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AN ELEMENTARY CALCULUS APPROACH TO THE REGULAR STURM-LIOUVILLE OPERATOR

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Abstract: It is known that the minimum and minimizer of the Rayleigh quotient provide the eigenvalues and eigenfunctions of a regular Srurm-Liouville operator. This expository paper uses elementary calculus method and some of the fundamental theorems of functional analysis, namely Arzela Ascoli's theorem and Lax-Milgram theorem to show the same result.

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1. Introduction

Sturm-Liouville theory, due to, Jacques C. Sturm and Joseph Liouville, goes back to 19^{th} century [1], [2]. There is a plenty of literature available on the theory, for example [3], [5], [6], [7], and citations therein. The Sturm-Liouville boundary value problem (SLBVP) consists of

$$-(p(x)u(x)')' + q(x)u(x) = \lambda s(x)u(x). \tag{1}$$

The coefficients satisfy $p(x) \in C^1[a,b]$, $q, r \in C^0[a,b]$ with $q \geq 0, s > 0$. We

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assume the Dirichlet boundary conditions throughout

$$u(a) = u(b) = 0.$$

and if the function $p(x) > 0$ on $[a, b]$, we call it a *regular Sturm-Liouville BVP*. The problem arises frequently in solving boundary value problems for several classes of partial differential equations [6] for example solving heat equation or wave equation by separation of variables. The other examples of Sturm Liouville boundary value problems being Hermite equations, Airy equations, Legendre equations etc. The one dimensional Schrödinger equation [7] that arises in quantum mechanics is given by

$$\frac{\hbar}{2m}\psi'' + (V(x) - E)\psi = 0,$$

where ψ is the wave function of the particle, $V(x)$ describes the potential field, m the mass of the particle and E is the energy level (eigenvalue) and $\hbar$ is the Plank's constant over 2π. Let y be the strength of an acoustic field (e.g. sonar) traveling down a waveguide formed naturally by the sea bed and the stratification of the water above it, ω a fixed frequency, λ the wavenumber, and let $c(x)$ represent the depth dependent speed of sound. The Sturm-Liouville differential equation governing the strength of acoustic field is taken from, see [7] is given by

$$y'' + \left(\frac{\omega}{c(x)^2} - \lambda^2\right) y = 0,$$

with boundary conditions $y(a) = 0 = y(b)$.

1.1. Sturm-Liouville Operator

The equation can be put into an operator form so as to use the spectral properties of the associated operator. The operator defined below is called a Sturm-Liouville operator

$$L(u) := \frac{-(p(x)u'(x))' + q(x)u(x)}{s(x)} = \lambda u(x). \tag{2}$$

The solution u is called the eigenfunction and the parameter λ is called the eigenvalue. The Sturm-Liouville theory is about understanding how many eigenvalues, λ, exist, what are the properties of the eigenfunctions and how the eigenfunctions vary when q, r, a or b are modified. We denote the eigenfunctions by u_k and their corresponding eigenvalues are denoted by λ_k. The above operator is symmetric [3], thus has real eigenvalues. It is known that the eigenvalues are simple, they are monotonically increasing, zeroes of eigenfuctions are separated, i.e. the k^{th} eigenfunction has exactly k zeros in $[a, b]$, between the zeros of u_k there lies a zero of u_{k+1}, the eigenvalues are real, the eigenfunctions forms a complete orthonormal system with respect to the weight function $w(x)$. Thus the eigenfunctions provide a generalized Fourier series for a dense subspace of $L_2[a, b]$, and hence for the space $L_2[a, b]$ itself. The inner product on the Hilbert space is assumed to be

$$< u, v >:= \int_a^b suv\, dx,$$

unless otherwise mentioned. The pair (λ_n, u_n), will be called an eigenpair in the sequel.

Example 1. The eigenfunctions of the following elementary differential equation

$$-u'' = ku$$

subject to Periodic boundary conditions $u(-\pi) = u(\pi),\ u'(-\pi) = u'(\pi)$ constitute the basis for the *Fourier series.* In fact any function $f \in L_2[-\pi, \pi]$ can be approximated by trigonometric functions which are the eigenfunctions of above equation yielding a Fourier series.

