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# A FIXED POINT RESULT FOR WEAKLY KANNAN TYPE CYCLIC CONTRACTIONS

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**Abstract:** In this article, we introduce the notion of Kannan type cyclic weakly contraction and derive the existence of fixed point for such mappings in the setup of complete metric spaces. Our result extend and improve some fixed point theorems in the literature.

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**Key Words:** fixed point, cyclic contraction mappings

## 1. Introduction and Preliminaries

It is well known that Banach's fixed point theorem for contraction mappings is one of the pivotal results in analysis. It has been used in many different fields of mathematics, but suffers from one major drawback i.e. in order to use the contractive condition, a self mapping $T$ must be Lipschitz continuous, with Lipschitz constant $L < 1$. In particular, $T$ must be continuous at all points of its domain.

A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

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Kannan [10], [11] proved the following result, giving an affirmative answer to above question.

**Theorem 1.1.** *If $T : X \to X$, where $(X, d)$ is a complete metric space, satisfies*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \tag{1.1}$$

*where $0 < k < \frac{1}{2}$ and $x, y \in X$, then $T$ has a unique fixed point.*

The mappings satisfying (1.1) are called Kannan type mappings.

Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [15] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contains contractions as a special case and he also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [2] and of Reich types [14]. Fixed point problems involving different type of contractive type inequalities have been studied by many authors (see [1]-[15] and references cited therein).

On the other hand, Kirk et al. [13] in 2003 introduce the following notion of cyclic representation and characterize the Banach Contraction Principle in the context of cyclic mapping.

**Definition 1.1.** [13] Let $X$ be a non-empty set and $T : X \to X$ an operator. By definition, $X = \cup_{i=1}^{m} X_i$ is a cyclic representation of $X$ with respect to $T$ if:

(a) $X_i$; $i = 1, \ldots, m$ are non-empty sets,

(b) $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

In this paper, we introduce the notion of cyclic weakly Kannan type contractions and then derive a fixed point theorem on such cyclic contractions in the framework of complete metric spaces.

## 2. Main Results

We introduce the notion of Kannan type cyclic weakly contraction in metric space.

Let $\Phi$ denote all monotone increasing continuous functions $\mu : [0, \infty) \to [0, \infty)$ with $\mu(t) = 0$ if and only if $t = 0$ and $\Psi$ denote all the lower semi-continuous functions $\psi : [0,\infty)^2 \to [0,\infty)$ with $\psi(t) > 0$ for $t \in (0,\infty)$ and $\psi(0) = 0$.

**Definition 2.1.** Let $(X,d)$ be a metric space, $m \in \mathbb{N}$, $A_1, A_2 \ldots, A_m$ nonempty subsets of $X$ and $Y = \cup_{i=1}^m A_i$. An operator $T : Y \to Y$ is called a Kannan type cyclic weakly contraction if

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $T$;

(2) $\mu(d(Tx,Ty)) \leq \mu(\frac{1}{2}[d(x,Tx) + d(y,Ty)]) - \psi(d(x,Tx), d(y,Ty))$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$, $\mu \in \Phi$ and $\psi \in \Psi$.

**Theorem 2.1.** *Let $(X,d)$ be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots$, $A_m$ nonempty closed subsets of $X$ and $Y = \cup_{i=1}^m A_i$. Suppose that $T$ is a Kannan type cyclic weakly contraction. Then, $T$ has a fixed point $z \in \cap_{i=1}^n A_i$.*

*Proof.* Let $x_0 \in X$. We can construct a sequence $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, hence the result. Indeed, we have $Tx_{n_0} = x_{n_0+1} = x_{n_0}$. So we assume that $x_{n+1} \neq x_n$ for any $n = 0, 1, 2, \ldots$. As $X = \cup_{i=1}^m A_i$, for any $n > 0$ there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since $T$ is a Kannan type cyclic weakly contraction, we have

$$\begin{aligned}
\mu(d(x_{n+1},x_n)) &= \mu(d(Tx_n, Tx_{n-1})) \\
&\leq \mu(\frac{1}{2}[d(x_n,Tx_n) + d(x_{n-1},Tx_{n-1})]) - \\
&\quad \psi(d(x_n,Tx_n), d(x_{n-1},Tx_{n-1})) \\
&= \mu(\frac{1}{2}[d(x_n,x_{n+1}) + d(x_{n-1},x_n)]) \\
&\quad -\psi(d(x_n,x_{n+1}), d(x_{n-1},x_n)) \qquad (2.1) \\
&\leq \mu(\frac{1}{2}[d(x_n,x_{n+1}) + d(x_{n-1},x_n)]).
\end{aligned}$$

Since $\mu$ is a non-decreasing function, for all $n = 1, 2 \ldots$, we have

$$d(x_{n+1},x_n) \leq d(x_n,x_{n-1}). \qquad (2.2)$$

Thus $\{d(x_{n+1},x_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Hence there exists $r \geq 0$ such that $d(x_{n+1},x_n) \to r$. Letting $n \to \infty$ in (2.2), we obtain that $\lim d(x_{n-1},x_{n+1}) = 2r$.

