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HYBRID ITERATIVE METHOD FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES IN HILBERT SPACES WITH APPLICATIONS

S. Imnang

Department of Mathematics and Statistics
Faculty of Science
Thaksin University
Phatthalung Campus, Phatthalung, 93110, THAILAND
and
Centre of Excellence in Mathematics, CHE
Si Ayutthaya Road, Bangkok 10400, THAILAND

Abstract: The purpose of this paper is to investigate the problem of finding a common element of the set of solutions of a general system of variational inequality problems for α-inverse-strongly monotone mappings, the set of solutions of mixed equilibrium problems and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Furthermore, we apply our main result with the problem of approximating a zero of a finite family of maximal monotone mappings in Hilbert spaces. Our main result extends and improves the recent results of Ceng, Wang and Yao [1] and many others.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and C be a nonempty

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closed convex subset of H. A mapping $T : C \to C$ is said to be *nonexpansive mapping* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The fixed point set of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

A mapping $A : C \to H$ is called *α-inverse-strongly monotone*, if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that every α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous.

For a given nonlinear operator $A : C \to H$, we consider the following variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \ge 0, \quad \forall x \in C. \tag{1.1}$$

The set of solutions of the variational inequality (1.1) is denoted by $VI(C, A)$. Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied and continued in the literature, see, Piri [11], Qin et al. [12], Shehu [13], Wangkeeree and Preechasilp [17], Yao et al. [19], Yao et al. [21] and relevant references cited therein.

Next, we focus on a general system of variational inequality problems [in short, GSVI] which is considered by Ceng et al. [1]: find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \ge 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where $A, B : C \to H$ are two nonlinear mappings, $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, then GSVI (1.2) reduces to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \ge 0, & \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.3}$$

which is defined by Verma [15], and is called the new system of variational inequalities. Further, if we add up the requirement that $x^* = y^*$, then problem (1.3) reduces to the classical variational inequality $VI(C, A)$. Ceng et al. [1] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of GSVI (1.2) for the α and β-inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Some related works, we refer to see [2, 4, 7, 8, 16, 20].

Recently, in 2012, Ceng et al. [2] considered an iterative method for the system of GSVI (1.2) and obtained a strong convergence theorem for the two different systems of GSVI (1.2) and the set of fixed points of a strict pseudo-contraction mapping in a real Hilbert space.

Let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper extended real-valued function and F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [3] considered the following mixed equilibrium problem (in short, MEP):

$$\text{Find } x \in C \text{ such that } F(x,y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.4}$$

The set of solution of MEP (1.4) is denoted by $MEP(F,\varphi)$. It is easy to see that x is a solution of MEP (1.4) implies that $x \in \mathrm{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$.

If $\varphi = 0$, then the MEP (1.4) becomes the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solution of (1.5) is denoted by $EP(F)$.

If $F = 0$, then the MEP (1.4) reduces to the convex minimization problem:

$$\text{Find } x \in C \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C.$$

If $\varphi = 0$ and $F(x,y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, where A is a mapping from C into H, then MEP (1.4) reduces to the classical variational inequality and $EP(F) = VI(C,A)$. For solving problem MEP (1.4), Ceng and Yao [3] introduced a hybrid iterative scheme for finding a common element of the set $MEP(F,\varphi)$ and the set of common fixed points of finite many nonexpansive mappings in a Hilbert space. Some related works, we refer to see [7, 13, 16, 19].

Motivated by the recent research work going on in this fascinating field. In this paper, we introduce a hybrid method for finding a common element of the set of solutions of GSVI (1.2) for α-inverse-strongly monotone mappings, the set of solutions of MEP (1.4) and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Furthermore, we apply our main result with the problem of approximating a zero of a finite family of maximal monotone mappings in a real Hilbert space. Our main result extends and improves the recent results of Ceng, Wang and Yao [1] and many others.

2. Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.1}$$

Obviously, this immediately implies that

$$\|(x - y) - (P_C x - P_C y)\|^2 \leq \|x - y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Recall that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} &\langle x - P_C x, y - P_C x\rangle \leq 0, \\ &\|x - y\|^2 \geq \|x - P_C x\|^2 + \|P_C x - y\|^2, \end{aligned} \tag{2.3}$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [5] for more details.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;

(A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) C is a bounded set.

In the sequel, we shall need to use the following lemmas.

Lemma 2.1. ([10]) *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$*

be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ *and* $x \in H$*, define a mapping* $T_r : H \to C$ *as follows.*

$$T_r(x) = \left\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C \right\}$$

for all $x \in H$*. Then the following conclusions hold:*

(1) *For each* $x \in H$*,* $T_r(x) \neq \emptyset$*;*

(2) T_r *is single-valued;*

(3) T_r *is firmly nonexpansive, i.e. for any* $x, y \in H$*,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(4) $F(T_r) = MEP(F, \varphi)$*;*

(5) $MEP(F, \varphi)$ *is closed and convex.*

Lemma 2.2. ([18]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. ([9]) *Let* $(H, \langle ., .\rangle)$ *be an inner product space. Then, for all* $x, y, z \in H$ *and* $\alpha, \beta, \gamma \in [0, 1]$ *with* $\alpha + \beta + \gamma = 1$*, we have*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 \\ - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.\end{aligned}$$

