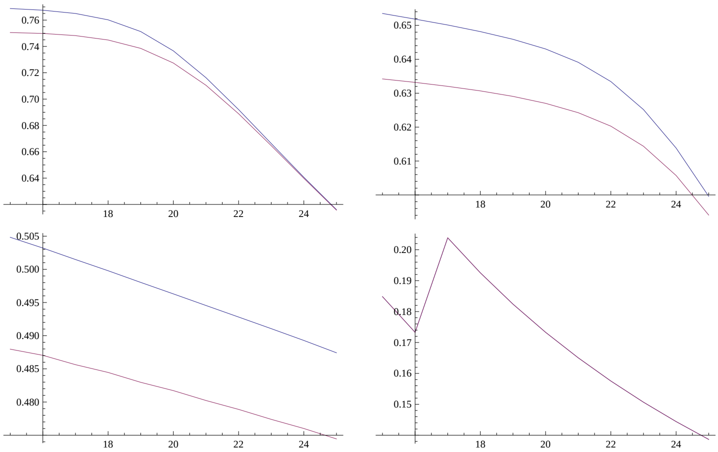


Figure 7: *Computation of* $h^{\#}(f)$ *and* $h^{\#}_{\mathsf{S}}(f)$ *(*$h^{\#}(f) \geq h^{\#}_{\mathsf{S}}(f)$*). On the left and top, we fix* $a = 1.99$ *and show the results for data size* $T = 5000000$*. We change the embedding dimension from* $m = 15$ *to* $25$*. The same computation is done for* $a = 1.75$ *(top and right),* $1.5$ *(bottom and left) and* $1.01$ *(bottom and right).*

given by the fact that permutation entropies are not a monotone family of the parameter value, because there are positive entropy maps such that almost every point is attracted by a periodic orbit. It is well–known that these maps have only one attractor which attracts almost all orbit and hence the estimations of permutation entropies can be done by iterating a single orbit.

**Puu's family**. The Puu family appears in a duopoly model in [18]. The model study can be reduced to analyze a one dimensional map given by $g_a(x) = x - \sqrt{\frac{x}{a}} + \sqrt{\sqrt{\frac{x}{a}} - x}$, $a \in [5.75, 6.25]$, which it is proved to be chaotic when the parameter value is close to 6.18 (see [11]). For the Puu family the results are summarized in Figures 12 to 15. Here we have not guarantee of the existence of a unique attractor (the map need not have negative Schwarzian derivative) and hence we compute the maximal value of permutation entropies was estimated to be the maximum among several orbit computations. Results reveal that permutation entropies are not monotone.

**Kopel's family**. Now, we consider the family of maps $g_a(x) = (f_a \circ f_4)(x)$, where $f_a(x)$ is the logistic family. These combination of logistic maps appear in a duopoly model so called Kopel's duopoly (see [15]). The map $g_a$ is not unimodal and therefore there are not good ways of computing its topological entropy. The results are summarized in Figures 16 to 19. Again we have had

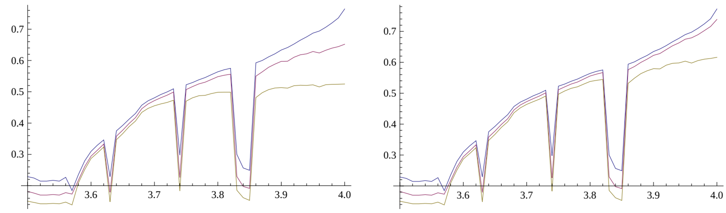


Figure 8: *Computation of* $h^{\#}(f)$ *for logistic family. On the left, we fix* $T =$ 500000 *and show the results for embedding dimension* $m = 15$ *(cyan),* 20 *(magenta) and* 25 *(yellow). The parameter value* $a$ *ranges the interval* $[3.5, 4]$ *with variation of* 0.01*. On the right, the same computation is done for* $T =$ 5000000.

