

We can proceed as in §§3.4 of [3] to get

Theorem 13. *The canonical projection $\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}} : M \otimes \mathcal{W}_{D^n \rightarrow M} \otimes \mathcal{W}_{D\{n\}_{n-1}}$ is an affine bundle over the vector bundle $(M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D\{n\}_{n-1}}) \rightarrow M \otimes \mathcal{W}_{D\{n\}_{n-1}}$.*

We have the following n -dimensional counterparts of Propositions 5, 6 and 7 in §§3.4 of [3].

Proposition 14. *For any $\alpha \in \mathbb{R}$, any $\gamma_+, \gamma_-, \gamma \in M \otimes \mathcal{W}_{D^n}$ and any $t \in M \otimes \mathcal{W}_D$ with*

$$\left(\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}}\right)(\gamma_+) = \left(\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}}\right)(\gamma_-)$$

and

$$\left(\text{id}_M \otimes \mathcal{W}_{i_{1 \rightarrow D}}\right)(t) = \left(\text{id}_M \otimes \mathcal{W}_{i_{1 \rightarrow D}}\right)(\gamma),$$

we have

$$\begin{aligned} \alpha(\gamma_+ \dot{-} \gamma_-) &= (\alpha \cdot \gamma_+) \dot{-} (\alpha \cdot \gamma_-) \\ \alpha \cdot (t \dot{+} \gamma) &= (\alpha \cdot t) \dot{+} \alpha \cdot \gamma \end{aligned}$$

Proposition 15. *The diagrams*

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^n \rightarrow M} \otimes \mathcal{W}_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}}) \otimes (M \otimes \mathcal{W}_{D^n}) & \rightarrow & M \otimes \mathcal{W}_D \\ \downarrow i & & \downarrow \\ \left((M \otimes \mathcal{W}_D) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}} (M \otimes \mathcal{W}_{D^n}) \right) \otimes \mathcal{W}_{D_m} & \rightarrow & (M \otimes \mathcal{W}_D) \otimes \mathcal{W}_{D_m} \\ (1 \leq i \leq n), & & \\ (M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D^n}) & \rightarrow & M \otimes \mathcal{W}_{D^n} \\ \downarrow i & & \downarrow i \\ \left((M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D^n}) \right) \otimes \mathcal{W}_{D_m} & \rightarrow & (M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_{D_m} \end{array}$$

$(1 \leq i \leq n)$ are commutative, where:

1. In the former diagram, the lower horizontal arrow represents

$$\left((\gamma_+, \gamma_-) \in (M \otimes \mathcal{W}_{D^n}) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}} (M \otimes \mathcal{W}_{D^n}) \mapsto (\gamma_+ \dot{-} \gamma_-) \in M \otimes \mathcal{W}_D \right)$$

$$\otimes \text{id}_{\mathcal{W}_{D_m}},$$

the upper horizontal arrow represents

$$(\gamma_+, \gamma_-) \in (M \otimes \mathcal{W}_{D^n}) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}} (M \otimes \mathcal{W}_{D^n}) \mapsto (\gamma_+ \dot{-} \gamma_-) \in M \otimes \mathcal{W}_D,$$

the left vertical arrow represents the composition of mappings

$$\begin{aligned} & (M \otimes \mathcal{W}_{D^n}) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}} (M \otimes \mathcal{W}_{D^n}) \\ & \xrightarrow{\left(\text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n, e) \in D^n \times D_m \mapsto (d_1, \dots, ed_i, \dots, d_n) \in D^n} \right)} \\ & \xrightarrow{\left(\text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n, e) \in D^n \times D_m \mapsto (d_1, \dots, ed_i, \dots, d_n) \in D^n} \right)} \\ & (M \otimes \mathcal{W}_{D^n \times D_m}) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1} \times D_m}} (M \otimes \mathcal{W}_{D^n \times D_m}) \\ & = ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_{D_m}) \times_{(M \otimes \mathcal{W}_{D\{n\}_{n-1}}) \otimes \mathcal{W}_{D_m}} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_{D_m}) \\ & = \left((M \otimes \mathcal{W}_{D^n}) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}} (M \otimes \mathcal{W}_{D^n}) \right) \otimes \mathcal{W}_{D_m}, \end{aligned}$$

and the right vertical arrow represents the composition of mappings

$$M \otimes \mathcal{W}_D \xrightarrow{\text{id}_M \otimes \mathcal{W}_{(d, e) \in D \times D_m \mapsto de \in D}} M \otimes \mathcal{W}_{D \times D_m} = (M \otimes \mathcal{W}_D) \otimes \mathcal{W}_{D_m};$$

