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COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MULTIFUNCTIONS

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Abstract: In this paper, we generalize the contractive multifunctions results given by H.E. Kunze et al, see [3]. Further generalization of collage theorem has also been discussed.

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1. Introduction and Preliminaries

In 1970, Covitz Nadler (see [7]) gave the following results "Multi-valued contraction mappings in generalized metric spaces". Using this result, H.E. Kunze et al (see [3]) introduce an iterative method involving projections that guarantees convergence, from any starting point $x_0 \in X$ to a point $x \in X_T$, the set of all fixed points of a multifunction operator T. Here multifunction operator $T : X \rightrightarrows Y$, is a set-valued mappings from a space X to the power set 2^Y.

We consider multifunctions T that satisfy the following contractivity condition:

There exists a $c \in [0, 1)$ such that $d_h(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$.

Here d_h denotes the Hausdorff metric. Then there exists a fixed point $\bar{x} \in X$ such that $\bar{x} \in T\bar{x}$, $\bar{x}$ is not necessarily unique.

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The purpose of this paper is to obtain common fixed point for pair of contractiv e maps in the setting of procedure given by H.E. Kunze et al [3]. Throughout this paper, T is a multifunction operator and X_T, the set of all fixed points of a multifunction operator T.

We recall some definitions of Hausdorff distance and some of their properties and results as well as properties of contractive functions.

Definition 1.1. (see [3]) Let $d(x, y)$ denote the Euclidean distance. We shall also let $H(X)$ denote the space of all compact subsets of X and $d_h(A, B)$ the Hausdorff distance between A and B, that is

$$d_h(A, B) = \max\left\{\max_{x\in A} d'(x, B), \max_{x\in B} d'(x, A)\right\},$$

where $d'(x, A)$ is the usual distance between the point x and the set A, i.e

$$d'(x, A) = \min_{y\in A} d(x, y).$$

In the following we will denote by $h(A, B) = \underset{x\in A}{max}\, d'(x, B)$.

It is well known that the space $(H(X), d_h)$ is a complete metric space if X is complete, see [5].

Lemma 1.1. (see [3]) *Let (X, d) be a metric space.*

1. For all $x, y \in X$, $C \subset X$, we have

$$d'(x, C) \leq d(x, y) + d'(y, C).$$

2. If $A \subset B$, then $h(C, A) \geq h(C, B)$ and

$$h(A, C) \leq h(B, C), \quad \textit{for all} \quad C \subset X.$$

3. For all $x, y \in X$, $A, B \subset X$, we have

$$d'(x, A) \leq d(x, y) + d'(y, B) + h(B, A).$$

4. For all $x \in X$ and $A, B \subset X$, we have

$$d'(x, A) \leq d'(x, B) + h(B, A)$$

5. Suppose that $(X, \|\|)$ be a real normed space and $E \subset X$ be a convex subset of X. Let $A_1, A_2, B_1, B_2 \subset E$, $\lambda_i \in [0, 1]$, and $\sum_i \lambda i = 1$.

Then

$$d_h(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) \leq \lambda_1 d_h(A_1, B_1) + \lambda_2 d_h(A_2, B_2).$$

6. Let $A_i, B_i \subset E$ *and* $\lambda_i \in [0,1]$ *for* $i = 1, 2, ..., N$, $\sum\limits_i \lambda_i = 1$.

Then

$$d_h\left(\sum_i \lambda_i A_i, \lambda_i B_i\right) \leq \sum_i \lambda_i d_h(A_i, B_i).$$

7. Let $A, B, C, \subset E$, $\lambda_1, \lambda_2 \in [0,1]$ *such that* $\lambda_1 + \lambda_2 = 1$. *Suppose that* A, B, C *are compact and* A *is convex.*

Then

$$d_h(A, \lambda_1 B + \lambda_2 C) \leq \lambda_1 d_h(A, B) + \lambda_2 d_h(A, C).$$

The following examples state how to calculate the Hausdorff distance when the sets are intervals.

Example 1.1. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$. Then

$$d_h(A, B) = \max\{\mid b_1 - a_1 \mid, \mid b_2 - a_2 \mid\}.$$

Theorem 1.1. (Banach) *Let* (X, d) *be a complete metric space. Also let* $T : X \to X$ *be a contraction mapping with contraction factor* $c \in [0, 1)$, *i.e. for all* $x, y \in X$, $d(Tx, Ty) \leq cd(x, y)$.

Then there exists a unique $\bar{x} \in X$ *such that* $\bar{x} = T\bar{x}$.

Moreover, for any $x \in X$, $d(T^n x, \bar{x}) \to 0$ *as* $n \to \infty$.

