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NOTES ON FUZZY ORDERED IDEALS AND FUZZY ORDERED FILTERS IN ORDERED Γ-GROUPOIDS

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Abstract: The fuzzy ordered ideals and fuzzy ordered filters in ordered Γ-groupoids are discussed in this note. A necessary and sufficient condition has been established for fuzzy ordered ideals and fuzzy ordered filters to be characteristic.

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1. Introduction and Prerequisites

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [14] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [14], several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: In 1971, Rosenfeld [9] was

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the first who studied fuzzy sets in the structure of groups. Fuzzy semigroups have been first considered by Kuroki [4, 5, 6, 7], and fuzzy ordered groupoids and ordered semigroups, by Kehayopulu and Tsingelis [2, 3]. In 2008, Shabir and Khan [12] defined fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterized ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. In 2009, Majumder and Sardar [8] studied fuzzy ideals and fuzzy ideal extensions in ordered semigroups. In 2010, Chinram and Saelee [1] studied fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups. In 2010, Shah and Rehman [13] introduced Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids which are in fact a generalization of ideals and bi-ideals of AG-groupoids and studied some characteristics of Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids.

As we know, Γ-groupoids are a generalization of groupoids. This paper is a sequel to our study [3] of fuzzification in ordered groupoids.

Our aim in this paper is fourfold.

1. To introduce the notions of fuzzy ordered ideals and fuzzy ordered filters in ordered Γ-groupoids.
2. To characterize the properties of fuzzy ordered ideals and fuzzy ordered filters in ordered Γ-groupoids.
3. To characterize the relationship between ordered ideals and fuzzy ordered ideals.
4. To characterize the relationship between ordered filters and fuzzy ordered filters.

To present the main theorems we discuss some elementary definitions that we use later.

Definition 1.1. Let M be a set. A *fuzzy subset* of M is an arbitrary mapping $f\colon M \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

Definition 1.2. Let M be a set and $A \subseteq M$. The *characteristic mapping* $f_A\colon M \to [0,1]$ defined via

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic mapping, f_A is a mapping of M into $\{0,1\} \subset [0,1]$. Hence f_A is a fuzzy subset of M.

Definition 1.3. Let M and Γ be any two nonempty sets. Then (M, Γ) is called a Γ-*groupoid* [11] if there exists a mappings $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$. A nonempty subset K of M is called a Γ-*subgroupoid* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Definition 1.4. A partially ordered Γ-groupoid $(M, \Gamma, \leq)$ is called an *ordered* Γ-*groupoid* [10] if for any $a, b, c \in M$ and $\gamma \in \Gamma, a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Definition 1.5. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A nonempty subset A of M is called an *ordered left ideal* of M if

L1. $M\Gamma A \subseteq A$.

L2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

A nonempty subset A of M is called an *ordered right ideal* of M if

R1. $A\Gamma M \subseteq A$.

R2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

A nonempty subset A of M is called an *ordered ideal* of M if it is both an ordered left and an ordered right ideal of M. That is,

I1. $M\Gamma A \subseteq A$ and $A\Gamma M \subseteq A$.

I2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

Definition 1.6. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A Γ-subgroupoid F of M is called an *ordered filter* of M if

F1. For any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $a, b \in F$.

F2. For any $b \in M$ and $a \in F, a \leq b$ implies $b \in F$.

Definition 1.7. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. For $H \subseteq M$, we define

$$(H] = \{t \in M \mid t \leq h \text{ for some } h \in H\}.$$

Definition 1.8. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A fuzzy subset f of M is called a *fuzzy ordered left ideal* of M if

FL1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.

FL2. $f(a\gamma b) \geq f(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered right ideal* of M if

FR1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.

FR2. $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered ideal* of M if it is both a fuzzy ordered left and a fuzzy ordered right ideal of M. That is,

FI1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.

FI2. $f(a\gamma b) \geq f(b)$ and $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.9. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A fuzzy subset f of M is called a *fuzzy ordered filter* of M if

FF1. For any $a, b \in M, a \leq b$ implies $f(a) \leq f(b)$.