2. In the latter diagram, the lower horizontal arrow represents

$$(t, \gamma) \in (M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D^n}) \mapsto t \dot{+} \gamma \in M \otimes \mathcal{W}_{D^n}$$

$$\otimes \text{id}_{\mathcal{W}_{D_m}},$$

the upper horizontal arrow represents

$$(t, \gamma) \in (M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D^n}) \mapsto t \dot{+} \gamma \in M \otimes \mathcal{W}_{D^n},$$

the left vertical arrow represents the composition of mappings

$$\begin{aligned} & (M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D^n}) \\ & \xrightarrow{\left(\text{id}_M \otimes \mathcal{W}_{(d, e) \in D \times D_m \mapsto ed \in D} \right) \times \left(\text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n, e) \in D^n \times D_m \mapsto (d_1, \dots, ed_i, \dots, d_n) \in D^n} \right)} \\ & (M \otimes \mathcal{W}_{D \times D_m}) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1} \times D_m}} (M \otimes \mathcal{W}_{D^n \times D_m}) \end{aligned}$$

$$\begin{aligned}
 &= ((M \otimes \mathcal{W}_D) \otimes \mathcal{W}_{D_m}) \times_{(M \otimes \mathcal{W}_{D\{n\}_{n-1}}) \otimes \mathcal{W}_{D_m}} ((M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_{D_m}) \\
 &= \left((M \otimes \mathcal{W}_D) \times_{M \otimes \mathcal{W}_{D\{n\}_{n-1}}} (M \otimes \mathcal{W}_{D^n}) \right) \otimes \mathcal{W}_{D_m},
 \end{aligned}$$

and the right vertical arrow represents the composition of mappings

$$\begin{aligned}
 &M \otimes \mathcal{W}_{D^n} \text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n, e) \in D^n \times D_m \mapsto (d_1, \dots, ed_i, \dots, d_n) \in D^n \times D_m} \otimes \mathcal{W}_{D^n \times D_m} \\
 &= (M \otimes \mathcal{W}_{D^n}) \otimes \mathcal{W}_{D_m}.
 \end{aligned}$$

Proposition 16. For any $\sigma \in \mathbf{S}_n$, any $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D^n}$ and any $t \in M \otimes \mathcal{W}_D$ with

$$(\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}})(\gamma_+) = (\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}})(\gamma_-)$$

and

$$(\text{id}_M \otimes \mathcal{W}_{i_{1 \rightarrow D}})(t) = (\text{id}_M \otimes \mathcal{W}_{i_{1 \rightarrow D^n}})(\gamma),$$

we have

$$\begin{aligned}
 (\gamma_+ \dot{-} \gamma_-)^\sigma &= \gamma_+ \dot{-} \gamma_- \\
 (t \dot{-} \gamma)^\sigma &= t \dot{-} \gamma^\sigma.
 \end{aligned}$$

Proposition 17. For any $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D^n}$ with

$$(\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}})(\gamma_+) = (\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}_{n-1} \rightarrow D^n}})(\gamma_-),$$

we have

$$\begin{aligned}
 &(\text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n) \in D^n \mapsto d_1 \dots d_n \in D})(\gamma_+ \dot{-} \gamma_-) \\
 &= (\text{id}_M \otimes \mathcal{W}_{i_{1 \rightarrow D}}) \left(\frac{1}{2} \mathbf{s}_1 \circ \mathbf{d}_1(\gamma_+) - \frac{1}{3} \mathbf{s}_1^2 \circ \mathbf{d}_1^2(\gamma_+) \dots - \frac{1}{n} \mathbf{s}_1^{n-1} \circ \mathbf{d}_1^{n-1}(\gamma_+) \right)
 \end{aligned}$$

3.1.2. Symmetric Forms

Definition 18. A symmetric D^n -form at $x \in E$ is a mapping $\omega_x : (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \rightarrow (E \otimes \mathcal{W}_D)_x^\perp$ subject to the following conditions:

1. We have

$$\omega_x(\alpha \cdot_i \gamma) = \alpha \omega_x(\gamma) \quad (1 \leq i \leq n)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and any $\alpha \in \mathbb{R}$.

2. The diagram

$$\begin{array}{ccc}
 (M \otimes \mathcal{W}_{D^n})_{\pi(x)} & \rightarrow & (M \otimes \mathcal{W}_{D^n})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\
 \omega_x \downarrow & & \downarrow \omega_x \otimes \text{id}_{\mathcal{W}_{D_m}} \\
 (E \otimes \mathcal{W}_D)_x & \xrightarrow{\text{id}_E \otimes \mathcal{W}_{\times_{D \times D_m \rightarrow D}}} & (E \otimes \mathcal{W}_D)_x \otimes \mathcal{W}_{D_m}
 \end{array}$$

(1 ≤ i ≤ n) is commutative, where the upper horizontal arrow is

$$\text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n, e) \in D^n \times D_m \mapsto (d_1, \dots, d_{i-1}, e d_i, d_{i+1}, \dots)} \in D^n \cdot$$

3. We have

$$\omega_x(\gamma^\sigma) = \omega_x(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ and any $\sigma \in \mathbf{S}_n$.