Lemma 2.4. ([14]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{b_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$*. Suppose* $x_{n+1} = (1 - b_n)y_n + b_n x_n$ *for all integers* $n \geq 1$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$*. Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.5. ([5]) *Demi-closedness principle. Assume that* T *is a nonexpansive self-mapping of a nonempty closed convex subset* C *of a real Hilbert space* H*. If* T *has a fixed point, then* $I - T$ *is demi-closed: that is, whenever* $\{x_n\}$ *is a sequence in* C *converging weakly to some* $x \in C$ *(for short,* $x_n \rightharpoonup x \in C$*), and the sequence* $\{(I - T)x_n\}$ *converges strongly to some* y *(for short,* $(I - T)x_n \to y$*), it follows that* $(I - T)x = y$*. Here* I *is the identity operator of* H*.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

In 2009, Kangtunyakarn and Suantai [6] introduced a new mapping called the S-mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$ with $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$. They defined the new mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,0} &= I,\\
U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I,\\
U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I,\\
U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I,\\
&\ \ \vdots\\
U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I,\\
S_n = U_{n,N} &= \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I.
\end{aligned}$$

The mapping S_n is called the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}$, $\alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Nonexpansivity of each T_i ensures the nonexpansivity of S_n.

Lemma 2.7. ([6]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $j = 1, 2, \ldots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0,1]$ and α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1, 2, \ldots, N$. Let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Lemma 2.8. ([6]) *Let C be a nonempty closed convex subset of a Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and for all $n \in \mathbb{N}$ and all $j \in \{1, 2, \ldots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0,1]$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Suppose $\alpha_i^{n,j} \to \alpha_i^j$ as $n \to \infty$ for all $i \in \{1, 3\}$ and all $j = 1, 2, 3, \ldots, N$. Let S and S_n be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then $\lim_{n\to\infty} \|S_n x - Sx\| = 0$ for every $x \in C$.*

Lemma 2.9. ([1]) *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.2) *if and only if x^* is a fixed of the mapping $G : C \to C$ defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \quad \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Throughout this paper, the set of fixed points of the mapping G is denoted by $GSVI(C, A, B)$.

3. Main Results

We are now in a position to state and prove our main results.

Lemma 3.1. *Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively and let $G : C \to C$ be defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)], \quad \forall x \in C.$$

If $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. Then G is nonexpansive.

Proof. For any $x, y \in C$, we have

$$\begin{aligned}
\|G(x) - G(y)\| &= \|P_C[P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)] \\
&\quad - P_C[P_C(y - \mu By) - \lambda AP_C(y - \mu By)]\|^2 \\
&\leq \|P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx) \\
&\quad - \big(P_C(y - \mu By) - \lambda AP_C(y - \mu By)\big)\| \\
&= \|(I - \lambda A)P_C(I - \mu B)x - (I - \lambda A)P_C(I - \mu B)y\|. \qquad (3.1)
\end{aligned}$$

It is well known that if $T : C \to H$ be ι-inverse-strongly monotone, then $I - \gamma T$ is nonexpansive for all $\gamma \in (0, 2\iota)$. By our assumption, we obtain $I - \lambda A$ and $I - \mu B$ are nonexpansive. It follows that $(I - \lambda A)P_C(I - \mu B)$ is nonexpansive. Therefore, from (3.1), we obtain immediately that the mapping G is nonexpansive. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive* *self-mappings of* C *such that* $\Omega = \bigcap_{i=1}^{N} F(T_i) \bigcap GSVI(C, A, B) \bigcap MEP(F, \varphi) \neq \emptyset$*. For all* $j \in \{1, 2, \dots, N\}$*, let* $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ *be such that* $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$*,* $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$*,* $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ *with* $0 < \eta_1 \leq \theta_1 < 1$*,* $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ *with* $0 < \eta_N \leq 1$ *and* $\{\alpha_2^{n,j}\}_{j=1}^{N}, \{\alpha_3^{n,j}\}_{j=1}^{N} \subset [0, \theta_2]$ *with* $0 \leq \theta_2 < 1$*. Let* S_n *be the S-mappings generated by* $T_1, T_2, \dots, T_N$ *and* $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$*. Assume that either (B1) or (B2) holds and that* v *is an arbitrary point in* C*. Let* $x_1 \in C$ *and* $\{x_n\}, \{u_n\}, \{y_n\}$ *be the sequences defined by*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \mu B u_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n P_C(y_n - \lambda A y_n), \quad n \geq 1, \end{cases}$$

where $\lambda \in (0, 2\alpha)$ *and* $\mu \in (0, 2\beta)$*. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$*;*

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$*;*

(C4) $\lim_{n\to\infty} |\alpha_1^{n+1,i} - \alpha_1^{n,i}| = 0$ *for all* $i \in \{1, 2, \dots, N\}$ *and* $\lim_{n\to\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| = 0$ *for all* $j \in \{2, 3, \dots, N\}$*.*

Then $\{x_n\}$ *converges strongly to* $\overline{x} = P_\Omega v$ *and* $(\overline{x}, \overline{y})$ *is a solution of GSVI (1.2), where* $\overline{y} = P_C(\overline{x} - \mu B\overline{x})$*.*