FF2. $f(a\gamma b) = \min\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.10. Let (M, Γ) be a Γ-groupoid. A fuzzy subset f of M is called a *fuzzy* Γ*-subgroupoid* of M if $f(a\gamma b) \geq \min\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.11. Let (M, Γ) be a Γ-groupoid and f a fuzzy subset of M. The mapping

$$f' \colon M \to [0,1] \text{ defined via } f'(x) = 1 - f(x)$$

is a fuzzy subset of M called the *complement* of f in S.

Definition 1.12. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A fuzzy subset f of M is called *prime* if $f(a\gamma b) \leq \max\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

2. Lemmas

Lemma 2.1. *If $(M, \Gamma, \leq)$ is an ordered Γ-groupoid and $\emptyset \neq A \subseteq M$, then the characteristic mapping $f_{(A]}$ is a fuzzy subset of M satisfying the following condition:*

$$x, y \in M, x \leq y \Rightarrow f_{(A]}(x) \geq f_{(A]}(y).$$

Proof. Obviously, $f_{(A]}$ is a fuzzy subset of M. Let $x, y \in M$ be such that $x \leq y$. If $y \notin (A]$, then $f_{(A]}(y) = 0$ and so $f_{(A]}(x) \geq 0 = f_{(A]}(y)$. Let $y \in (A]$. Then $f_{(A]}(y) = 1$. Since $y \in (A]$, there exists $a \in A$ such that $y \leq a$. We have $x \leq a$, whence $x \in (A]$. Then $f_{(A]}(x) = 1$ and so $f_{(A]}(x) = 1 \geq 1 = f_{(A]}(y)$. $\square$

Lemma 2.2. *Let (M, Γ) be a Γ-groupoid and f a fuzzy subset of M. Then the following statements are equivalent:*

(i) $f(x\gamma y) = \min\{f(x), f(y)\}$ *for all* $x, y \in M$ *and* $\gamma \in \Gamma$.

(ii) $f^{\square}(x\gamma y) = \max\{f^{\square}(x), f^{\square}(y)\}$ *for all* $x, y \in M$ *and* $\gamma \in \Gamma$.

Proof. Assume that $f(x\gamma y) = \min\{f(x), f(y)\}$ for $x, y \in M$ and $\gamma \in \Gamma$. Without loss of generality, we may assume that $f(x\gamma y) = f(x)$. Then $f(x) \leq f(y)$, so $f^{\square}(x\gamma y) = 1 - f(x\gamma y) = 1 - f(x) = f^{\square}(x)$ and $f^{\square}(x) = 1 - f(x) \geq 1 - f(y) = f^{\square}(y)$. Hence $f^{\square}(x\gamma y) = \max\{f^{\square}(x), f^{\square}(y)\}$.

Conversely, assume that $f^{\square}(x\gamma y) = \max\{f^{\square}(x), f^{\square}(y)\}$ for $x, y \in M$ and $\gamma \in \Gamma$. Without loss of generality, we may assume that $f^{\square}(x\gamma y) = f^{\square}(x)$. Then $f^{\square}(x) \geq f^{\square}(y)$, so $1 - f(x\gamma y) = 1 - f(x)$ and $1 - f(x) \geq 1 - f(y)$. Thus $f(x\gamma y) = f(x)$ and $f(x) \leq f(y)$. Hence $f(x\gamma y) = \min\{f(x), f(y)\}$. $\square$

3. Main Results

Proposition 3.1. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $\emptyset \neq A \subseteq M$. Then $A = (A]$ if and only if the fuzzy subset f_A of M has the following property:*

$$x, y \in M, x \leq y \Rightarrow f_A(x) \geq f_A(y).$$

Proof. It follows from Lemma 2.1.

