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ON THE TOPOLOGICALLY MIXING OF A TUPLE OF OPERATORS

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Abstract: In this paper we give conditions under which a tuple of commutative bounded linear operators, acting on a Banach space, satisfying the property of topologically mixing.

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1. Introduction

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let

$$\mathcal{F}_{\mathbf{T}} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \geq 0,\ i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_{\mathbf{T}}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic.

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Definition 1.2. A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called topologically mixing if for any given open sets U and V, there exist positive integers $M(1)$, ..., $M(n)$ such that

$$T_1^{m(1)} ... T_n^{m(n)}(U) \cap V \neq \emptyset, \quad \forall m(i) \geq M(i),\ i = 1, ..., n.$$

A topologically mixing operator satisfies a much stronger condition holds for every large $M_1, ..., M_n$. Roughly speaking, the iterates of any open set become well spread throughout the space. A nice criterion namely the Hypercyclicity Criterion is used in the proof of our main theorem. It was developed independently by Kitai([8]), Gethner and shapiro ([7]). This criterion has used to show that hypercyclic operators arise within the class of composition operators([4]), weighted shifts ([9]), adjoints of multiplication operators ([5]),and adjoints of subnormal and hyponormal operators ([3]),and Hereditarily operators ([2]), and topologically mixing operators ([6]). The formulation of the Hypercyclicty Criterion in the following theorem was given by J. Bes ([1]). For some topics we refer to [1–17].

2. Main Result

A nice criterion namely the Hypercyclicity Criterion is used in the proof of our main theorem. Note that, all operators in this paper are commutative operators.

Theorem 2.1. (The Hypercyclicity Criterion) *Suppose that X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n acting on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that:*

1. $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$,

2. there exist a sequence of functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.

Remember that a strictly increasing sequence of positive integers $\{n_k\}$ is said to be syndetic if $\sup \{n_{k+1} - n_k\} < \infty$. The following theorem is a generalization of Theorem 1.1 in [6] for tuples.

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an $n-$tuple of operators acting on a Frechet space (X, d). If $\mathcal{T}$ satisfies the Hypercyclicity Criterion for syndetic sequences $\{m_{j(i)}\}_j$ for $j = 1, ..., n$, then $\mathcal{T}$ is topologically mixing.*

Proof. Let U and V be any open sets in X, We will show that there are positive integers $M_1, ..., M_n$ such that

$$T_1^{m_1}T_2^{m_2}...T_n^{m_n}(U)\cap V\neq\phi,\quad \forall m_i\geq M_i,\quad i=1,...,n.$$

Note that the sequences $\{m_{j(i)}\}_j$ $(i=1,...,n)$ in the Hypercyclicity Criterion are syndetic, thus there exist some positive integers $M_0(i)$, $i=1,...,n$ such that for all $j\geq 0$ we have

$$m_{(j+1)i}-m_{ji}\leq M_0(i),\quad i=1,...,n.$$

For each $i=1,...,n$ let $0\leq r_i\leq M_0(i)$ and consider open sets $V_{r_1,...,r_n}$ such that $T_1^{r_1}T_2^{r_2}...T_n^{r_n}(V_{r_1,...,r_n})=V$.

Let Y and Z be the dense sets that are given in the Hypercyclicity Criterion. Choose $x\in U\cap Y$, and take $\epsilon>0$ such that ball $B(x,\epsilon)\subset U$. Also, for each $r_i=0,1,2,...,M_0(i)$ $(i=0,1,2,...,n)$ take $z_{r_1,...,r_n}\in V_{r_1,...,r_n}\cap Z$. Assume that ϵ is small enough such that the ball $B(z_{r_1,...,r_n},2\epsilon)\subset V_{r_1,...,r_n}$. For simplicity we denote $d(z,0)$ by $\parallel z\parallel$. Let j_0 be large enough such that for all $j\geq j_0$ and $r_i=0,1,2,...,M_0(i)$ for $i=0,1,2,...,n$:

$$\|T_1^{m_{j(1)}}T_2^{m_{j(2)}}...T_n^{m_{j(n)}}x\|\leq\epsilon,$$

$$\|S_j(z_{r_1,...,r_n})\|<\epsilon,$$

$$\|T_1^{m_{j(1)}}T_2^{m_{j(2)}}...T_n^{m_{j(n)}}S_j(z_{r_1,...,r_n})-z_{r_1,...,r_n}\|<\epsilon.$$

Set $M_i=m_{k_0(i)}$ for $i=1,...,n$. For each $i=1,...,n$, let $p_i\geq M_i$. Then there are some $m_{j(i)}$ with $j\geq j_0$ and $0\leq s_i\leq M_0(i)$ such that $p_i=m_{j(i)}+s_i$ for $i=1,...,n$. For each $i=1,...,n$, define $x_{p_i}=x+S_j(z_{s_1,...,s_n})$. Then $\parallel x_{p_i}-x\parallel=\parallel S_j(z_{s_1,...,s_n})\parallel<\epsilon$. Hence $x_{p_i}\in B(x,\epsilon)\subset U$. Also, we have

$$T_1^{p_1}...T_n^{p_n}(x_{p_i})=T_1^{s_1}...T_n^{s_n}(T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(x+S_j(z_{s_1,...,s_n})).$$

Note that

$$T_1^{m_{j(1)}}T_2^{m_{j(2)}}...T_n^{m_{j(n)}}(x+S_j(z_{s_1,...,s_n}))\in V,$$

because

$$\begin{aligned}||T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(x &+ S_j(z_{s_1,...,s_n}))-z_{s_1,...,s_n}||\leq||T_1^{m_{j(1)}}...T_n^{m_{j(n)}}(x)||\\ &+ ||T_1^{m_{j(1)}}...T_n^{m_{j(n)}}S_j(z_{s_1,...,s_n})-z_{s_1,...,s_n}||<2\epsilon.\end{aligned}$$

Therefore,

$$T_1^{m_{j(1)}}T_2^{m_{j(2)}}...T_n^{m_{j(n)}}(x+S_j(z_{s_1,...,s_n}))\in B(z_{s_1,...,s_n}),2\epsilon)\subset V_{s_1,...,s_n}$$

and so

$$T_1^{p_1}T_2^{p_2}...T_n^{p_n}(x_{p_i}))\in T_1^{s_1}T_2^{s_2}...T_n^{s_n}(V_{s_1,...,s_n})=V.$$

Now since $x_{p_i}\in U$, thus $T_1^pT_2^q(U)\cap V\neq\emptyset$. This completes the proof. □

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