Theorem 1.2. (Collage Theorem, see [2], [4]) *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a contraction mapping with contraction factor* $c \in [0, 1)$.

Then for any $x \in X$,

$$d(x, \bar{x}) \leq \frac{1}{1-c} d(x, Tx),$$

where $\bar{x}$ *is the fixed point of* T.

Theorem 1.3. (Anti-Collage Theorem, see [9]) *Assume the conditions of the Collage Theorem above.*

Then for any $x \in Y$,

$$d(x, \bar{x}) \geq \frac{1}{1+c} d(x, Tx).$$

Theorem 1.4. (Covitz-Nadler, see [6], [7]) *Let (X, d) be a complete metric space and suppose that $T : X \to H(X)$ be a set valued contraction mapping, i.e. $d_h(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$, and $c \in [0, 1)$.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in T\bar{x}$.

Note that the fixed point $\bar{x}$ is not necessarily unique.

2. Common Fixed Point Theorems

In this section we shall prove common fixed point theorem for pair of contractio n multifunctions.

The following theorem is extends and improves the Theorem 6 from [3].

Theorem 2.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to H(X)$ be a pair of contraction multifunctions such that*

$$d_h(T_1(x), T_2(y)) \leq Kd(x, y)$$

for all $x, y \in X$, with $K \in [0, 1)$.

Then:

1. For all $x_0 \in X$ there exists a common point $\bar{x} \in X$ such that

$$(x_{2n+2}) = P(x_{2n+1}) \to \bar{x}$$

and

$$x_{2n+3} = P(x_{2n+2}) \to \bar{x},$$

when $n \to +\infty$.

2. $\bar{x}$ is a common fixed point, that is, $\bar{x} \in T_1\bar{x}$ and $\bar{x} \in T_2\bar{x}$.

Proof. Starting from a point $x_0 \in X$, take the projection $P(x_0)$ of the point on the set T_1x_0.

Therefore

$$d'(x_0, T_1x_0) = d(x_0, P(x_0)).$$

Let $x_1 = P(x_0)$ and take the projection of x_1 on the set T_1x_1. We have

$$\begin{aligned} d(x_2, x_1) =& d(P(x_1), x_1) = d'(x_1, T_1x_1) \\ =& d'(P(x_0), T_1x_1) \\ \leq& h(T_1x_0, T_1x_1) \end{aligned}$$

$$\begin{aligned}&\leq cd(x_0, x_1)\\&= cd'(x_0, T_1x_0),\end{aligned}$$

and for each $n \in N$, we have

$$\begin{aligned}d(x_{2n+1}, x_{2n+2}) &= d'(x_{2n+1}, T_1x_{2n+1})\\&\leq c^{2n+1}d'(x_0, T_1x_0)\\&= c^{2n+1}d(x_0, x_1).\end{aligned}$$

Then for all $n, m \in N, n_0 \leq (2n+1) < m$, we have

$$\begin{aligned}d(x_{2n+1}, x_m) &\leq \sum_{i=2n+1}^{m-1} d(x_i, x_{i+1})\\&\leq \sum_{i=2n+1}^{m-1} c^{2n+1+i}d(x_0, x_1)\\&= c^{2n+1}d(x_0, x_1)\sum_{i=o}^{m-1} c^i\\&= c^{2n+1}\left(\frac{1-c^m}{1-c}\right)d(x_0, x_1)\\&\leq \frac{c^{n_0}}{1-c}d(x_0, x_1).\end{aligned}$$

Therefore the sequence $\{x_{2n+1}\}$ is a Cauchy sequence in X.
Since X is complete, there exists an $\bar{x} \in X$ such that

$$x_{2n+1} \to \bar{x}.$$

Since

$$\begin{aligned}d'(\bar{x}, T_1\bar{x}) &\leq d(\bar{x}, x_{2n+1}) + d'(x_{2n+1}, T_1x_{2n+1}) + h(T_1x_{2n+1}, T_1\bar{x})\\&\leq d(\bar{x}, x_{2n+1}) + c^{2n+1}d(x_0, T_1x_0) + cd(x_{2n+1}, \bar{x})\\&= (1+c)d(\bar{x}, x_{2n+1}) + c^{2n+1}d(x_0, T_1x_0),\end{aligned}$$

it follows that $\bar{x} \in T_1\bar{x}$.

Similarly it can be established that $\bar{x} \in T_2\bar{x}$, that is $\bar{x}$ is a common fixed point of pair of T_1 and T_2.