Conversely, let $x \in (A]$. Then there exists $a \in A$ such that $x \leq a$. By the hypothesis, we have $f_A(x) \geq f_A(a)$. Since $a \in A$, we have $f_A(a) = 1$. Then $f_A(x) = 1$ and so $x \in A$. Hence $(A] \subseteq A$. We can easily see that $A \subseteq (A]$. Therefore $A = (A]$. $\square$

Proposition 3.2. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $\emptyset \neq L \subseteq M$. Then L is an ordered left ideal of M if and only if the fuzzy subset f_L is a fuzzy ordered left ideal of M.*

Proof. By the definition of characteristic mapping, f_L is a mapping of M into $\{0,1\} \subset [0,1]$. Hence f_L is a fuzzy subset of M.

FL1: Let $x, y \in M$ be such that $x \leq y$. If $y \notin L$, then $f_L(y) = 0$ and so $f_L(x) \geq 0 = f_L(y)$. Let $y \in L$. Then $f_L(y) = 1$. Since $x \leq y \in L$ and L is an ordered left ideal of M, we have $x \in L$. Thus $f_L(x) = 1$ and so $f_L(x) = 1 \geq 1 = f_L(y)$.

FL2: Let $x, y \in M$ and $\gamma \in \Gamma$. If $y \notin L$, then $f_L(y) = 0$ and so $f_L(x\gamma y) \geq 0 = f_L(y)$. Let $y \in L$. Then $f_L(y) = 1$. Since $x\gamma y \in M\Gamma L \subseteq L$, we have $f_L(x\gamma y) = 1$. Thus $f_L(x\gamma y) = 1 \geq 1 = f_L(y)$.

Therefore the fuzzy subset f_L is a fuzzy ordered left ideal of M.

Conversely, we shall show that L is an ordered left ideal of M.

L1: Let $x \in M, y \in L$ and $\gamma \in \Gamma$. Since f_L is a fuzzy ordered left ideal of M, we have $f_L(x\gamma y) \geq f_L(y)$. Since $y \in L$, we have $f_L(y) = 1$. Thus $f_L(x\gamma y) = 1$ and so $x\gamma y \in L$.

L2: Let $x \in M$ and $y \in L$ be such that $x \leq y$. Since f_L is a fuzzy ordered left ideal of M, we have $f_L(x) \geq f_L(y)$. Since $y \in L$, we have $f_L(y) = 1$. Thus $f_L(x) = 1$ and so $x \in L$.

Therefore L is an ordered left ideal of M. □

Corollary 3.3. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $\emptyset \neq I \subseteq M$. Then I is an ordered ideal of M if and only if the fuzzy subset f_I is a fuzzy ordered ideal of M.*

Proposition 3.4. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $\emptyset \neq F \subseteq M$. Then F is an ordered filter of M if and only if the fuzzy subset f_F is a fuzzy ordered filter of M.*

Proof. We shall show that the fuzzy subset f_F is a fuzzy ordered filter of M.

FF1: Let $x, y \in M$ be such that $x \leq y$. If $x \notin F$, then $f_F(x) = 0$ and so $f_F(x) = 0 \leq f_F(y)$. Let $x \in F$. Then $f_F(x) = 1$. Since $y \geq x \in F$ and F is an ordered filter of M, we have $y \in F$. Thus $f_F(y) = 1$ and so $f_F(x) = 1 \leq 1 = f_F(y)$.

FF2: Let $x, y \in M$ and $\gamma \in \Gamma$. If $x\gamma y \notin F$, then $f_F(x\gamma y) = 0$. Since $x\gamma y \notin F$, we have $x \notin F$ or $y \notin F$. Thus $f_F(x) = 0$ or $f_F(y) = 0$, so $\min\{f_F(x), f_F(y)\} = 0$. Hence $f_F(x\gamma y) = 0 = \min\{f_F(x), f_F(y)\}$. Let $x\gamma y \in F$. Then $f_F(x\gamma y) = 1$. Since $x\gamma y \in F$, we have $x \in F$ and $y \in F$. Thus $f_F(x) = 1$ and $f_F(y) = 1$, so $\min\{f_F(x), f_F(y)\} = 1$. Hence $f_F(x\gamma y) = 1 = \min\{f_F(x), f_F(y)\}$.