4. We have

$$\omega_x((\text{id}_M \otimes \mathcal{W}_{(d_1, \dots, d_n) \in D^n \mapsto (d_1, \dots, d_{i-2}, d_{n-1}, d_n)} \in D^{n-1}))(\gamma)) = 0$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^{n-1}})_{\pi(x)}$.

Notation 19. We denote by $\mathbb{S}^{D^n}(\pi)$ the totality of symmetric D^n -forms at $x \in E$. We denote by $\mathbb{S}^{D^n}(E)$ the set-theoretic union of $\mathbb{S}_x^{D^n}(\pi)$'s for all $x \in E$. The canonical projection $\mathbb{S}^{D^n}(E) \rightarrow E$ is obviously a vector bundle.

Proposition 20. Let $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$. Then we have

$$\omega(\mathbf{s}_i(\gamma)) = 0 \quad (1 \leq i \leq n + 1)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$.

Proof. For any $\alpha \in \mathbb{R}$, we have

$$\omega(\mathbf{s}_i(\gamma)) = \omega(\alpha \cdot \mathbf{s}_i(\gamma)) = \alpha \omega(\mathbf{s}_i(\gamma))$$

Letting $\alpha = 0$, we have the desired conclusion. □

3.1.3. The Theorem

The following proposition will be used in the proof of Proposition 3.6.

Proposition 21. Let $\nabla_x \in \mathbb{J}_x^{D^n}(\pi)$, $t \in (M \otimes \mathcal{W}_D)_{\pi(x)}$ and $\gamma, \gamma_+, \gamma_- \in (M \otimes \mathcal{W}_{D^n})_{\pi(x)}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}}_{n-1} \rightarrow D^n}\right) (\gamma_+) = \left(\text{id}_M \otimes \mathcal{W}_{i_{D\{n\}}_{n-1} \rightarrow D^n}\right) (\gamma_-).$$

Then we have

$$\begin{aligned} \nabla_x(\gamma_+) \dot{-} \nabla_x(\gamma_-) &= (\underline{\pi}_{n,1}(\nabla_x)) (\gamma_+ \dot{-} \gamma_-) \\ (\pi_{n,1}(\nabla_x)) (t) \dot{+} \nabla_x(\gamma) &= \nabla_x(t \dot{+} \gamma) \end{aligned}$$

Proof. It is an easy exercise of affine geometry to show that the conated two formulas are equivalent. Here we deal only with the former in case of $n = 2$, leaving the general treatment safely to the reader. We have

$$\begin{aligned} &(\text{id}_E \otimes \mathcal{W}_{(d_1, d_2) \in D^2 \mapsto d_1 d_2 \in D}) (\nabla_x(\gamma_+) \dot{-} \nabla_x(\gamma_-)) \\ &= \left(\nabla_x(\gamma_+) \frac{1}{1} - \nabla_x(\gamma_-)\right) \frac{1}{2} (\pi_{2,1}(\mathbf{d}_1)) (\nabla_x(\gamma_+)) \\ &[\text{By Proposition 17}] \\ &= \nabla_x\left(\left(\gamma_+ \frac{1}{1} - \gamma_-\right) \frac{1}{2} (\pi_{2,1}(\mathbf{d}_1)) (\gamma_+)\right) \\ &= \nabla_x\left(\left(\text{id}_M \otimes \mathcal{W}_{(d_1, d_2) \in D^2 \mapsto d_1 d_2 \in D}\right) (\gamma_+ \dot{-} \gamma_-)\right) \\ &[\text{By Proposition 17}] \\ &= (\text{id}_E \otimes \mathcal{W}_{(d_1, d_2) \in D^2 \mapsto d_1 d_2 \in D}) (\pi_{2,1}(\nabla_x)(\gamma_+ \dot{-} \gamma_-)) \\ &[\text{By Proposition 13}] \end{aligned}$$

□

Proposition 22. Let $\nabla_x^+, \nabla_x^- \in \mathbb{J}_x^{n+1}(\pi)$ with

$$\pi_{n+1,n}(\nabla_x^+) = \pi_{n+1,n}(\nabla_x^-).$$

Then the assignment $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \mapsto \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)$ belongs to $\mathbb{S}_x^{D^{n+1}}(\pi)$.

Proof. 1. Since

$$\begin{aligned} &(\pi \otimes \text{id}_{\mathcal{W}_D}) (\nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)) \\ &= (\pi \otimes \text{id}_{\mathcal{W}_{D^{n+1}}}) (\nabla_x^+(\gamma)) \dot{-} (\pi \otimes \text{id}_{\mathcal{W}_{D^{n+1}}}) (\nabla_x^-(\gamma)) \\ &[\text{By Proposition 8}] \\ &= 0, \end{aligned}$$

$\nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)$ belongs in $(E \otimes \mathcal{W}_D)_x^\perp$.