Proof. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. It follows from Lemma 2.9 that

$$x^* = P_C[P_C(x^* - \mu B x^*) - \lambda A P_C(x^* - \mu B x^*)].$$

Put $y^* = P_C(x^* - \mu B x^*)$ and $t_n = P_C(y_n - \lambda A y_n)$, then $x^* = P_C(y^* - \lambda A y^*)$ and

$$x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n.$$

By nonexpansiveness of $I - \lambda A$, $I - \mu B$, P_C and T_{r_n}, we have

$$\begin{aligned} \|t_n - x^*\|^2 &= \|P_C(I - \lambda A) y_n - P_C(I - \lambda A) y^*\|^2 \\ &\leq \|y_n - y^*\|^2 = \|P_C(I - \mu B) u_n - P_C(I - \mu B) x^*\|^2 \\ &\leq \|u_n - x^*\|^2 = \|T_{r_n} x_n - T_{r_n} x^*\|^2 \leq \|x_n - x^*\|^2, \end{aligned} \tag{3.2}$$

which, implies that

$$\|x_{n+1} - x^*\| = \|a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n - x^*\|$$

$$\begin{aligned}
&\le a_n\|v-x^*\|+b_n\|x_n-x^*\|+(1-a_n-b_n)\|t_n-x^*\| \\
&\le a_n\|v-x^*\|+b_n\|x_n-x^*\|+(1-a_n-b_n)\|x_n-x^*\| \\
&\le \max\{\|v-x^*\|,\|x_1-x^*\|\}.
\end{aligned}$$

Thus, $\{x_n\}$ is bounded. Consequently, the sequences $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{Ay_n\}$, $\{Bu_n\}$ and $\{S_n t_n\}$ are also bounded. Also, observe that

$$\begin{aligned}
\|t_{n+1}-t_n\| &= \|P_C(y_{n+1}-\lambda Ay_{n+1})-P_C(y_n-\lambda Ay_n)\| \\
&\le \|y_{n+1}-y_n\| \\
&= \|P_C(u_{n+1}-\mu Bu_{n+1})-P_C(u_n-\mu Bu_n)\| \\
&\le \|u_{n+1}-u_n\|. \qquad (3.3)
\end{aligned}$$

On the other hand, from $u_n = T_{r_n}x_n \in \mathrm{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \mathrm{dom}\varphi$, we have

$$F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0,\quad \forall y\in C, \qquad (3.4)$$

and

$$F(u_{n+1},y)+\varphi(y)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0,\quad \forall y\in C. \qquad (3.5)$$

Putting $y=u_{n+1}$ in (3.4) and $y=u_n$ in (3.5), we have

$$F(u_n,u_{n+1})+\varphi(u_{n+1})-\varphi(u_n)+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle \ge 0,$$

and

$$F(u_{n+1},u_n)+\varphi(u_n)-\varphi(u_{n+1})+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle \ge 0.$$

From the monotonicity of F, we obtain that

$$\left\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \ge 0,$$

and hence

$$\left\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-x_n-\frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \ge 0.$$

Then, we have

$$\|u_{n+1}-u_n\|^2 \le \left\langle u_{n+1}-u_n,x_{n+1}-x_n+(1-\frac{r_n}{r_{n+1}})(u_{n+1}-x_{n+1})\right\rangle$$

$$\leq \|u_{n+1} - u_n\| \bigg\{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \bigg\},$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.6}$$

It follows from (3.3) and (3.6) that

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.7}$$

Let $x_{n+1} = b_n x_n + (1 - b_n) z_n$. Then, we obtain

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - b_{n+1} x_{n+1}}{1 - b_{n+1}} - \frac{x_{n+1} - b_n x_n}{1 - b_n} \\ &= \frac{a_{n+1} v + (1 - a_{n+1} - b_{n+1}) S_{n+1} t_{n+1}}{1 - b_{n+1}} - \frac{a_n v + (1 - a_n - b_n) S_n t_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}} (v - S_{n+1} t_{n+1}) + \frac{a_n}{1 - b_n} (S_n t_n - v) + S_{n+1} t_{n+1} - S_n t_n. \end{aligned} \tag{3.8}$$

Next, we estimate $\|S_{n+1} t_{n+1} - S_n t_n\|$.

For each $k \in \{2, 3, \ldots, N\}$, we have

$$\begin{aligned} &\|U_{n+1,k} t_n - U_{n,k} t_n\| = \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} t_n + \alpha_2^{n+1,k} U_{n+1,k-1} t_n + \alpha_3^{n+1,k} t_n \\ &\quad - \alpha_1^{n,k} T_k U_{n,k-1} t_n - \alpha_2^{n,k} U_{n,k-1} t_n - \alpha_3^{n,k} t_n\| \\ &= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} t_n - T_k U_{n,k-1} t_n) \\ &\quad + (\alpha_1^{n+1,k} - \alpha_1^{n,k}) T_k U_{n,k-1} t_n + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) t_n \\ &\quad + \alpha_2^{n+1,k} (U_{n+1,k-1} t_n - U_{n,k-1} t_n) + (\alpha_2^{n+1,k} - \alpha_2^{n,k}) U_{n,k-1} t_n\| \\ &\leq \alpha_1^{n+1,k} \|U_{n+1,k-1} t_n - U_{n,k-1} t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} t_n\| \\ &\quad + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|t_n\| + \alpha_2^{n+1,k} \|U_{n+1,k-1} t_n - U_{n,k-1} t_n\| \\ &\quad + |\alpha_2^{n+1,k} - \alpha_2^{n,k}| \|U_{n,k-1} t_n\| \\ &= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k}) \|U_{n+1,k-1} t_n - U_{n,k-1} t_n\| \\ &\quad + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} t_n\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|t_n\| \\ &\quad + |\alpha_2^{n+1,k} - \alpha_2^{n,k}| \|U_{n,k-1} t_n\| \\ &\leq \|U_{n+1,k-1} t_n - U_{n,k-1} t_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}| \|T_k U_{n,k-1} t_n\| \end{aligned}$$