Therefore the fuzzy subset f_F is a fuzzy ordered filter of M.

Conversely, let $x, y \in F$ and $\gamma \in \Gamma$. Since f_F is a fuzzy ordered filter of M, we have $f_F(x\gamma y) = \min\{f_F(x), f_F(y)\}$. Suppose $x\gamma y \notin F$. Then $f_F(x\gamma y) = 0$,

so $\min\{f_F(x), f_F(y)\} = 0$. Thus $f_F(x) = 0$ or $f_F(y) = 0$, so $x \notin F$ or $y \notin F$. It is impossible, hence $x\gamma y \in F$. Therefore F is a Γ-subgroupoid of M.
F1: Let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in F$. Since f_F is a fuzzy ordered filter of M, we have $f_F(x\gamma y) = \min\{f_F(x), f_F(y)\}$. Since $x\gamma y \in F$, we have $f_F(x\gamma y) = 1$. Thus $\min\{f_F(x), f_F(y)\} = 1$ and so $f_F(x) = f_F(y) = 1$. Hence $x, y \in F$.
F2: Let $x \in F$ and $y \in M$ be such that $x \leq y$. Since f_F is a fuzzy ordered filter of M, we have $f_F(x) \leq f_F(y)$. Since $x \in F$, we have $f_F(x) = 1$. Thus $f_F(y) = 1$ and so $y \in F$.

Therefore F is an ordered filter of M. □

Proposition 3.5. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and f a fuzzy subset of M. Then f is a fuzzy ordered filter of M if and only if the complement $f^{\complement}$ of f is a prime fuzzy ordered ideal of M.*

Proof. We shall show that the complement $f^{\complement}$ of f is a prime fuzzy ordered ideal of M.
FI1: Let $x, y \in M$ be such that $x \leq y$. Since f is a fuzzy ordered filter of M, we have $f(x) \leq f(y)$. Then $-f(x) \geq -f(y)$, so $f^{\complement}(x) = 1 - f(x) \geq 1 - f(y) = f^{\complement}(y)$.
FI2: Let $x, y \in M$ and $\gamma \in \Gamma$. Since f is a fuzzy ordered filter of M, we have $f(x\gamma y) = \min\{f(x), f(y)\}$. By Lemma 2.2, we have $f^{\complement}(x\gamma y) = \max\{f^{\complement}(x), f^{\complement}(y)\}$. Thus $f^{\complement}(x\gamma y) \geq f^{\complement}(x)$ and $f^{\complement}(x\gamma y) \geq f^{\complement}(y)$.

Hence the complement $f^{\complement}$ of f is a fuzzy ordered ideal of M.

In the proof of FI2, we have that $f^{\complement}(x\gamma y) \leq \max\{f^{\complement}(x), f^{\complement}(y)\}$ for all $x, y \in M$ and $\gamma \in \Gamma$. Therefore the complement $f^{\complement}$ of f is a prime fuzzy ordered ideal of M.

Conversely, we shall show that f is a fuzzy ordered filter of M.
FF1: Let $x, y \in M$ be such that $x \leq y$. Since $f^{\complement}$ is a fuzzy ordered ideal of M, we have $f^{\complement}(x) \geq f^{\complement}(y)$. Then $1 - f(x) \geq 1 - f(y)$ and so $f(x) \leq f(y)$.
FF2: Let $x, y \in M$ and $\gamma \in \Gamma$. Since $f^{\complement}$ is a prime fuzzy ordered ideal of M, we have $f^{\complement}(x\gamma y) \geq f^{\complement}(x), f^{\complement}(x\gamma y) \geq f^{\complement}(y)$, and $f^{\complement}(x\gamma y) \leq \max\{f^{\complement}(x), f^{\complement}(y)\}$. Hence $f^{\complement}(x\gamma y) = \max\{f^{\complement}(x), f^{\complement}(y)\}$. By Lemma 2.2, we have $f(x\gamma y) = \min\{f(x), f(y)\}$.

Therefore f is a fuzzy ordered filter of M. □

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