Letting $n \to \infty$ in (2.1), using the continuity of $\mu$ and lower semi-continuity of $\psi$, we obtain that $\mu(r) \leq \mu(r) - \psi(r,r)$. This implies that $\psi(r,r) \leq 0$ by the continuity of $\psi$, which is a contradiction unless $r = 0$. Thus we proved that

$$d(x_{n+1},x_n) \to 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. For this, we prove the following claim first:

(A) For every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$, then $d(x_r, x_q) < \epsilon$.

Assume the contrary of (A). Thus there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying $d(x_{r_n}, x_{q_n}) \geq \epsilon$.

Now, we take $n > 2m$. Then corresponding to $q_n \geq n$, we can choose $r_n$ in such that it is a smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1(m)$ and $d(x_{r_n}, x_{q_n}) \geq \epsilon$. Therefore, $d(x_{r_{n-m}}, x_{q_n}) < \epsilon$. By using the triangular inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{r_{n-m}}) + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}) \\ &< \epsilon + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}). \end{aligned}$$

Letting $n \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we have

$$\lim d(x_{q_n}, x_{r_n}) = \epsilon. \tag{2.3}$$

Again by triangular inequality,

$$\begin{aligned} \epsilon &\leq d(x_{q_n}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}). \end{aligned}$$

Letting $n \to \infty$ and using $d(x_{n+1}, x_n) \to 0$, we have

$$\lim d(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{2.4}$$

As $x_{q_n}$ and $x_{r_n}$ lie in different adjacently labeled sets $A_i$ and $A_{i+1}$ for certain $1 \leq i \leq m$, using the fact $T$ is a Kannan type cyclic weakly contraction, we have

$$\begin{aligned} \mu(\epsilon) &\leq \mu(d(x_{q_{n+1}}, x_{r_{n+1}})) \\ &= \mu(d(Tx_{q_n}, Tx_{r_n})) \\ &\leq \mu(\frac{1}{2}[d(x_{q_n}, Tx_{q_n}) + d(x_{r_n}, Tx_{r_n})]) - \end{aligned}$$

$$
\begin{aligned}
& \psi(d(x_{q_n}, Tx_{q_n}), d(x_{r_n}, Tx_{r_n})) \\
= \ & \mu(\frac{1}{2}[d(x_{q_n}, x_{q_{n+1}}) + d(x_{r_n}, x_{r_{n+1}})]) - \\
& \psi(d(x_{q_n}, x_{q_{n+1}}), d(x_{r_n}, x_{r_{n+1}})). \qquad (2.5)
\end{aligned}
$$

On taking $n \to \infty$ in (2.5), using continuity of $\mu$, and lower semi-continuity of $\psi$, we get that $\epsilon = 0$, which is contradiction with $\epsilon > 0$. Hence (A) is proved.

Using (A), we shall show that $\{x_n\}$ is a Cauchy sequence in $Y$. Fix $\epsilon > 0$. By (A), we can find $n_0 \in \mathbb{N}$ such that $r, q \geq n_0$ with $r - q \equiv 1(m)$

$$d(x_r, x_q) \leq \frac{\epsilon}{2}. \qquad (2.6)$$

Since $\lim d(x_n, x_{n+1}) = 0$, we can also find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{2m}, \qquad (2.7)$$

for any $n \geq n_1$. Assume that $r, s \geq max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k(m)$. Hence $s - r + t = 1(m)$ for $t = m - k + 1$. So, we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \ldots + d(x_{s+1}, x_s). \qquad (2.8)$$

Using (2.6), (2.7) and (2.8), we have

$$d(x_r, x_s) \leq \frac{\epsilon}{2} + j \times \frac{\epsilon}{2m} \leq \frac{\epsilon}{2} + m \times \frac{\epsilon}{2m} = \epsilon. \qquad (2.9)$$

Hence $\{x_n\}$ is a Cauchy sequence in $Y$. Since $Y$ is closed in $X$, then $Y$ is also complete and there exists $x \in Y$ such that $\lim x_n = x$.