2. For any $\alpha \in \mathbb{R}$ and any natural number i with $1 \leq i \leq n + 1$, we have

$$\begin{aligned} & \nabla_x^+(\alpha \cdot \gamma) \dot{-} \nabla_x^-(\alpha \cdot \gamma) \\ &= \alpha \cdot \nabla_x^+(\gamma) \dot{-} \alpha \cdot \nabla_x^-(\gamma) \\ &= \alpha(\nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)), \end{aligned}$$

which implies that the assignment

$$\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \mapsto \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma) \in (E \otimes \mathcal{W}_D)_x^+$$

abides by the first condition in Definition 18.

3. To see that the assignment abides by the second condition in Definition 18, it suffices to note that the diagram

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} & \xrightarrow{i} & (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \downarrow & & \downarrow \\ (E \otimes \mathcal{W}_{D^{n+1}})_x \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} & \xrightarrow{i} & (E \otimes \mathcal{W}_{D^{n+1}})_x \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1}})_x \\ (E \otimes \mathcal{W}_{D^{n+1}})_x & & \otimes \mathcal{W}_{D_m} \\ \downarrow & & \downarrow \\ (E \otimes \mathcal{W}_D)_x & & (E \otimes \mathcal{W}_D)_x \otimes \mathcal{W}_{D_m} \end{array}$$

$(1 \leq i \leq n + 1)$

is commutative, where the upper horizontal arrow is

$$\text{id}_{M \otimes \mathcal{W}_{D^{n+1}}} \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} : (d_1, \dots, d_{n+1}, e) \in D^{n+1} \times D_m \mapsto (d_1, \dots, d_{i-1}, ed_i, d_{i+1}, \dots, d_{n+1}) \in D^{n+1},$$

the middle horizontal arrow is the mapping

$$\begin{aligned} & \left((E \otimes \mathcal{W}_{D^{n+1}})_x \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1}})_x \right) \times \\ & \left((E \otimes \mathcal{W}_{D^{n+1}})_x \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1}})_x \right) \times \\ & \left(\text{id}_{E \otimes \mathcal{W}_{(d_1, \dots, d_{n+1}, e) \in D^{n+1} \times D_m \mapsto (d_1, \dots, d_{i-1}, ed_i, d_{i+1}, \dots, d_{n+1}) \in D^{n+1}}} \right) \times \\ & \left(\text{id}_{E \otimes \mathcal{W}_{(d_1, \dots, d_{n+1}, e) \in D^{n+1} \times D_m \mapsto (d_1, \dots, d_{i-1}, ed_i, d_{i+1}, \dots, d_{n+1}) \in D^{n+1}}} \right) \\ & \left((E \otimes \mathcal{W}_{D^{n+1} \times D_m}) \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1} \times D_m}) \right) \\ & = \left((E \otimes \mathcal{W}_{D^{n+1}}) \times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1}}) \right) \otimes \mathcal{W}_{D_m}, \end{aligned}$$

the lower horizontal arrow is

$$\text{id}_E \otimes \mathcal{W}_{\times_{D \times D_m \rightarrow D}},$$

the upper left vertical arrow is

$$\begin{aligned} \gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} &\mapsto \\ (\nabla_x^+(\gamma), \nabla_x^-(\gamma)) \in (E \otimes \mathcal{W}_{D^{n+1}})_x &\times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1}})_x, \end{aligned}$$

the lower left vertical arrow is

$$\begin{aligned} (\gamma^+, \gamma^-) \in (E \otimes \mathcal{W}_{D^{n+1}})_x &\times_{E \otimes \mathcal{W}_{D\{n+1\}_n}} (E \otimes \mathcal{W}_{D^{n+1}})_x \rightarrow \gamma^+ - \gamma^- \\ &\in (E \otimes \mathcal{W}_D)_x, \end{aligned}$$

the upper right vertical arrow is obtained from the upper left vertical arrow by multiplication of $\otimes \text{id}_{\mathcal{W}_{D^m}}$ from the right, and the lower right vertical arrow is obtained from the lower left vertical arrow by multiplication of $\otimes \text{id}_{\mathcal{W}_{D^m}}$ from the right. The upper square is commutative by the third condition in Definition 1, while the lower square is commutative by Proposition 15, so that the outer square is also commutative, which is no other than the second condition in Definition 18.

4. For any $\sigma \in \mathbf{S}_{n+1}$, we have

$$\begin{aligned} &(\nabla_x^+(\gamma) - \nabla_x^-(\gamma))^\sigma - \nabla_x^-(\gamma^\sigma) \\ &= (\nabla_x^+(\gamma))^\sigma \dot{-} (\nabla_x^-(\gamma))^\sigma \\ &= \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma), \end{aligned}$$

which implies that the assignment abides by the third condition in Definition 18.