$$\begin{aligned}
&+|\alpha_3^{n+1,k}-\alpha_3^{n,k}|\|t_n\|+|(\alpha_1^{n,k}-\alpha_1^{n+1,k})+(\alpha_3^{n,k}-\alpha_3^{n+1,k})|\|U_{n,k-1}t_n\| \\
&\le \|U_{n+1,k-1}t_n-U_{n,k-1}t_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|\|T_kU_{n,k-1}t_n\| \\
&\quad +|\alpha_3^{n+1,k}-\alpha_3^{n,k}|\|t_n\|+|\alpha_1^{n,k}-\alpha_1^{n+1,k}|\|U_{n,k-1}t_n\| \\
&\quad +|\alpha_3^{n,k}-\alpha_3^{n+1,k}|\|U_{n,k-1}t_n\| \\
&= \|U_{n+1,k-1}t_n-U_{n,k-1}t_n\|+|\alpha_1^{n+1,k}-\alpha_1^{n,k}|(\|T_kU_{n,k-1}t_n\|+\|U_{n,k-1}t_n\|) \\
&\quad +|\alpha_3^{n+1,k}-\alpha_3^{n,k}|(\|t_n\|+\|U_{n,k-1}t_n\|). \qquad (3.9)
\end{aligned}$$

It follow from (3.9) that

$$\begin{aligned}
&\|S_{n+1}t_n-S_nt_n\|=\|U_{n+1,N}t_n-U_{n,N}t_n\| \\
&\le \|U_{n+1,1}t_n-U_{n,1}t_n\|+\sum_{j=2}^{N}|\alpha_1^{n+1,j}-\alpha_1^{n,j}|(\|T_jU_{n,j-1}t_n\|+\|U_{n,j-1}t_n\|) \\
&\quad +\sum_{j=2}^{N}|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|t_n\|+\|U_{n,j-1}t_n\|) \\
&= |\alpha_1^{n+1,1}-\alpha_1^{n,1}|\|T_1t_n-t_n\| \\
&\quad +\sum_{j=2}^{N}|\alpha_1^{n+1,j}-\alpha_1^{n,j}|(\|T_jU_{n,j-1}t_n\|+\|U_{n,j-1}t_n\|) \\
&\quad +\sum_{j=2}^{N}|\alpha_3^{n+1,j}-\alpha_3^{n,j}|(\|t_n\|+\|U_{n,j-1}t_n\|).
\end{aligned}$$

This together with the condition $(C4)$, we obtain

$$\lim_{n\to\infty}\|S_{n+1}t_n-S_nt_n\|=0. \qquad (3.10)$$

It follows from (3.7) that

$$\begin{aligned}
\|S_{n+1}t_{n+1}-S_nt_n\| &\le \|t_{n+1}-t_n\|+\|S_{n+1}t_n-S_nt_n\| \\
&\le \|x_{n+1}-x_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\
&\quad +\|S_{n+1}t_n-S_nt_n\|. \qquad (3.11)
\end{aligned}$$

By (3.8) and (3.11), we have

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\le \frac{a_{n+1}}{1-b_{n+1}}\|v-S_{n+1}t_{n+1}\|+\frac{a_n}{1-b_n}\|S_nt_n-v\|$$

$$
\begin{aligned}
&+\|S_{n+1}t_{n+1}-S_nt_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{a_{n+1}}{1-b_{n+1}}\|v-S_{n+1}t_{n+1}\|+\frac{a_n}{1-b_n}\|S_nt_n-v\|\\
&+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\|\\
&+\|S_{n+1}t_n-S_nt_n\|.
\end{aligned}
$$

This together with (C1)-(C3) and (3.10), we obtain that

$$
\limsup_{n\to\infty}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\le 0.
$$

Hence, by Lemma 2.4, we get $\|x_n-z_n\|\to 0$ as $n\to\infty$. Consequently,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-b_n)\|z_n-x_n\|=0. \tag{3.12}
$$

From (C3), (3.3) and (3.6), we also have $\|u_{n+1}-u_n\|\to 0$, $\|t_{n+1}-t_n\|\to 0$ and $\|y_{n+1}-y_n\|\to 0$, as $n\to\infty$.

Since

$$
x_{n+1}-x_n=a_n(v-x_n)+(1-a_n-b_n)(S_nt_n-x_n),
$$

therefore

$$
\|S_nt_n-x_n\|\to 0 \quad \text{as} \quad n\to\infty. \tag{3.13}
$$

Next, we prove that $\lim_{n\to\infty}\|x_n-u_n\|=0$. From Lemma 2.1(3), we have

$$
\begin{aligned}
\|u_n-x^*\|^2&=\|T_{r_n}x_n-T_{r_n}x^*\|^2\le\langle T_{r_n}x_n-T_{r_n}x^*,x_n-x^*\rangle\\
&=\langle u_n-x^*,x_n-x^*\rangle=\frac{1}{2}\big\{\|u_n-x^*\|^2+\|x_n-x^*\|^2-\|x_n-u_n\|^2\big\}.
\end{aligned}
$$

Hence

$$
\|u_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-u_n\|^2. \tag{3.14}
$$