Example 2. The following example of regular Sturm-Liouville BVP arises while solving a partial differential equations, using separation of variables

$$-u''(x) = \lambda u(x), \quad u(0) = 0 = u(L).$$

The eigenvalues for this problem are $\lambda_n = \frac{n^2\pi^2}{L},\ n = 1, 2, \cdots$ and the eigenfunctions are $u_n(x) = \sin \frac{n\pi x}{L}$. So $(n^2\pi^2, \sin n\pi)$ are eigenpairs. The completeness of the eigenfunctions provides a *Fourier Sine series.*

Example 3. Taking the boundary conditions $u(0) = 0 = u'(1)$ in the above equation, one obtains $\lambda_n = \frac{(2n-1)^2\pi^2}{4}$ and $u_n = \sin \frac{2n-1}{2}x$, where $n = 0, 1, 2 \cdots$. These eigenfunctions constitute a Fourier Sine series for functions in $L^2([0, 1])$.

In general, the completeness of eigenfunctions can be used to solve equations like

$$L(u) = f$$

where the function $f \in L_2[a, b]$ since $f(x)$ can be expanded as a generalized Fourier series

$$f(x) = \sum_n \alpha_n u_n.$$

This implies that

$$u := \sum_n \frac{c_n u_n}{\lambda_n}$$

solves the equation, $\lambda_n \neq 0$, which can be avoided. For, adding a c which is not an eigenvalue to the operator we get an operator $L_c := L + c$ such that $(L + c)u = Lu + cu = \lambda u + cu = (\lambda + c)u$. Thus the operator L and L_c have the same eigenfunctions.

An elementary calculus based approach, theorem (4.1), is provided based on some of the fundamental theorems of functional analysis.

1.2. Rayleigh Quotient

Rayleigh quotient due to Rayleigh and Courant-Fischer provides the variational characterization of the eigenvalues. The eigenvalue problem can be recast using a bilinear form $S(u, v)$ and Rayleigh quotient $R(u)$ [7, 6].

$$S(u, v) := \int_a^b \left((-pu')' + qu\right) vdx,$$

$$R(u) := \frac{S(u, u)}{< u, u >}. \tag{3}$$

where u and v range over some admissible functions satisfying the boundary conditions. Integration by parts and using the boundary conditions this can be written as

$$S(u, v) = \int_a^b (pu'v' + quv)dx.$$

It is clear that $R(u_n) = \lambda_n = S(u, u)$ for the eigenpairs (λ_n, u_n) of the SLBVP (1) [6]. The above equation shows that that all the eigenvalues of the SLBVP considered above have nonnegative eigenvalues which is the necessary and sufficient condition for the operator L, defined by (2), to be positive semidefinite. The quadratic form and Rayleigh quotient are defined for a much wider class of functions than $C^2[a, b]$ [7]. In fact we only need continuous functions for which u' exists almost everywhere such that $\int_a^b |u'|^2 dx < \infty$. This suggests a weaker formulation of (1).

2. Weaker Formulation

The SLBVP (1) can be put into a weaker equivalent form on a suitable Hilbert space. The Lax-Milgram theorem then guarantee a weak solution of SLBVP. Consider the following form of the SLBVP.

$$-(pu')' + qu = r \tag{4}$$

with homogeneous boundary conditions $u(a) = 0 = u(b)$. Let $p \in C^1[a,b]$, $q\ ,r \in C[a,b]$, such that $p(x) > 0$ and $q(x) \geq 0$ be given bounded functions. Multiplying the equation by v and integrate by parts, we obtain

$$\int_a^b (pu'v' + quv)dx = \int_a^b rvdx \tag{5}$$

for all $v \in C^1[a,b]$ with $v(a) = 0 = v(b)$. Conversely if $u \in C^2[a,b]$ satisfies (5), by integration by parts we obtain

$$\int_a^b [(pu')' - qu + r]vdx = 0,$$

for all $v \in C^1[a,b]$ such that $v(a) = 0 = v(b)$. Using the arguments shown as in [4]