It remains to show that the assignment abides by the fourth condition in Definition 18, which follows directly from the second condition in Definition 18 and the assumption that $\hat{\pi}_{n+1,n}(\nabla_x^+) = \hat{\pi}_{n+1,n}(\nabla_x^-)$. □

Proposition 23. *Let $\nabla_x \in \mathbb{J}_x^{D^{n+1}}(\pi)$ and $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$. Then the assignment $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \mapsto \omega(\gamma) \dot{+} \nabla_x(\gamma)$ belongs to $\mathbb{J}_x^{D^{n+1}}(\pi)$.*

Proof. 1. Since

$$\begin{aligned} &(\pi \otimes \text{id}_{\mathcal{W}_{D^{n+1}}}) (\omega(\gamma) \dot{+} \nabla_x(\gamma)) \\ &= (\pi \otimes \text{id}_{\mathcal{W}_D}) (\omega(\gamma)) \dot{+} (\pi \otimes \text{id}_{\mathcal{W}_{D^{n+1}}}) (\nabla_x(\gamma)) \end{aligned}$$

[By Proposition 12]
 $= \gamma,$

the assignment

$$\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \mapsto \omega(\gamma) \dot{+} \nabla_x(\gamma)$$

stands to the first condition in Definition 1.

2. For any $\alpha \in \mathbb{R}$ and any natural number i with $1 \leq i \leq n+1$, we have

$$\begin{aligned} & \omega(\alpha \cdot_i \gamma) \dot{+} \nabla_x(\alpha \cdot_i \gamma) \\ &= \alpha \omega(\gamma) \dot{+} \alpha \cdot_i \nabla_x(\gamma) \\ &= \alpha \cdot_i (\omega(\gamma) \dot{+} \nabla_x(\gamma)), \end{aligned}$$

so that the assignment stands to the second condition in Definition 1.

3. To see that the assignment acquiesces to the third condition in Definition 1, it suffices to note that the commutative diagram

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} & \xrightarrow{i} & (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \downarrow & & \downarrow \\ (E \otimes \mathcal{W}_D)_x \times_{E \otimes \mathcal{W}_{D^{n+1}}} (E \otimes \mathcal{W}_{D^{n+1}})_x & \xrightarrow{i} & \left((E \otimes \mathcal{W}_D) \times_E (E \otimes \mathcal{W}_{D^{n+1}})_x \right) \otimes \mathcal{W}_{D_m} \\ & & \downarrow \\ & & (E \otimes \mathcal{W}_{D^{n+1}})_x \otimes \mathcal{W}_{D_m} \\ (1 \leq i \leq n+1) & & \end{array}$$

is commutative, where the upper horizontal arrow is

$$\text{id}_M \otimes \mathcal{W}_{\binom{\cdot}{i}_{D^{n+1} \times D_m}},$$

the middle horizontal arrow is the composition of mappings

$$\begin{aligned} & (E \otimes \mathcal{W}_D) \times_M (E \otimes \mathcal{W}_{D^{n+1}}) \\ & \xrightarrow{(\text{id}_M \otimes \mathcal{W}_{\times_{D \times D_m \rightarrow D}}) \times \left(\text{id}_M \otimes \mathcal{W}_{\binom{\cdot}{i}_{D^{n+1} \times D_m}} \right)} \\ & (E \otimes \mathcal{W}_{D \times D_m}) \times_{E \otimes \mathcal{W}_{D_m}} (E \otimes \mathcal{W}_{D^{n+1} \times D_m}) \end{aligned}$$

$$= \left((E \otimes \mathcal{W}_D) \times_M (E \otimes \mathcal{W}_{D^{n+1}}) \right) \otimes \mathcal{W}_{D^m},$$

the lower horizontal arrow is

$$\text{id}_E \otimes \mathcal{W} \begin{pmatrix} \cdot \\ i \end{pmatrix}_{D^{n+1} \times D^m},$$

the upper left vertical arrow is

$$\gamma \in M \otimes \mathcal{W}_{D^{n+1}} \mapsto (\omega_x(\gamma), \nabla_x(\gamma)) \in (E \otimes \mathcal{W}_D) \times_E (E \otimes \mathcal{W}_{D^{n+1}}),$$

the lower left vertical arrow is

$$(t, \gamma) \in (E \otimes \mathcal{W}_D) \times_E (E \otimes \mathcal{W}_{D^{n+1}}) \mapsto t \dot{+} \gamma \in (E \otimes \mathcal{W}_{D^{n+1}}),$$

the upper right vertical arrow is the upper left vertical arrow multiplied by $\otimes \text{id}_{\mathcal{W}_{D^m}}$ from the right, and the lower right vertical arrow is the lower left vertical arrow multiplied by $\otimes \text{id}_{\mathcal{W}_{D^m}}$ from the right. The upper square is commutative by the third condition in Definition 1 and the second condition in Definition 18, while the lower square is commutative by Proposition 15, so that the outer square is also commutative, which is no other than the third condition in Definition 1.