From Lemma 2.3, (3.2) and (3.14), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|t_n-x^*\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|u_n-x^*\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\big[\|x_n-x^*\|^2-\|x_n-u_n\|^2\big]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2-(1-a_n-b_n)\|x_n-u_n\|^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(1-a_n-b_n)\|x_n-u_n\|^2 &\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\le a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
$$

From the conditions (C1), (C2) and (3.12), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.15}$$

Since

$$\|S_nt_n-u_n\|\le\|S_nt_n-x_n\|+\|x_n-u_n\|,$$

it follows from (3.13) and (3.15) that

$$\lim_{n\to\infty}\|S_nt_n-u_n\|=0. \tag{3.16}$$

Next, we show that $\|Ay_n-Ay^*\|\to 0$ and $\|Bu_n-Bx^*\|\to 0$ as $n\to\infty$. From (3.2), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|t_n-x^*\|^2\\
&= a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\big[\|y_n-y^*\|^2+\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2\big]\\
&\le a_n\|v-x^*\|^2+\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|t_n-x^*\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|y_n-y^*\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2\\
&\le a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad+(1-a_n-b_n)\big[\|u_n-x^*\|^2+\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2\big],
\end{aligned}
$$

$$\leq a_n\|v-x^*\|^2+\|x_n-x^*\|^2 + (1-a_n-b_n)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2.$$

Therefore, we have

$$-(1-a_n-b_n)\lambda(\lambda-2\alpha)\|Ay_n-Ay^*\|^2 \leq a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|,$$

and

$$-(1-a_n-b_n)\mu(\mu-2\beta)\|Bu_n-Bx^*\|^2 \leq a_n\|v-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.$$

This together with (3.12), (C1) and (C2), we obtain

$$\|Ay_n-Ay^*\|\to 0 \ \text{ and } \ \|Bu_n-Bx^*\|\to 0 \ \text{ as } \ n\to\infty. \tag{3.17}$$

Next, we prove that $\|S_nt_n-t_n\|\to 0$ as $n\to\infty$. From (2.1) and nonexpansiveness of $I-\mu B$, we get

$$\begin{aligned}\|y_n-y^*\|^2 &= \|P_C(u_n-\mu Bu_n)-P_C(x^*-\mu Bx^*)\|^2\\ &\leq \langle (u_n-\mu Bu_n)-(x^*-\mu Bx^*), y_n-y^*\rangle\\ &= \frac{1}{2}\big[\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)\|^2+\|y_n-y^*\|^2\\ &\quad -\|(u_n-\mu Bu_n)-(x^*-\mu Bx^*)-(y_n-y^*)\|^2\big]\\ &\leq \frac{1}{2}\big[\|u_n-x^*\|^2+\|y_n-y^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\big].\end{aligned}$$

By (3.2), we obtain

$$\begin{aligned}\|y_n-y^*\|^2 &\leq \|u_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2\\ &\leq \|x_n-x^*\|^2-\|(u_n-x^*)-(y_n-y^*)\|^2\\ &\quad +2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle-\mu^2\|Bu_n-Bx^*\|^2.\end{aligned}$$

Hence,

$$\|x_{n+1}-x^*\|^2 \leq a_n\|v-x^*\|^2+b_n\|x_n-x^*\|^2+(1-a_n-b_n)\|y_n-y^*\|^2$$

$$
\begin{aligned}
&\leq a_n\|v-x^*\|^2 + b_n\|x_n-x^*\|^2 + (1-a_n-b_n)\big[\|x_n-x^*\|^2 \\
&\quad - \|(u_n-x^*)-(y_n-y^*)\|^2 \\
&\quad + 2\mu\langle (u_n-x^*)-(y_n-y^*), Bu_n-Bx^*\rangle - \mu^2\|Bu_n-Bx^*\|^2\big] \\
&\leq a_n\|v-x^*\|^2 + \|x_n-x^*\|^2 \\
&\quad - (1-a_n-b_n)\|(u_n-x^*)-(y_n-y^*)\|^2 \\
&\quad + (1-a_n-b_n)2\mu\|(u_n-x^*)-(y_n-y^*)\|\|Bu_n-Bx^*\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-a_n-b_n)\|(u_n-x^*)-(y_n-y^*)\|^2 \\
&\leq a_n\|v-x^*\|^2 + (1-a_n-b_n)2\mu\|(u_n-x^*)-(y_n-y^*)\|\|Bu_n-Bx^*\| \\
&\quad + (\|x_n-x^*\| + \|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|.
\end{aligned}
$$

This together with (C1), (3.12) and (3.17), we obtain

$$
\|(u_n-x^*)-(y_n-y^*)\| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.18}
$$

From Lemma 2.6 and (2.2), it follows that

$$
\begin{aligned}
&\|(y_n-t_n)+(x^*-y^*)\|^2 = \|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*) \\
&\quad - \big[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\big] + \lambda(Ay_n-Ay^*)\|^2 \\
&\leq \|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*) - \big[P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\big]\|^2 \\
&\quad + 2\lambda\langle Ay_n-Ay^*, (y_n-t_n)+(x^*-y^*)\rangle \\
&\leq \|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\|^2 - \|P_C(y_n-\lambda Ay_n)-P_C(y^*-\lambda Ay^*)\|^2 \\
&\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\| \\
&\leq \|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\|^2 - \|S_nP_C(y_n-\lambda Ay_n)-S_nP_C(y^*-\lambda Ay^*)\|^2 \\
&\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\| \\
&\leq \|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*) \\
&\quad - (S_nt_n-x^*)\|\big[\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\| + \|S_nt_n-x^*\|\big] \\
&\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\| \\
&= \|u_n-S_nt_n+x^*-y^*-(u_n-y_n) \\
&\quad - \lambda(Ay_n-Ay^*)\|\big[\|(y_n-\lambda Ay_n)-(y^*-\lambda Ay^*)\| + \|S_nt_n-x^*\|\big] \\
&\quad + 2\lambda\|Ay_n-Ay^*\|\|(y_n-t_n)+(x^*-y^*)\|.
\end{aligned}
$$