$$(pu')' - qu + r = 0.$$

This implies that u satisfies the equation (4). Thus (5) indeed formulates (4) in a weaker form. Let

$$F(v) \quad := \quad \int_a^b rvdx,$$

then an equivalent formulation can be put into the following bilinear form

$$S(u,v) = F(v), \tag{6}$$

for all $v \in C^1[a,b]$ such that $v(a) = 0 = v(b)$, where $F(v)$ is a bounded linear functional on $C^1[a,b]$. The equivalence works for all $v \in C^1[a,b]$ which is not complete with respect to L_2 norm. It is therefore important to introduce the Sobolev space $H^1[a,b]$ of functions having weak derivatives so the Lax-Milgram theorem [8] can be applied which guarantees the existence of a unique solution of (6), a weak solution of (4).

Definition 1. A function $u \in L_2[a,b]$ is said to have a weak derivative $u' \in L_2[a,b]$ if

$$\int_a^b uv'dx = -\int_a^b u'vdx,$$

for all $v \in C^1[a,b]$ with $v(a) = 0 = v(b)$.

It turns out that the linear space of such functions is a Hilbert space as given by the following theorem taken from [4].

Theorem 2.1. *The linear space*

$$H^1[a,b] := \{u \in L_2[a,b] : u' \in L_2[a,b]\}$$

with the scalar product

$$(u,v)_{H^1} := \int_a^b (uv + u'v')dx$$

is a Hilbert space.

The following theorem is taken from [4]

Theorem 2.2. *The linear space*

$$H_0^1[a,b] := \{u \in H^1[a,b] : u(a) = 0 = u(b)\}$$

is a complete subspace of $H^1[a,b]$.

Thus we have a formulation of a weak solution as in [4].

Definition 2. A function $u \in H_0^1[a,b]$ is called a weak solution to the boundary value problem (4) if (6) is satisfied for all $v \in H_0^1[a,b]$.

It turns out that S is bounded and strictly coercive if $p > 0$ and $q \geq 0$, [4]. Also $F(v)$ is a bounded linear functional on $H_0^1[a,b]$, for

$$|F(v)| \leq ||r||_{L_2}||v||_{L^2} \leq ||r||_{L_2}||v||_{H^1}.$$

Under these conditions, we have the following theorem from functional analysis [4], [8], [9].

Theorem 2.3. (Lax-Milgram) *There exists a unique* $u \in H_0^1[a,b]$ *such that*

$$S(u,v) = F(v)$$

for all $v \in H_0^1[a,b]$.

The following lemma shows that each weak solution of such a boundary value problem is also a classical solution [4].

Lemma 1. *Each weak solution to the boundary value problem (4) is also a classical solution; i.e. it is twice continuously differentiable.*

The following lemma is standard lemma found on texts, it is presented here to fit into our settings.

Lemma 2. *Let q, r be as in (4). Suppose $u \in H_0^1[a,b]$, satisfies and satisfies lemma (5) but only for those functions $v \in H_0^1[a,b]$ and such that*

$$\int_a^b v(x)f_1(x)dx = \int_a^b v(x)f_2(x)dx = ... = \int_a^b v(x)f_n(x)dx = 0$$

for some continuous functions $f_1, ..., f_n$ satisfying $f(a) = f(b) = 0$ and

$$\int_a^b f_i(x)f_k(x)dx = 0 \qquad for \quad i \neq k.$$

Then u is a classical solution of

$$u'' + qu - r = \sum_{i=1}^{n} \gamma_i f_i \tag{7}$$

for some choice of constants $\gamma_1, \gamma_2, ...\gamma_n$.