4. For any $\sigma \in \mathbf{S}_{n+1}$, we have

$$\begin{aligned} & \omega(\gamma^\sigma) \dot{+} \nabla_x(\gamma^\sigma) \\ &= \omega(\gamma) \dot{+} (\nabla_x(\gamma))^\sigma \\ &= (\omega(\gamma) \dot{+} \nabla_x(\gamma))^\sigma, \end{aligned}$$

so that the assignment stands to the fourth condition in Definition 1.

5. That the assignment $\gamma \in (M \otimes \mathcal{W}_{D^{n+1}})_{\pi(x)} \mapsto \omega(\gamma) \dot{+} \nabla_x(\gamma)$ stands to the first condition of Definition 2 follows from the simple fact that the image of the assignment under $\hat{\pi}_{n+1,n}$ coincides with $\hat{\pi}_{n+1,n}(\nabla_x)$, which is consequent upon Proposition 20.

6. It remains to show that the assignment abides by the second condition in Definition 2, which follows directly from fourth condition in Definition 18 and the second condition in Definition 2.

□

Now we are in a position to give a definition.

Definition 24. 1. For any $\nabla_x^+, \nabla_x^- \in \mathbb{J}^{n+1}(\pi)$ with

$$\pi_{n+1,n}(\nabla_x^+) = \pi_{n+1,n}(\nabla_x^-),$$

we define $\nabla_x^+ \dot{-} \nabla_x^- \in \mathbb{S}_x^{D_{n+1}}(\pi)$ to be

$$(\nabla_x^+ \dot{-} \nabla_x^-)(\gamma) = \nabla_x^+(\gamma) \dot{-} \nabla_x^-(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)}$.

2. For any $\omega \in \mathbb{S}_x^{D_{n+1}}(\pi)$ and any $\nabla_x \in \mathbb{J}^{n+1}(\pi)$, we define $\omega \dot{+} \nabla_x \in \mathbb{J}_x^{n+1}(\pi)$ to be

$$(\omega \dot{+} \nabla_x)(\gamma) = \omega(\gamma) \dot{+} \nabla_x(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)}$.

With these two operations depicted in the above definition, it is easy to see that

Theorem 25. (cf. Theorem 1.1 of [5]). *The bundle $\pi_{n+1,n} : \mathbb{J}^{n+1}(\pi) \rightarrow \mathbb{J}^n(\pi)$ is an affine bundle over the vector bundle $\mathbb{J}^n(\pi) \times_{\mathbb{E}} \mathbb{S}^{D_{n+1}}(\pi) \rightarrow \mathbb{J}^n(\pi)$.*

Proof. This follows from Theorem 13. □

3.2. The Theorem in the Third Approach

3.2.1. Affine Bundles

Now we turn to another kind of affine bundles, for which we can proceed in the same way as in Subsubsection 3.1.1.

Lemma 26. *The diagram*

$$\begin{array}{ccc} D_n & \xrightarrow{i_{D_n \rightarrow D_{n+1}}} & D_{n+1} \\ i_{D_n \rightarrow D_{n+1}} \downarrow & & \downarrow \Psi_{D_{n+1}} \\ D_{n+1} & \xrightarrow{\Phi_{D_{n+1}}} & D_{n+1} \oplus D \end{array}$$

is a quasi-colimit diagram, where $i : D_n \rightarrow D_{n+1}$ is the canonical injection, $\Phi_{D_{n+1}}(d) = (d, 0)$ and $\Psi_{D_{n+1}}(d) = (d, d^{n+1})$.

This implies directly that

Proposition 27. Given $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D_{n+1}}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_+) = \left(\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_-),$$

there exists unique $\gamma \in M \otimes \mathcal{W}_{D_{n+1} \oplus D}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{\Psi_{D_{n+1}}}\right)(\gamma) = \gamma_+ \text{ and}$$

$$\left(\text{id}_M \otimes \mathcal{W}_{\Phi_{D_{n+1}}}\right)(\gamma) = \gamma_-$$

Notation 28. Under the same notation as in the above proposition, we denote

$$\left(\text{id}_M \otimes \mathcal{W}_{\Xi_{D_{n+1}}}\right)(\gamma)$$

by $\gamma_+ \dot{-} \gamma_-$, where $\Xi_{D_{n+1}} : D \rightarrow D^{n+1} \oplus D$ is the mapping

$$d \in D \mapsto (0, \dots, 0, d) \in D^{n+1} \oplus D.$$

From the very definition of $\dot{-}$, we have

Proposition 29. Let φ be a mapping from M into M' . Given $\gamma_+, \gamma_- \in M \otimes \mathcal{W}_{D_{n+1}}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_+) = \left(\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_-),$$

we have

$$\left(\text{id}_{M'} \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)\left(\left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_+)\right)$$

$$= \left(\text{id}_{M'} \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)\left(\left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_-)\right)$$

and

$$\left(\varphi \otimes \text{id}_{\mathcal{W}_D}\right)(\gamma_+ \dot{-} \gamma_-)$$

$$= \left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_+) \dot{-} \left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}}\right)(\gamma_-)$$