This together with (3.16), (3.18) and (3.17), we obtain $\|(y_n-t_n)+(x^*-y^*)\| \to 0$ as $n\to\infty$. This together with (3.13), (3.15) and (3.18), we obtain that

$$
\|S_nt_n-t_n\| \leq \|S_nt_n-x_n\| + \|x_n-u_n\| + \|(u_n-y_n)-(x^*-y^*)\|
$$

$$+\|(y_n - t_n) + (x^* - y^*)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.19}$$

Next, we show that

$$\limsup_{n\to\infty}\langle v - \overline{x}, x_n - \overline{x}\rangle \le 0,$$

where $\overline{x} = P_\Omega v$.

Indeed, since $\{t_n\}$ and $\{S_n t_n\}$ are two bounded sequences in C, we can choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \rightharpoonup z \in C$ and

$$\limsup_{n\to\infty}\langle v - \overline{x}, S_n t_n - \overline{x}\rangle = \lim_{i\to\infty}\langle v - \overline{x}, S_{n_i} t_{n_i} - \overline{x}\rangle.$$

Since $\lim_{n\to\infty}\|S_n t_n - t_n\| = 0$, we obtain that $S_{n_i} t_{n_i} \rightharpoonup z$ as $i \to \infty$.

Next, we show that $z \in \Omega$.

(a) We first show $z \in \bigcap_{i=1}^N F(T_i)$.

We can assume that $\alpha_1^{n,j} \to \alpha_1^j \in (0,1)$ and $\alpha_1^{n,N} \to \alpha_1^N \in (0,1]$ as $n \to \infty$ for all $j \in \{1, 2, \ldots, N-1\}$ and $\alpha_3^{n,j} \to \alpha_3^j \in [0,1)$ as $n \to \infty$ for $j = 1, 2, \ldots, N$. Let S be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ where $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, for $j = 1, 2, \ldots, N$. From Lemma 2.8, we have $\|S_n t_n - S t_n\| \to 0$ as $n \to \infty$. Since

$$\|S t_n - t_n\| \le \|S t_n - S_n t_n\| + \|S_n t_n - t_n\|,$$

it follows by (3.19) that $\|S t_n - t_n\| \to 0$ as $n \to \infty$.

Since $t_{n_i} \rightharpoonup z$ and $\|S t_n - t_n\| \to 0$, we obtain by Lemma 2.5 and Lemma 2.7 that $z \in F(S) = \bigcap_{i=1}^N F(T_i)$.

(b) Now, we show that $z \in GSVI(C, A, B)$.

Since

$$\|t_n - x_n\| \le \|S_n t_n - t_n\| + \|S_n t_n - x_n\|,$$

it follows from (3.19) and (3.13) that $\|t_n - x_n\| \to 0$ as $n \to \infty$. Furthermore, by Lemma 3.1, we have $G : C \to C$ is nonexpansive. Then, we have

$$\begin{aligned}
\|t_n - G(t_n)\| &= \|P_C(y_n - \lambda A y_n) - G(t_n)\| \\
&= \|P_C\big[P(u_n - \mu B u_n) - \lambda A P(u_n - \mu B u_n)\big] - G(t_n)\| \\
&= \|G(u_n) - G(t_n)\| \le \|u_n - t_n\| \\
&\le \|u_n - x_n\| + \|x_n - t_n\|,
\end{aligned}$$

which implies $\|t_n - G(t_n)\| \to 0$ as $n \to \infty$. Again by Lemma 2.5, we have $z \in GSVI(C, A, B)$.

(c) We show that $z \in MEP(F, \varphi)$. Since $t_{n_i} \rightharpoonup z$ and $\|x_n - t_n\| \to 0$, we obtain that $x_{n_i} \rightharpoonup z$. From $\|u_n - x_n\| \to 0$, we also obtain that $u_{n_i} \rightharpoonup z$. By using the same argument as that in the proof of [10, Theorem 3.1, pp. 1825], we can show that $z \in MEP(F, \varphi)$. Therefore there holds $z \in \Omega$.

On the other hand, it follows from (2.3) and $S_{n_i}t_{n_i} \rightharpoonup z$ as $i \to \infty$ that

$$\limsup_{n\to\infty}\langle v-\overline{x}, x_n-\overline{x}\rangle = \limsup_{n\to\infty}\langle v-\overline{x}, S_nt_n-\overline{x}\rangle = \lim_{i\to\infty}\langle v-\overline{x}, S_{n_i}t_{n_i}-\overline{x}\rangle$$
$$= \langle v-\overline{x}, z-\overline{x}\rangle \le 0. \tag{3.20}$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-\overline{x}\|^2 &= \langle a_nv+b_nx_n+(1-a_n-b_n)S_nt_n-\overline{x}, x_{n+1}-\overline{x}\rangle\\ &= a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + b_n\langle x_n-\overline{x}, x_{n+1}-\overline{x}\rangle\\ &\quad + (1-a_n-b_n)\langle S_nt_n-\overline{x}, x_{n+1}-\overline{x}\rangle\end{aligned}$$