Proof. Let $v \in H_0^1[a,b]$ be an arbitrary function. Define

$$\bar{v}(x) := v(x) - \sum_{i=1}^{n} f_i(x)\frac{\int_a^b f_i(x)v(x)dx}{\int_a^b f_i^2 dx} = v(x) - \sum_{i=1}^{n} c_i f_i(x),$$

where

$$c_i = \frac{\int_a^b f_i(x)v(x)dx}{\int_a^b f_i^2(x)dx}.$$

For every $k \in \{1, ..., n\}$, we have

$$\int_a^b \bar{v}(x)f_k(x)dx = 0$$

because of the orthogonality of the $f_k{}'s$. Therefore $\bar{v}$ is admissible in the assumption and we have

$$\int_a^b [(pu')' + qu - r]\bar{v}dx = 0. \tag{8}$$

Since,

$$v = \bar{v} + \sum_{i=1}^{n} c_i f_i$$

we have,

$$\int_a^b [(pu')' + qu - r]v dx = \int_a^b [(pu')' + qu - r]\bar{v} dx + \sum_{i=1}^{n} c_i \int_a^b [(pu')' + qu - r] f_i dx$$

Setting

$$\beta_i := \int_a^b [(pu')' + qu - r] f_i dx$$

and using (8) we have

$$\int_a^b [(pu')' + qu - r]v dx = \sum_{i=1}^{n} c_i \beta_i = \sum_{i=1}^{n} \beta_i \frac{\int_a^b f_i v dx}{\int_a^b f_i^2(x) dx} = \sum_{i=1}^{n} \gamma_i \int_a^b f_i v dx,$$

where

$$\gamma_i := \frac{\beta_i}{\int_a^b f_i^2(x) dx}.$$

This implies

$$\int_a^b [u'' + qu - r - \sum_{i=1}^{n} \gamma_i f_i] v dx = 0$$

for an arbitrary function $v \in H_0^1[a, b]$. Then lemma (2.3) implies the existence of a unique weak solution and theorem (1) implies that its actually a classical solution of

$$(pu')' + qu - r - \sum_{i=1}^{n} \gamma_i f_i = 0.$$

i.e.

$$(pu')' + qu - r = \sum_{i=1}^{n} \gamma_i f_i.$$

□

3. Arzelá-Ascoli's Theorem

Let X be a compact metric space and let $C(X)$ denote the space of continuous functions on X equipped with *sup* norm.

Theorem 3.1. (Arzelá-Ascoli's Theorem) *If a sequence $u_n \in C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*

Here bounded means $||u_n(x)|| \leq C$ for each $x \in X$ and each n. Equicontinuous means that to every $\epsilon > 0$ there exists a $\delta > 0$ such that for $x, y \in X$

$$|x - y| < \delta \implies |u_n(x) - u_n(y)| < \epsilon,$$

for all n. Since the solutions are expected to be of class $C^2[a, b]$, we need to have a norm on the space. The space $C^2([a, b])$ is naturally equipped with the norm

$$||u||_{C^2} = \sup_{x\in[a,b]} \{|u(x)|\} + \sup_{x\in[a,b]} \{|u'(x)|\} + \sup_{x\in[a,b]} \{|u''(x)|\}.$$

Equivalently,

$$||u||_{C^2} = ||u||_{C^0} + ||u'||_{C^0} + ||u''||_{C^0}.$$

We will use the following special case of (Arzelá-Ascoli theorem) [9].

Theorem 3.2. (Arzelá-Ascoli's Theorem, Special Form) *Let $u_n \in C^2[a, b]$ be such that*

$$\|u_n\|_{C^2} \leq M,$$

where M is a constant. Then there is a subsequence $(u_{n_k})_k$ and $u \in C^1[a, b]$: $\lim_k (u_{n_k})_k = u$ in the C^1 norm.

Proof. Since $u_n \in C^2[a, b]$, we have

$$||u_n||_{C^2} = \sup_{x\in[a,b]} \{|u_n(x)|\} + \sup_{x\in[a,b]} \{|u_n'(x)|\}| + \sup_{x,\in[a,b]} \{|u_n''(x)|\} \leq M,$$

and that they are uniformly bounded in C^1 norm too. Equicontinuity follows from,

$$|u_n(x) - u_n(y)| \;\; = \;\; |\int_x^y u_n'(s)ds| \leq \int_x^y |u_n'(s)|ds \leq M|x - y|,$$

by choosing $\delta = \epsilon/M$. So it has a uniformly convergent subsequence. Let it be $u_{n_k} \to u$. Again

$$|u_{n_k}'(x) - u_{n_k}'(y)| = |\int_x^y u_{n_k}''(s)ds| \leq \int_x^y |u_{n_k}''(s)|ds \leq M|x - y|.$$