It is easy to see that

Proposition 30. 1. We have

$$\alpha \gamma_+ \dot{-} \alpha \gamma_- = \alpha^{n+1}(\gamma_+ \dot{-} \gamma_-) \quad (1 \leq i \leq n+1)$$

for any $\alpha \in \mathbb{R}$ and any $\gamma_{\pm} \in M \otimes \mathcal{W}_{D_{n+1}}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_+) = \left(\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}}\right)(\gamma_-).$$

2. The diagram

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \times (M \otimes \mathcal{W}_{D_{n+1}}) & \rightarrow & \left((M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \times (M \otimes \mathcal{W}_{D_{n+1}}) \right) \\ \downarrow & & \downarrow \\ M \otimes \mathcal{W}_D & \rightarrow & M \otimes \mathcal{W}_{D \times D_m} \end{array}$$

commutes, where the upper horizontal arrow is the composition of mappings

$$\begin{aligned} & (M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \times (M \otimes \mathcal{W}_{D_{n+1}}) \\ & \xrightarrow{(\text{id}_M \otimes \mathcal{W}_{\times D_{n+1} \times D_m \rightarrow D_{n+1}}) \times (\text{id}_M \otimes \mathcal{W}_{\times D_{n+1} \times D_m \rightarrow D_{n+1}})} \\ & (M \otimes \mathcal{W}_{D_{n+1} \times D_m})_{M \otimes \mathcal{W}_{D_n \times D_m}} \times (M \otimes \mathcal{W}_{D_{n+1} \times D_m}) \\ & = \left((M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \times (M \otimes \mathcal{W}_{D_{n+1}}) \right) \otimes \mathcal{W}_{D_m}, \end{aligned}$$

the lower horizontal arrow is

$$(M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \otimes \mathcal{W}_{d \in D_m \mapsto d^n \in D_m},$$

the left vertical arrow is

$$(\gamma_+, \gamma_-) \in (M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \times (M \otimes \mathcal{W}_{D_{n+1}}) \mapsto \gamma_+ \dot{-} \gamma_- \in M \otimes \mathcal{W}_D,$$

and the right vertical arrow is

$$\begin{aligned} & (\gamma_+, \gamma_-) \in (M \otimes \mathcal{W}_{D_{n+1}})_{M \otimes \mathcal{W}_{D_n}} \times (M \otimes \mathcal{W}_{D_{n+1}}) \mapsto \gamma_+ \dot{-} \gamma_- \in M \otimes \mathcal{W}_D \\ & \otimes \text{id}_{\mathcal{W}_{D_m}}. \end{aligned}$$

Lemma 31. The diagram

$$\begin{array}{ccccc} 1 & & \xrightarrow{i_{1 \rightarrow D}} & & D \\ i_{1 \rightarrow D_{n+1}} \downarrow & & & & \downarrow \Xi_{D_{n+1}} \\ D_{n+1} & & \xrightarrow{\Phi_{D_{n+1}}} & & D_{n+1} \oplus D \end{array}$$

is a quasi-colimit diagram, where $i_{1 \rightarrow D_{n+1}}$ is the canonical injection.

This implies at once that

Proposition 32. Given $t \in M \otimes \mathcal{W}_D$ and $\gamma \in M \otimes \mathcal{W}_{D_{n+1}}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{i_1 \rightarrow D_{n+1}} \right) (\gamma) = \left(\text{id}_M \otimes \mathcal{W}_{i_1 \rightarrow D} \right) (t),$$

there exists a unique function $\tilde{\gamma} : D_{n+1} \oplus D \rightarrow M$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{\Phi_{D_{n+1}}} \right) (\tilde{\gamma}) = \gamma$$

and

$$\left(\text{id}_M \otimes \mathcal{W}_{\Xi_{D_{n+1}}} \right) (\tilde{\gamma}) = t.$$

Notation 33. Under the same notation as in the above proposition, we denote

$$\left(\text{id}_M \otimes \mathcal{W}_{\Psi_{D_{n+1}}} \right) (\tilde{\gamma})$$

by $t \dot{+} \gamma$.

From the very definition of $\dot{+}$ we have

Proposition 34. Let φ be a mapping of M into M' . Given $t \in M \otimes \mathcal{W}_D$ and $\gamma \in M \otimes \mathcal{W}_{D_{n+1}}$ with

$$\left(\text{id}_M \otimes \mathcal{W}_{i_1 \rightarrow D_{n+1}} \right) (\gamma) = \left(\text{id}_M \otimes \mathcal{W}_{i_1 \rightarrow D} \right) (t),$$

we have

$$\left(\text{id}_{M'} \otimes \mathcal{W}_{i_1 \rightarrow D_{n+1}} \right) \left((\varphi \otimes \text{id}_{\mathcal{W}_D}) (t) \right) = \left(\text{id}_{M'} \otimes \mathcal{W}_{i_1 \rightarrow D_{n+1}} \right) \left(\left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}} \right) (\gamma) \right)$$

and

$$\left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}} \right) (t \dot{+} \gamma) = (\varphi \otimes \text{id}_{\mathcal{W}_D}) (t) \dot{+} \left(\varphi \otimes \text{id}_{\mathcal{W}_{D_{n+1}}} \right) (\gamma)$$

Now we have the following affine bundle theorem.