$$\begin{aligned}&\le a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + \frac{1}{2}b_n(\|x_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2)\\ &\quad + \frac{1}{2}(1-a_n-b_n)(\|t_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2)\\ &\le a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + \frac{1}{2}b_n(\|x_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2)\\ &\quad + \frac{1}{2}(1-a_n-b_n)(\|x_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2)\\ &= a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle + \frac{1}{2}(1-a_n)(\|x_n-\overline{x}\|^2+\|x_{n+1}-\overline{x}\|^2),\end{aligned}$$

which implies that

$$\|x_{n+1}-\overline{x}\|^2 \le (1-a_n)\|x_n-\overline{x}\|^2 + 2a_n\langle v-\overline{x}, x_{n+1}-\overline{x}\rangle.$$

It follows from Lemma 2.2 and (3.20) that $\{x_n\}$ converges strongly to $\overline{x}$. This completes the proof. □

Let $N=1$, $T_1=S$, $\alpha_2^{n,1}=\alpha_3^{n,1}=0$, $\varphi=0$, $F(x,y)=0$ for all $x,y\in C$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 3.2, then $u_n=P_Cx_n=x_n$. By Theorem 3.2, we obtain the following result.

Corollary 3.3. [1, Theorem 3.1] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let S be a nonexpansive self-mapping of C such that $\Omega = F(S)\bigcap GSVI(C,A,B) \neq \emptyset$.*

Assume that v is an arbitrary point in C. Let $x_1 \in C$ and $\{x_n\}$ and $\{y_n\}$ be the sequences generated by

$$\begin{cases} y_n = P_C(x_n - \mu B x_n), \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S P_C(y_n - \lambda A y_n), \quad n \geq 1. \end{cases}$$

If $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and the sequences $\{a_n\}$, $\{b_n\}$ are as in Theorem 3.2, then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$ and $(\overline{x}, \overline{y})$ is a solution of GSVI (1.2)*, where $\overline{y} = P_C(\overline{x} - \mu B \overline{x})$.*

4. Applications

In this section, we apply Theorem 3.2 with three strong convergence theorems in a real Hilbert space.

We recall that a mapping $T : C \to C$ is called strictly pseudocontractive if there exists some k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we have

$$\|(I - T)x - (I - T)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay\rangle + \|Ax - Ay\|^2.$$

Hence we have

$$\langle x - y, Ax - Ay\rangle \geq \frac{1 - k}{2}\|Ax - Ay\|^2.$$

Then, A is $\frac{1-k}{2}$-inverse-strongly monotone.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let the mappings $T, V : C \to C$ be strictly pseudocontractive with constants k, l, respectively. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C such that $\Omega = \bigcap_{i=1}^N F(T_i) \bigcap GSVI(C, I - T, I - V) \bigcap MEP(F, \varphi) \neq \emptyset$. For all $j \in \{1, 2, \ldots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in$*

$[0,1]$, $\alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j}=1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1}\subset[\eta_1,\theta_1]$ *with* $0<\eta_1\leq\theta_1<1$, $\{\alpha_1^{n,N}\}\subset[\eta_N,1]$ *with* $0<\eta_N\leq 1$ *and* $\{\alpha_2^{n,j}\}_{j=1}^{N},\{\alpha_3^{n,j}\}_{j=1}^{N}\subset[0,\theta_2]$ *with* $0\leq\theta_2<1$. *Let* S_n *be the* S*-mappings generated by* $T_1,T_2,\ldots,T_N$ *and* $\alpha_1^{(n)},\alpha_2^{(n)},\ldots,\alpha_N^{(n)}$. *Assume that either (B1) or (B2) holds and that* v *is an arbitrary point in* C. *Let* $x_1\in C$ *and* $\{x_n\},\{u_n\},\{y_n\}$ *be the sequences defined by*

$$\begin{cases} F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \quad \forall y\in C,\\ y_n=(1-\mu)u_n+\mu Vu_n,\\ x_{n+1}=a_nv+b_nx_n+(1-a_n-b_n)S_n\big((1-\lambda)y_n+\lambda Ty_n\big), \quad n\geq 1,\end{cases}$$

where $\lambda\in(0,1-k)$ *and* $\mu\in(0,1-l)$. *Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty}a_n=0$ *and* $\sum_{n=1}^{\infty}a_n=\infty$;

(C2) $0<\liminf_{n\to\infty}b_n\leq\limsup_{n\to\infty}b_n<1$;

(C3) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}|r_{n+1}-r_n|=0$;

(C4) $\lim_{n\to\infty}|\alpha_1^{n+1,i}-\alpha_1^{n,i}|=0$ *for all* $i\in\{1,2,\ldots,N\}$ *and* $\lim_{n\to\infty}|\alpha_3^{n+1,j}-\alpha_3^{n,j}|=0$ *for all* $j\in\{2,3,\ldots,N\}$.

Then $\{x_n\}$ *converges strongly to* $\overline{x}=P_\Omega v$ *and* $(\overline{x},\overline{y})$ *is a solution of problem (1.2), where* $\overline{y}=(1-\mu)\overline{x}+\mu V\overline{x}$.