Now, we shall prove that $x$ is a fixed point of $T$. As $Y = \cup_{i=1}^{m} A_i$ is a cyclic representation of $Y$ with respect to $T$, the sequence $\{x_n\}$ has infinite terms in each $A_i$ for $i = \{1, 2, \ldots, m\}$. Suppose that $x \in A_i$, $Tx \in A_{i+1}$ and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_i$. By using the contractive condition, we can obtain

$$
\begin{aligned}
\mu(d(x_{n_k+1}, Tx)) &= \mu(d(Tx_{n_k}, Tx)) \\
&\leq \mu(\frac{1}{2}[d(x_{n_k}, Tx_{n_k}) + d(x, Tx)]) - \\
&\quad \psi(d(x_{n_k}, Tx_{n_k}), d(x, Tx)) \\
&= \mu(\frac{1}{2}[d(x_{n_k}, x_{n_k+1}) + d(x, Tx)]) - \\
&\quad \psi(d(x_{n_k}, x_{n_k+1}), d(x, Tx)).
\end{aligned}
$$

Letting $n \to \infty$ and using continuity of $\mu$ and lower semi-continuity of $\psi$, we have

$$\mu(d(x, Tx)) \le \mu(\frac{1}{2}d(x, Tx)) - \psi(0, d(x, Tx)),$$

which is a contradiction unless $d(x, Tx) = 0$. Hence $x$ is a fixed point of $T$.

Now, we shall prove the uniqueness of fixed point. Suppose that $x_1$ and $x_2$ ($x_1 \neq x_2$) are two fixed points of $T$. Using the contractive condition and continuity of $\mu$ and lower semi continuity of $\psi$, we have

$$\begin{aligned} \mu(d(x_1, x_2)) &= \mu(d(Tx_1, Tx_2)) \\ &\le \mu(\frac{1}{2}[d(x_1, Tx_1) + d(x_2, Tx_2)]) - \psi(d(x_1, Tx_1), d(x_2, Tx_2)) \\ &= \mu(\frac{1}{2}[d(x_1, x_1) + d(x_2, x_2)]) - \psi(d(x_1, x_1), d(x_2, x_2)) \\ &= 0, \end{aligned}$$

which is a contradiction. Hence the result. □

If $\mu(a) = a$, then we have the following result.

**Corollary 2.2.** *Let $(X, d)$ be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots,$ $A_m$ nonempty closed subsets of $X$ and $Y = \cup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ be an operator such that:*

*(1) $\cup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $T$;*

*(2) $d(Tx, Ty) \le \frac{1}{2}[d(x, Tx) + d(y, Ty)] - \psi(d(x, Tx), d(y, Ty))$*

*for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$ and $\psi \in \Psi$. Then, $T$ has a fixed point $z \in \cap_{i=1}^n A_i$.*

If $\psi(a, b) = (\frac{1}{2} - k)(a + b)$, where $k \in [0, \frac{1}{2})$, we have the following result.

**Corollary 2.3.** *Let $(X, d)$ be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots,$ $A_m$ nonempty closed subsets of $X$ and $Y = \cup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ be an operator such that*

*(1) $\cup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $T$;*

*(2) there exists $k \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \le k[d(x, Tx) + d(y, Ty)]$*

*for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$. Then, $T$ has a fixed point $z \in \cap_{i=1}^n A_i$.*

Other consequences of our results are the given in the following, for mappings involving contractions of integral type.

Denote by $\Lambda$ the set of functions $\mu\colon [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(h1) $\mu$ is a Lebesgue-integrable mapping on each compact of $[0,\infty)$;

(h2) for any $\epsilon > 0$, we have $\int_0^\epsilon \mu(t) > 0$.

**Corollary 2.4.** *Let $(X,d)$ be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots,$ $A_m$ nonempty closed subsets of $X$ and $Y = \cup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ be an operator such that*

*(1) $\cup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $T$;*

*(2) there exists $k \in [0, \frac{1}{2})$ such that*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq k \int_0^{d(x,Tx)+d(y,Ty)} \alpha(s)ds$$

*for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$ and $\alpha \in \Lambda$. Then, $T$ has a fixed point $z \in \cap_{i=1}^n A_i$.*

If we take $A_i = X$, $i = 1, 2, \ldots, m$, we obtain the following result.

**Corollary 2.5.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a mapping such that*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq k \int_0^{d(x,Tx)+d(y,Ty)} \alpha(s)ds$$

*for any $xy \in X$, $k \in [0, \frac{1}{2})$ and $\alpha \in \Lambda$. Then, $T$ has a fixed point $z \in \cap_{i=1}^n A_i$.*

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