So u_{n_k} has a uniformly convergent sequence too. Let it be $u_{n_{k_p}} \to v$. We show that $u' = v$. Consider

$$\frac{u_{n_{k_p}}(x+h) - u_{n_{k_p}}(x)}{h} = \frac{1}{h}\int_0^h u'_{n_{k_p}}(x+t)dt.$$

Taking limit on both sides as $p \to \infty$, because of uniform convergence

$$\frac{u(x+h) - u(x)}{h} = \frac{1}{h}\int_0^h v(x+t)dt,$$

taking limit as $h \to 0$, since v is continuous

$$u'(x) = v(x).$$

This implies that the sequence u_n has a convergent subsequence in C^1 norm. Hence the original theorem of Arzelá-Ascoli implies the existence of a convergent subsequence. □

Now the main theorem that shows the existence of eigenpair is given below based on these classical results.

4. Variational Principle

An elementary calculus based proof for the existence of eigenpair is presented using variational characterizations of eigenvalues, Rayleigh quotient, a special form of Arzelá-Ascoli's theorem and Lax-Milgram theorem. This provides the weak solution of the SLBVP (1), using theorem lemma 5 and lemma 2, this turns out to be twice continuously differentiable solution of (1). For simplicity we consider $p = 1$ in the SLBVP (1).

Theorem 4.1. (Variational Principle)

$$\lambda_n = min\ R(u),$$

for $n = 1, 2 \cdots$, over all twice continuously differentiable functions orthogonal to first $n-1$ eigenfunctions such that $u(a) = u(b) = 0$. Also the minimizer is u_n.

For $n = 1$, the orthogonality condition is dropped, so λ_1 is the global minimum of $R(u)$ over $C_0^2[a,b]$.

Proof. Let

$$\Lambda = inf\left\{R(u) : u \in C^2[a,b],\ u(a) = 0 = u(b),\ u^{\perp}\{u_1, u_2, \cdots u_{n-1}\}\right\}.$$

Since $R(u) \geq 0$, it is clear that the infimum of $R(u)$ exists. By the definition of Λ we can choose a sequence $\{v_m\}_{m=1}^{\infty}$ in $C^2[a,b]$ such that $< v_m, u_1 >=< v_m, u_2 >= ... < v_m, u_{n-1} >= 0$, such that $R(v_m) \to \Lambda$. Without the loss of generality let us assume that

$$< v_m, v_m >= 1.$$

By Arzela-Ascoli theorem there exists subsequence, yet denoted by $\{v_m\}$, of this sequence that converges to some limit $v \in C^1$ in the C^1 norm as in theorem 3.2. So $v \in H_0^1[a,b]$ as $C^1[a,b]$ is dense in $H_0^1[a,b]$. Before completing the proof of the theorem, the following lemma is needed.

Lemma 3. *We claim that for all $f \in H_0^1[a,b]$, this v satisfies*

$$\int_a^b v(-f'' + qf - \Lambda sf)dx = 0.$$

Since $v_m \to v$ uniformly, it follows that $< v, v >= 1$, i.e. v is not identically equal to zero.

$$\begin{aligned} \int_a^b v(-f'' + qf - \Lambda sf)dx &= 0 \\ \Longrightarrow \int_a^b (f'v' + qfv - \Lambda svf)dx &= 0 \end{aligned}$$

This is the weaker formulation (6)

$$S(f, v) = F(f)$$

for all functions $f \in H_0^1[a,b]$ satisfying conditions of lemma (2) for the choice function $f_i = su_i$. Thus lemma (2) implies that $v \in C^2[a,b]$ and satisfies

$$-v'' + qv - \Lambda sv = s(\gamma_1 u_1 + \gamma_2 u_2 + ... + \gamma_{n-1}u_{n-1}) \tag{9}$$

for a suitable choice of the $\gamma_i's$. In case if $n = 1$, the right side of above is understood to be zero. So we have $-v'' + qv - \Lambda sv = 0$. If $n > 1$, since $v_m \to v$ uniformly on $[a,b]$, we have

$$< v, u_1 >=< v, u_2 >= ... < v, u_{n-1} >= 0.$$

Multiplying (9) by u_j and integrating from a to b, for $j = 1, ..., n-1$, using the orthogonality and equivalent weaker formulation $S(u_i, v) = F(u_i)$,

$$\gamma_j = \int_a^b (u_j'v' + qu_jv - \Lambda svu_j)dx = 0.$$

So the right side of (9) vanishes identically.