Proof. Put $A=I-T$ and $B=I-V$. Then A is $\frac{1-k}{2}$-inverse-strongly monotone and B is $\frac{1-l}{2}$-inverse-strongly monotone, respectively. We have

$$P_C(u_n-\mu Bu_n)=(1-\mu)u_n+\mu Vu_n$$

and

$$P_C(y_n-\lambda Ay_n)=(1-\lambda)y_n+\lambda Ty_n.$$

Therefore, the conclusion follows immediately from Theorem 3.2. □

Theorem 4.2. *Let* H *be a real Hilbert space. Let* F *be a function from* $H\times H$ *to* $\mathbb{R}$ *satisfying (A1)-(A5) and* $\varphi:H\to\mathbb{R}\bigcup\{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $A:H\to H$ *be* α*-inverse-strongly monotone mapping. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpansive self-mappings of* H *such that* $\Omega=\bigcap_{i=1}^N F(T_i)\bigcap A^{-1}0\bigcap MEP(F,\varphi)\neq\emptyset$. *For all* $j\in\{1,2,\ldots,N\}$, *let* $\alpha_j^{(n)}=(\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j})$ *be such that* $\alpha_1^{n,j},\alpha_2^{n,j},\alpha_3^{n,j}\in[0,1]$, $\alpha_1^{n,j}+\alpha_2^{n,j}+\alpha_3^{n,j}=1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1}\subset[\eta_1,\theta_1]$ *with* $0<\eta_1\leq\theta_1<1$, $\{\alpha_1^{n,N}\}\subset[\eta_N,1]$ *with* $0<\eta_N\leq 1$ *and* $\{\alpha_2^{n,j}\}_{j=1}^{N},\{\alpha_3^{n,j}\}_{j=1}^{N}\subset[0,\theta_2]$ *with* $0\leq\theta_2<1$. *Let* S_n *be the* S*-mappings generated by* $T_1,T_2,\ldots,T_N$ *and* $\alpha_1^{(n)},\alpha_2^{(n)},\ldots,\alpha_N^{(n)}$.

Assume that either (B1) or (B2) holds and that v is an arbitrary point in H. Let $x_1 \in H$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined by

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ y_n = u_n - \lambda A u_n, \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n(y_n - \lambda A y_n), \quad n \geq 1, \end{cases}$$

where $\lambda \in (0, 2\alpha)$. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;
(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$;
(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(C4) $\lim_{n\to\infty} |\alpha_1^{n+1,i} - \alpha_1^{n,i}| = 0$ *for all* $i \in \{1, 2, \ldots, N\}$ *and* $\lim_{n\to\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| = 0$ *for all* $j \in \{2, 3, \ldots, N\}$.
Then $\{x_n\}$ converges strongly to $\overline{x} = P_\Omega v$.

Proof. Put $\lambda = \mu$, $C = H$, $B = A$ and $P_H = I$ in Theorem 3.2. By using the same argument as that in the proof of [1, Theorem 4.1, pp. 388-389], we can show that $A^{-1}0 = GSVI(C, A, B) = VI(A, H)$ and

$$\text{problem (1.2)} \Leftrightarrow \text{problem (1.3)} \Leftrightarrow VI(A, H).$$

Thus, by Theorem 3.2 we obtain the desired result. □

Recall that the resolvent of the maximal monotone mapping $B : H \to 2^H$ is defined by $J_r^B = (I + rB)^{-1}$ for all $r > 0$, it is known that $F(J_r^B) = B^{-1}0$ and J_r^B is nonexpansive.

Theorem 4.3. *Let H be a real Hilbert space. Let F be a function from $H \times H$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : H \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : H \to H$ be α-inverse-strongly monotone mapping and let $B_1, B_2, \ldots, B_N : H \to 2^H$ be maximal monotone mappings such that $\Omega = \bigcap_{i=1}^N B_i^{-1}0 \bigcap A^{-1}0 \bigcap MEP(F, \varphi) \neq \emptyset$. For all $i \in \{1, 2, \ldots, N\}$ and $r_i > 0$, let $J_{r_i}^{B_i}$ be the resolvents of B_i. For all $j \in \{1, 2, \ldots, N\}$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,N}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N, \{\alpha_3^{n,j}\}_{j=1}^N \subset [0, \theta_2]$ with $0 \leq \theta_2 < 1$. Let S_n be the S-mappings generated by $J_{r_1}^{B_1}, J_{r_2}^{B_2}, \ldots, J_{r_N}^{B_N}$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Assume that either (B1) or (B2) holds and that v is an arbitrary point in H. Let $x_1 \in H$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined* *by*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ y_n = u_n - \lambda A u_n, \\ x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n(y_n - \lambda A y_n), & n \geq 1, \end{cases}$$

where $\lambda \in (0, 2\alpha)$*. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$*;*

(C3) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$*;*

(C4) $\lim_{n\to\infty} |\alpha_1^{n+1,i} - \alpha_1^{n,i}| = 0$ *for all* $i \in \{1, 2, \ldots, N\}$ *and* $\lim_{n\to\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| = 0$ *for all* $j \in \{2, 3, \ldots, N\}$*.*

Then $\{x_n\}$ *converges strongly to* $\overline{x} = P_\Omega v$*.*

Proof. For all $i = 1, 2, \ldots, N$ and $r_i > 0$, we have $F(J_{r_i}^{B_i}) = B_i^{-1}0$. Putting $P_H = I$ and $T_i = J_{r_i}^{B_i}$ for all $i = 1, 2, \ldots, N$, by Theorem 4.2, we obtain the desired result. ☐

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