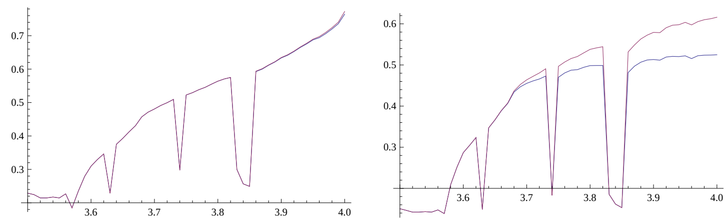


Figure 9: *Computation of* $h^{\#}(f)$ *for logistic family. On the left, we fix* $m = 15$ *and show the results for* $T = 500000$ *and* 5000000*. The parameter value* $a$ *ranges the interval* $[3.5, 4]$ *with variation of* 0.01*. On the right, the same computation is done for* $m = 25$.

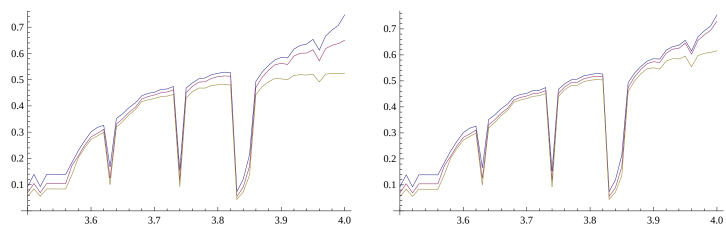


Figure 10: *Computation of* $h^{\#}_{\mathsf{S}}(f)$ *for logistic family. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(cyan),* 20 *(magenta) and* 25 *(yellow). The parameter value* $a$ *ranges the interval* $[3.5, 4]$ *with variation of* 0.01*. On the right, the same computation is done for* $T = 5000000$.

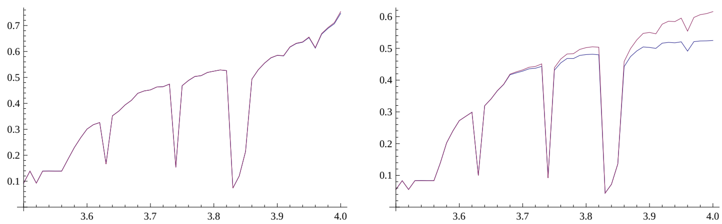


Figure 11: *Computation of* $h_S^{\#}(f)$ *for logistic family. On the left, we fix* $m = 15$ *and show the results for embedding dimension* $T = 500000$ *and* $5000000$. *The parameter value* $a$ *ranges the interval* $[3.5, 4]$ *with variation of* $0.01$. *On the right, the same computation is done for* $m = 25$.

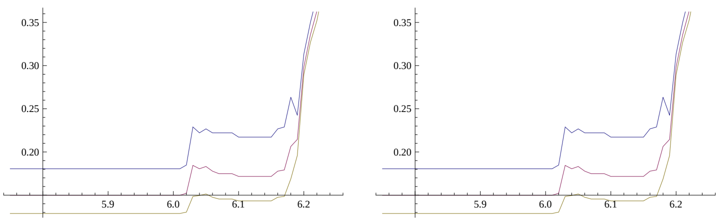


Figure 12: *Computation of* $h^{\#}(f)$ *for Puu's family. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(cyan),* $20$ *(magenta) and* $25$ *(yellow). The parameter value* $a$ *ranges the interval* $[5.75, 6.25]$ *with variation of* $0.01$. *On the right, the same computation is done for* $T = 5000000$.

