COMMON FIXED POINT THEOREMS FOR
CONTRACTIVE MULTIFUNCTIONS

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Abstract: In this paper, we generalize the contractive multifunctions results given by H.E. Kunze et al, see [3]. Further generalization of collage theorem has also been discussed.

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1. Introduction and Preliminaries

In 1970, Covitz Nadler (see [7]) gave the following results “Multi-valued contraction mappings in generalized metric spaces”. Using this result, H.E. Kunze et al (see [3]) introduce an iterative method involving projections that guarantees convergence, from any starting point \( x_0 \in X \) to a point \( x \in X_T \), the set of all fixed points of a multifunction operator \( T \). Here multifunction operator \( T : X \Rightarrow Y \), is a set-valued mappings from a space \( X \) to the power set \( 2^Y \).

We consider multifunctions \( T \) that satisfy the following contractivity condition:

\[
\text{There exists a } c \in [0, 1) \text{ such that } d_h(Tx, Ty) \leq cd(x, y) \text{ for all } x, y \in X.
\]

Here \( d_h \) denotes the Hausdorff metric. Then there exists a fixed point \( \bar{x} \in X \) such that \( \bar{x} \in T\bar{x} \), \( \bar{x} \) is not necessarily unique.

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The purpose of this paper is to obtain common fixed point for pair of contractive maps in the setting of procedure given by H.E. Kunze et al [3]. Throughout this paper, $T$ is a multifunction operator and $X_T$, the set of all fixed points of a multifunction operator $T$.

We recall some definitions of Hausdorff distance and some of their properties as well as properties of contractive functions.

**Definition 1.1.** (see [3]) Let $d(x, y)$ denote the Euclidean distance. We shall also let $H(X)$ denote the space of all compact subsets of $X$ and $d_h(A, B)$ the Hausdorff distance between $A$ and $B$, that is

$$d_h(A, B) = \max \left\{ \max_{x \in A} d'(x, B), \max_{x \in B} d'(x, A) \right\},$$

where $d'(x, A)$ is the usual distance between the point $x$ and the set $A$, i.e

$$d'(x, A) = \min_{y \in A} d(x, y).$$

In the following we will denote by $h(A, B) = \max_{x \in A} d'(x, B)$.

It is well known that the space $(H(X), d_h)$ is a complete metric space if $X$ is complete, see [5].

**Lemma 1.1.** (see [3]) Let $(X, d)$ be a metric space.

1. For all $x, y \in X$, $C \subset X$, we have

$$d'(x, C) \leq d(x, y) + d'(y, C).$$

2. If $A \subset B$, then $h(C, A) \geq h(C, B)$ and

$$h(A, C) \leq h(B, C), \quad \text{for all } C \subset X.$$

3. For all $x, y \in X$, $A, B \subset X$, we have

$$d'(x, A) \leq d(x, y) + d'(y, B) + h(B, A).$$

4. For all $x \in X$ and $A, B \subset X$, we have

$$d'(x, A) \leq d'(x, B) + h(B, A)$$

5. Suppose that $(X, ||||)$ be a real normed space and $E \subset X$ be a convex subset of $X$. Let $A_1, A_2, B_1, B_2 \subset E$, $\lambda_i \in [0, 1]$, and $\sum_i \lambda_i = 1$. 
Then
\[ d_h(\lambda_1 A_1 + \lambda_2 A_2, \lambda_1 B_1 + \lambda_2 B_2) \leq \lambda_1 d_h(A_1, B_1) + \lambda_2 d_h(A_2, B_2). \]

6. Let \( A_i, B_i \subset E \) and \( \lambda_i \in [0,1] \) for \( i = 1, 2, ..., N \), \( \sum \lambda_i = 1 \).
Then
\[ d_h \left( \sum_i \lambda_i A_i, \lambda_i B_i \right) \leq \sum_i \lambda_i d_h(A_i, B_i). \]

7. Let \( A, B, C, \subset E, \lambda_1, \lambda_2 \in [0,1] \) such that \( \lambda_1 + \lambda_2 = 1 \). Suppose that
\( A, B, C \) are compact and \( A \) is convex.
Then
\[ d_h(A, \lambda_1 B + \lambda_2 C) \leq \lambda_1 d_h(A, B) + \lambda_2 d_h(A, C). \]

The following examples state how to calculate the Hausdorff distance when
the sets are intervals.

**Example 1.1.** Let \( A = [a_1, a_2] \) and \( B = [b_1, b_2] \). Then
\[ d_h(A, B) = \max\{|b_1 - a_1|, |b_2 - a_2|\}. \]

**Theorem 1.1.** (Banach) Let \( (X, d) \) be a complete metric space. Also let
\( T : X \to X \) be a contraction mapping with contraction factor \( c \in [0,1) \), i.e. for
all \( x, y \in X \), \( d(Tx, Ty) \leq cd(x, y) \).
Then there exists a unique \( \bar{x} \in X \) such that \( \bar{x} = Tx \).
Moreover, for any \( x \in X \), \( d(T^n x, \bar{x}) \to 0 \) as \( n \to \infty \).