$$-v'' + qv - \Lambda s v = 0.$$

This proves that Λ is an eigenvalue and v is an eigenfunction. Since $\Lambda = inf\{R(u) : u^{\perp}\{u_1, u_2, \cdots u_{n-1}\}\}$, $\lambda_n = R(u_n) \geq \Lambda$. On the other hand using

$$\begin{aligned} R(v_m) &= S(v_m, v_m) \\ &= S(v_m - v, v_m - v) + 2S(v_m, v) - S(v, v) \\ &\geq 2S(v_m, v) - S(v, v). \end{aligned}$$

Taking the limit

$$R(v) \geq S(v, v).$$

So

$$\Lambda \geq S(v, v) = \lambda_n.$$

Thus $(\Lambda, v) = (\lambda_n, u_n)$, is the n^th and eigenpair. □

Proof of Lemma 3: Let f be an arbitrary function in $H_0^1[a,b]$, such that $< f, u_1 >=< f, u_2 >= \cdots < f, u_{n-1} >= 0$. Let's minimize the Rayleigh quotient in the bigger space $H_0^1[a,b]$ which contains $C^2[a,b]$. It follows that either (i) $R(v_m + f) \geq \Lambda$ or (ii) $R(v_m + f) \leq \Lambda$, depending on $f \in H_0^1[a,b]$. Integrating the middle term by parts and using $v_m(a) = 0 = v_m(b)$, we get

$$\begin{aligned} R(v_m + f) &= \frac{S(v_m, v_m) + S(f, f) + 2\int_a^b (v_m' f' + v_m f q) dx}{\int_a^b s(v_m + f)^2 dx} \\ &= \frac{S(v_m, v_m) + S(f, f) + 2\int_a^b v_m(-f'' + fq) dx}{\int_a^b s(v_m + f)^2 dx} \end{aligned}$$

Case(i), Let $R(v_m + f) \geq \Lambda$, then

$$S(v_m, v_m) + S(f, f) + 2\int_a^b v_m(-f'' + fq) dx \geq \Lambda \int_a^b s(v_m + f)^2 dx.$$

$$S(v_m, v_m) - \Lambda + S(f, f) - \Lambda \int_a^b s f^2 dx + 2\int_a^b v_m(-f'' + qf - \Lambda s f) dx \geq 0.$$

Taking limit as $v_m \to v$, since $R(v_m) = S(v_m, v_m) \to \Lambda$,

$$S(f, f) - \Lambda \int_a^b s f^2 dx + 2\int_a^b v(-f'' + qf - \Lambda s f) dx \geq 0.$$

Replacing f by $-\epsilon f$ in the above makes the first two terms of order ϵ^2 and the second term of order ϵ. Thus the lemma (3) follows by taking sufficiently small value of ϵ.

Case (ii), Let $R(v_m + f) \leq \Lambda$, repeating the above we arrive at

$$S(f,f) - \Lambda \int_a^b sf^2 dx + 2\int_a^b v(-f'' + qf - \Lambda sf)dx \leq 0.$$

The same argument forces the second integral to be zero. Thus we have lemma 3. □

The following are some of the standard results about the eigenfunctions [6], [7].

Theorem 4.2. *(i) The eigenfunctions belonging to distinct eigenvalues are orthogonal with respect to the weight function.*

(ii) The eigenvalues λ_k are simple. (i.e there do not exist two linearly independent eigenfunctions with the same eigenvalue). In other words the geometric multiplicity of the eigenvalues of the Sturm Liouville problem is one.

(iii)The eigenvalues can be ordered as an increasing sequence tending to infinity.

(iv)The eigenfunctions are complete set of functions with respect to the inner product defined above.

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