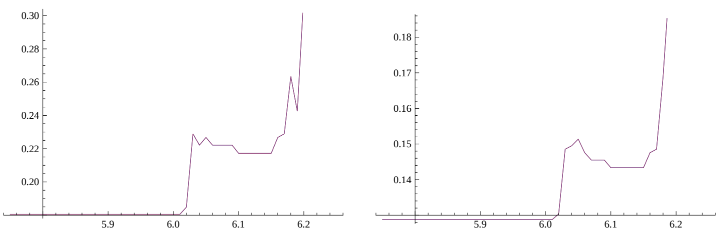


Figure 13: *Computation of* $h^{\#}(f)$ *for Puu's family. On the left, we fix* $m = 15$ *and show the results for* $T = 500000$ *and* $5000000$. *The parameter value* $a$ *ranges the interval* $[5.75, 6.25]$ *with variation of* $0.01$. *On the right, the same computation is done for* $m = 25$.

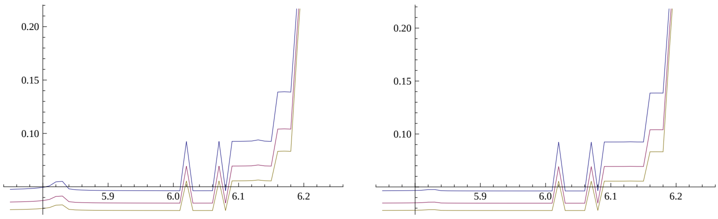


Figure 14: *Computation of* $h^{\#}_{\mathsf{S}}(f)$ *for Puu's family. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(cyan),* $20$ *(magenta) and* $25$ *(yellow). The parameter value* $a$ *ranges the interval* $[5.75, 6.25]$ *with variation of* $0.01$*. On the right, the same computation is done for* $T = 5000000$*.*

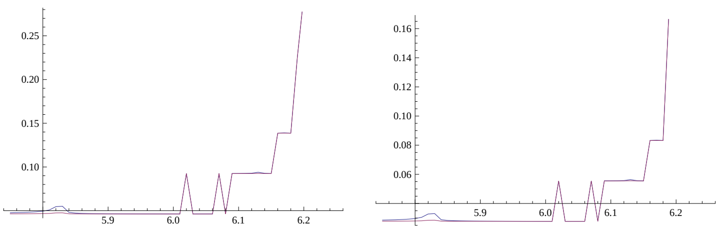


Figure 15: *Computation of* $h^{\#}_{\mathsf{S}}(f)$ *for Puu's family. On the left, we fix* $m = 15$ *and show the results for data length* $T = 500000$ *and* $5000000$*. The parameter value* $a$ *ranges the interval* $[5.75, 6.25]$ *with variation of* $0.01$*. On the right, the same computation is done for* $m = 25$*.*

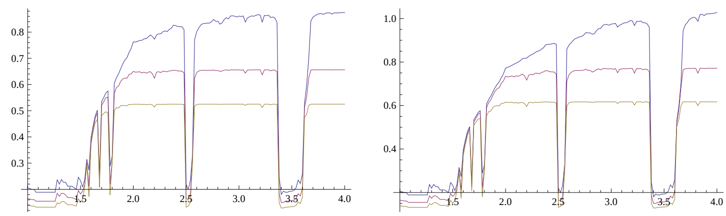

Figure 16: *Computation of* $h^{\#}(f)$ *for Kopel's family. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(cyan),* 20 *(magenta) and* 25 *(yellow). The parameter value* $a$ *ranges the interval* $[1, 4]$ *with variation of* 0.02*. On the right, the same computation is done for* $T = 5000000$*.*

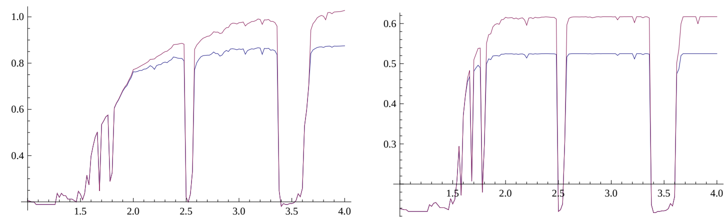

Figure 17: *Computation of* $h^{\#}(f)$ *for Kopel's family. On the left, we fix* $m = 15$ *and show the results for data length* $T = 500000$ *and* 5000000*. The parameter value* $a$ *ranges the interval* $[1, 4]$ *with variation of* 0.02*. On the right, the same computation is done for* $m = 25$*.*

to test the values for several orbits because the maps can have two attractors.