**Theorem 1.2.** (Collage Theorem, see \( [2], [4] \)) Let \( (X, d) \) be a complete
metric space and \( T : X \to X \) a contraction mapping with contraction factor
\( c \in [0,1) \).
Then for any \( x \in X \),
\[ d(x, \bar{x}) \leq \frac{1}{1 - c} d(x, Tx), \]
where \( \bar{x} \) is the fixed point of \( T \).

**Theorem 1.3.** (Anti-Collage Theorem, see \( [9] \)) Assume the conditions of
the Collage Theorem above.
Then for any \( x \in Y \),
\[ d(x, \bar{x}) \geq \frac{1}{1 + c} d(x, Tx). \]
Theorem 1.4. (Covitz-Nadler, see [6], [7]) Let \((X, d)\) be a complete metric space and suppose that \(T : X \to H(X)\) be a set valued contraction mapping, i.e. \(d_h(Tx, Ty) \leq cd(x, y)\) for all \(x, y \in X\), and \(c \in [0, 1)\).

Then there exists \(\bar{x} \in X\) such that \(\bar{x} \in T\bar{x}\).

Note that the fixed point \(\bar{x}\) is not necessarily unique.

2. Common Fixed Point Theorems

In this section we shall prove common fixed point theorem for pair of contraction multifunctions.

The following theorem is extends and improves the Theorem 6 from [3].

Theorem 2.1. Let \((X, d)\) be a complete metric space and \(T_1, T_2 : X \to H(X)\) be a pair of contraction multifunctions such that

\[d_h(T_1(x), T_2(y)) \leq Kd(x, y)\]

for all \(x, y \in X\), with \(K \in [0, 1)\).

Then:

1. For all \(x_0 \in X\) there exists a common point \(\bar{x} \in X\) such that

\[(x_{2n+2}) = P(x_{2n+1}) \to \bar{x}\]

and

\[x_{2n+3} = P(x_{2n+2}) \to \bar{x},\]

when \(n \to +\infty\).

2. \(\bar{x}\) is a common fixed point, that is, \(\bar{x} \in T_1\bar{x}\) and \(\bar{x} \in T_2\bar{x}\).

Proof. Starting from a point \(x_0 \in X\), take the projection \(P(x_0)\) of the point on the set \(T_1x_0\).

Therefore

\[d'(x_0, T_1x_0) = d(x_0, P(x_0)).\]

Let \(x_1 = P(x_0)\) and take the projection of \(x_1\) on the set \(T_1x_1\). We have

\[d(x_2, x_1) = d(P(x_1), x_1) = d'(x_1, T_1x_1)\]

\[= d'(P(x_0), T_1x_1)\]

\[\leq h(T_1x_0, T_1x_1)\]
\[
\begin{align*}
&\leq cd(x_0, x_1) \\
&= cd'(x_0, T_1 x_0),
\end{align*}
\]
and for each \( n \in N \), we have
\[
\begin{align*}
d(x_{2n+1}, x_{2n+2}) &= d'(x_{2n+1}, T_1 x_{2n+1}) \\
&\leq c^{2n+1} d'(x_0, T_1 x_0) \\
&= c^{2n+1} d(x_0, x_1).
\end{align*}
\]
Then for all \( n, m \in N, n_0 \leq (2n + 1) < m \), we have
\[
\begin{align*}
d(x_{2n+1}, x_m) &\leq \sum_{i=2n+1}^{m-1} d(x_i, x_{i+1}) \\
&\leq \sum_{i=2n+1}^{m-1} c^{2n+1+i} d(x_0, x_1) \\
&= c^{2n+1} d(x_0, x_1) \sum_{i=0}^{m-1} c^i \\
&= c^{2n+1} \left( \frac{1 - c^m}{1 - c} \right) d(x_0, x_1) \\
&\leq \frac{c^{n_0}}{1 - c} d(x_0, x_1).
\end{align*}
\]
Therefore the sequence \( \{x_{2n+1}\} \) is a Cauchy sequence in \( X \).
Since \( X \) is complete, there exists an \( \bar{x} \in X \) such that
\( x_{2n+1} \to \bar{x} \).

Since
\[
\begin{align*}
d'(\bar{x}, T_1 \bar{x}) &\leq d(\bar{x}, x_{2n+1}) + d'(x_{2n+1}, T_1 x_{2n+1}) + h(T_1 x_{2n+1}, T_1 \bar{x}) \\
&\leq d(\bar{x}, x_{2n+1}) + c^{2n+1} d(x_0, T_1 x_0) + cd(x_{2n+1}, \bar{x}) \\
&= (1 + c) d(\bar{x}, x_{2n+1}) + c^{2n+1} d(x_0, T_1 x_0),
\end{align*}
\]
it follows that \( \bar{x} \in T_1 \bar{x} \).
Similarly it can be established that \( \bar{x} \in T_2 \bar{x} \), that is \( \bar{x} \) is a common fixed point of pair of \( T_1 \) and \( T_2 \).
This completes the proof of the theorem. \( \square \)
Theorem 2.2. Let $(X, d)$ be a complete metric space. Then $X_{T_1}$ and $X_{T_2}$ are complete.