This completes the proof of the theorem. □

Theorem 2.2. *Let (X, d) be a complete metric space. Then X_{T_1} and X_{T_2} are completes.*

Proof. Let $x_{2n+1} \in X_{T_1}$ be a Cauchy sequence of points of X_{T_1}.

Since $x_{2n+1} \in X$ and X is complete then there exists a points $\bar{x} \in X$ such that $d(x_{2n+1}, \bar{x}) \to 0$.

Now we have

$$\begin{aligned} d(\bar{x}, T_1\bar{x}) \leq & d(\bar{x}, x_{2n+1}) + d'(x_{2n+1}, T_1x_{2n+1}) + h(T_1x_{2n+1}, T_1\bar{x}) \\ \leq & 2d(\bar{x}, x_{2n+1}). \end{aligned}$$

Similarly it can be proved that for X_{T_2}. □

Remark 2.1. Let (X, d) be a compact metric space.

In this case it is easy to prove that X_{T_1} and X_{T_2} are compact.

Now, we will generalize some results in [3], for pair of contraction multifunctions.

Theorem 2.3. (Generalized and Extended Collage Theorem) *Let (X, d) be a complete metric space and $T_1, T_2 : X \to H(X)$ be a pair of contraction multifunctions with contractivity factor $c \in [0, 1)$.*

Then for all $x \in X$ there exists a common fixed point $\bar{x}$ such that either

$$d(x_0, \bar{x}) \leq \frac{d'(x, T_1x)}{1 - c}$$

or

$$d(x_0, \bar{x}) \leq \frac{d'(x, T_2x)}{1 - c}$$

so that either

$$d'(x, X_{T_1}) \leq \frac{d'(x, T_1x)}{1 - c}$$

or

$$d'(x, X_{T_2}) \leq \frac{d'(x, T_2x)}{1 - c}.$$

Proof. For any $x \in X$, let $x_0 = x$.

By the use of Theorem 2.1, there exists a point $\bar{x} \in X_{T_1}$ such that the projection scheme $P(x_{2n+1}) \to \bar{x}$.

Then

$$d(x_0, \bar{x}) \leq \sum_{i=1}^{2n+1} d(x_i, x_{i-1}) + d(x_{2n+1}, \bar{x})$$

$$
\begin{aligned}
&= \sum_{i=1}^{2n+1} d(P(x_{i-1}), x_{i-1}) + d(x_{2n+1}, \bar{x}) \\
&\leq \sum_{i=o}^{2n} c^i d(x_1, x_0) + d(x_{2n+1}, \bar{x}) \\
&\leq \sum_{i=o}^{2n} c^i d'(x_0, T_1 x_0) + d(x_{2n+1}, \bar{x}) \\
&\leq \frac{1}{1-c} d'(x_0, T_1 x_0) + d(x_{2n+1}, \bar{x}).
\end{aligned}
$$

Taking the limit $n \to \infty$, we have the desired result.

It follows similar that $\bar{x}$ is the common fixed point of pair of T_1 and T_2, i.e. for $\bar{x} \in X_{T_2}$ and projection scheme $P(x_{2n+2}) \to \bar{x}$. We have

$$
d(x_0, \bar{x}) \leq \frac{1}{1-c} d'(x_0, T_2 x_0) + d(x_{2n+2}, \bar{x}).
$$

So, we obtain the desired result, if $n \to \infty$.

This completes the proof of the Theorem. □

Theorem 2.4. (Generalized and Extended Anti-Collage Theorem) *Let (X, d) be a complete metric space and $T_1, T_2 : X \to H(X)$ be a pair of contraction multifunctions with contractivity factor $c \in [0, 1)$.*

Let

$$
X_{T_1} = \{x \in X : x \in T_1 x\}
$$

and

$$
X_{T_2} = \{x \in X : x \in T_2 x\}
$$

be the sets of all fixed points of T_1 and T_2, respectively.

Then

$$
d'(x, T_1 x) \leq (1 + c) d'(x, X_{T_1})
$$

and

$$
d'(x, T_2 x) \leq (1 + c) d'(x, X_{T_2}).
$$

Proof. From Lemma 1.1, for all $y \in X_{T_1}$, we have:

$$
\begin{aligned}
d'(x, T_1 x) &\leq d(x, y) + d'(y, T_1 x) \\
&\leq d(x, y) + h(T_1 y, T_1 x) \\
&\leq (1 + c) d(x, y).
\end{aligned}
$$

The desired result follows by taking the infimum.
Similarly we can established that for all $y \in X_{T_2}$ we have

$$d'(x, T_2x) \leq (1 + c)d(x, y).$$

This completes the proof of the theorem. □

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