In order to check how good is the convergence of permutation entropy to topological entropy in the case of the Logistic map we have run our program for orbits of 100 millions points with the logistic parameter running from 3.5 to 4 with steps of 0.01 and an fixed embedding dimension of 30. The programm took 290:24:49 to finish the computation in the supercomputer Hipatia. Results are showed in Figure 20.

## 3. Conclusion and Open Problems

We have shown how permutation entropies can be computed in an effective way and can be a useful tool to measure the dynamical complexity. Permutation entropies are easy to program in a computer and our code is good enough to

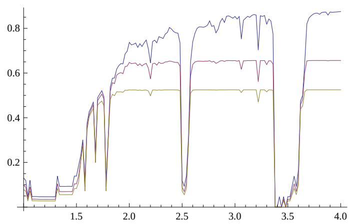


Figure 18: *Computation of* $h^{\#}_{\mathsf{S}}(f)$ *for Kopel's family. On the left, we fix* $T = 500000$ *and show the results for embedding dimension* $m = 15$ *(cyan),* $20$ *(magenta) and* $25$ *(yellow). The parameter value* $a$ *ranges the interval* $[1, 4]$ *with variation of* $0.02$*. On the right, the same computation is done for* $T = 5000000$.

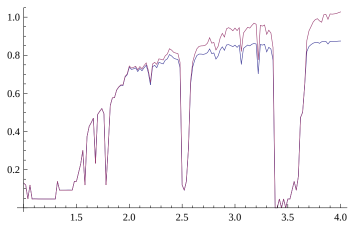


Figure 19: *Computation of* $h^{\#}_{\mathsf{S}}(f)$ *for Kopel's family. On the left, we fix* $m = 15$ *and show the results for* $T = 500000$ *and* $5000000$*. The parameter value* $a$ *ranges the interval* $[1, 4]$ *with variation of* $0.02$*. On the right, the same computation is done for* $m = 25$.

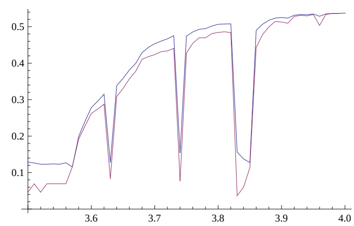


Figure 20: *Computation of* $h^{\#}_{\mathsf{S}}(f)$ *and* $h^{\#}(f)$ *for logistic family with embedding dimension* $m = 30$ *and* $T = 10^8$*. The parameter value* $a$ *ranges the interval* $[3.5, 4]$ *with variation of* $0.01$.

compute the number of permutations for quite long orbits. However, there are some improvements which are needed. The first one is convergence, which is slow. This is important to decide whether a map have positive permutation entropies, impling that topological entropy is positive and therefore some chaotic properties. Of course, we may have maps with zero permutation entropies but positive topological entropy, and therefore it is interesting to check whether the equality

$$h(f) = \sup_{x \in I} h^*(\mathrm{Orb}(x, f))$$

holds for any continuous interval map (recall Misiurewicz's example [17]). Since the structure of $\omega$–limit sets of infinite zero entropy maps is the same for both, piecewise monotone and non piecewise monotone, we may conclude that $\sup_{x \in I} h^*(\mathrm{Orb}(x, f)) = 0$ will imply that $h(f) = 0$. On the other hand, since positive topological entropy maps have horseshoes (see e.g. [2, Chapter 4]), this imply that $h(f) > 0$ will imply that $\sup_{x \in I} h^*(\mathrm{Orb}(x, f)) > 0$.

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