Proof. Let $x_{2n+1} \in X_{T_1}$ be a Cauchy sequence of points of $X_{T_1}$. Since $x_{2n+1} \in X$ and $X$ is complete then there exists a point $\overline{x} \in X$ such that $d(x_{2n+1}, \overline{x}) \to 0$.

Now we have

$$d(\overline{x}, T_1 \overline{x}) \leq d(\overline{x}, x_{2n+1}) + d'(x_{2n+1}, T_1 x_{2n+1}) + h(T_1 x_{2n+1}, T_1 \overline{x})$$

$$\leq 2d(\overline{x}, x_{2n+1}).$$

Similarly it can be proved that for $X_{T_2}$. $\square$

Remark 2.1. Let $(X, d)$ be a compact metric space. In this case it is easy to prove that $X_{T_1}$ and $X_{T_2}$ are compact.

Now, we will generalize some results in [3], for pair of contraction multifunctions.

Theorem 2.3. (Generalized and Extended Collage Theorem) Let $(X, d)$ be a complete metric space and $T_1, T_2 : X \to H(X)$ be a pair of contraction multifunctions with contractivity factor $c \in [0, 1]$.

Then for all $x \in X$ there exists a common fixed point $\overline{x}$ such that either

$$d(x_0, \overline{x}) \leq \frac{d'(x, T_1 x)}{1 - c}$$

or

$$d(x_0, \overline{x}) \leq \frac{d'(x, T_2 x)}{1 - c}$$

so that either

$$d'(x, X_{T_1}) \leq \frac{d'(x, T_1 x)}{1 - c}$$

or

$$d'(x, X_{T_2}) \leq \frac{d'(x, T_2 x)}{1 - c}.$$

Proof. For any $x \in X$, let $x_0 = x$.

By the use of Theorem 2.1, there exists a point $\overline{x} \in X_{T_1}$ such that the projection scheme $P(x_{2n+1}) \to \overline{x}$.

Then

$$d(x_0, \overline{x}) \leq \sum_{i=1}^{2n+1} d(x_i, x_{i-1}) + d(x_{2n+1}, \overline{x})$$
\[ \begin{align*}
&= \sum_{i=1}^{2n+1} d(P(x_{i-1}), x_{i-1}) + d(x_{2n+1}, \bar{x}) \\
&\leq \sum_{i=1}^{2n} c^i d(x_1, x_0) + d(x_{2n+1}, \bar{x}) \\
&\leq \sum_{i=0}^{2n} c^i d(x_0, T_1 x_0) + d(x_{2n+1}, \bar{x}) \\
&\leq \frac{1}{1 - c} d'(x_0, T_1 x_0) + d(x_{2n+1}, \bar{x}).
\end{align*} \]

Taking the limit \( n \to \infty \), we have the desired result.

It follows similar that \( \bar{x} \) is the common fixed point of pair of \( T_1 \) and \( T_2 \), i.e. for \( \bar{x} \in X_{T_2} \) and projection scheme \( P(x_{2n+2}) \to \bar{x} \). We have

\[ d(x_0, \bar{x}) \leq \frac{1}{1 - c} d'(x_0, T_2 x_0) + d(x_{2n+2}, \bar{x}). \]

So, we obtain the desired result, if \( n \to \infty \).

This completes the proof of the Theorem. \( \square \)

**Theorem 2.4.** (Generalized and Extended Anti-Collage Theorem) Let \((X, d)\) be a complete metric space and \( T_1, T_2 : X \to H(X) \) be a pair of contraction multifunctions with contractivity factor \( c \in [0, 1) \).

Let

\[ X_{T_1} = \{x \in X : x \in T_1 x\} \]

and

\[ X_{T_2} = \{x \in X : x \in T_2 x\} \]

be the sets of all fixed points of \( T_1 \) and \( T_2 \), respectively.

Then

\[ d'(x, T_1 x) \leq (1 + c)d'(x, X_{T_1}) \]

and

\[ d'(x, T_2 x) \leq (1 + c)d'(x, X_{T_2}). \]

**Proof.** From Lemma 1.1, for all \( y \in X_{T_1} \), we have:

\[ d'(x, T_1 x) \leq d(x, y) + d'(y, T_1 x) \]

\[ \leq d(x, y) + h(T_1 y, T_1 x) \]

\[ \leq (1 + c)d(x, y). \]
The desired result follows by taking the infimum. Similarly we can establish that for all $y \in X_{T_2}$ we have

$$d'(x, T_2x) \leq (1 + c)d(x, y).$$

This completes the proof of the theorem. \hfill \Box

References