Theorem 35. The canonical projection

$$\text{id}_M \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}} : M \otimes \mathcal{W}_{D_{n+1}} \rightarrow M \otimes \mathcal{W}_{D_n}$$

is an affine bundle over the vector bundle $(M \otimes \mathcal{W}_D) \times_M (M \otimes \mathcal{W}_{D_n}) \rightarrow M \otimes \mathcal{W}_{D_n}$.

3.2.2. Symmetric Forms

Definition 36. A symmetric D_n -form at $x \in E$ is a mapping $\omega_x : (M \otimes \mathcal{W}_{D_n})_{\pi(x)} \rightarrow (E \otimes \mathcal{W}_D)_x^\perp$ subject to the following conditions:

1. For any $\gamma \in (M \otimes \mathcal{W}_{D_n})_{\pi(x)}$ and any $\alpha \in \mathbb{R}$, we have

$$\omega_x(\alpha\gamma) = \alpha^n \omega_x(\gamma)$$

2. The diagram

$$\begin{array}{ccc} (M \otimes \mathcal{W}_{D_n})_{\pi(x)} & \xrightarrow{\text{id}_M \otimes \mathcal{W}_{\times D_n \times D_m \rightarrow D_n}} & (M \otimes \mathcal{W}_{D_n})_{\pi(x)} \otimes \mathcal{W}_{D_m} \\ \omega_x \downarrow & & \downarrow \omega_x \otimes \text{id}_{\mathcal{W}_{D_m}} \\ (E \otimes \mathcal{W}_D)_x & \xrightarrow{\text{id}_E \otimes \mathcal{W}_{(d,e) \in D \times D_m \rightarrow \mathbb{R}^n \in D}} & (E \otimes \mathcal{W}_D)_x \otimes \mathcal{W}_{D_m} \end{array}$$

is commutative.

3. For any simple polynomial ρ of $d \in D_n$ and any $\gamma \in (M \otimes \mathcal{W}_{D_l})_{\pi(x)}$ with $\dim_n \rho = l < n$, we have

$$\omega((M \otimes \mathcal{W}_\rho)(\gamma)) = 0$$

Notation 37. We denote by $\mathbb{S}^{D_n}(\pi)$ the totality of symmetric D_n -forms at $x \in E$. We denote by \mathbb{S}^{D_n} the set-theoretic union of $\mathbb{S}_x^{D_n}(\pi)$'s for all $x \in E$. The canonical projection $\mathbb{S}^{D_n}(\pi) \rightarrow E$ is obviously a vector bundle.

3.2.3. The Theorem

Now we turn to a variant of Theorem 25, for which we can proceed as in 3.1.3, so that proofs of the following results are omitted or merely indicated.

Proposition 38. Let $\nabla^+, \nabla^- \in \mathbb{J}_x^{D_{n+1}}(\pi)$ with $\pi_{n+1,n}(\nabla^+) = \pi_{n+1,n}(\nabla^-)$. Then the assignment $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)} \mapsto \nabla^+(\gamma) - \nabla^-(\gamma) \in (E \otimes \mathcal{W}_D)_x$ belongs to $\mathbb{S}_x^{D_{n+1}}(\pi)$.

Proposition 39. Let $\nabla \in \mathbb{J}_x^{D_{n+1}}(\pi)$ and $\omega \in \mathbb{S}_x^{D_{n+1}}(\pi)$. Then the assignment $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)} \mapsto \omega(\gamma) \dot{+} \nabla(\gamma) \in (E \otimes \mathcal{W}_{D_{n+1}})_x$ belongs to $\mathbb{J}_x^{D_{n+1}}(\pi)$.

- Notation 40.** 1. For any $\nabla^+, \nabla^- \in \mathbb{J}^{D_{n+1}}(\pi)$ with

$$\hat{\pi}_{n+1,n}(\nabla^+) = \hat{\pi}_{n+1,n}(\nabla^-),$$

we define $\nabla^+ \dot{-} \nabla^- \in \mathbb{S}^{D_{n+1}}(\pi)$ to be

$$(\nabla^+ \dot{-} \nabla^-)(\gamma) = \nabla^+(\gamma) \dot{-} \nabla^-(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)}$.

- For any $\nabla \in \mathbb{J}_x^{D_{n+1}}(\pi)$ and any $\omega \in \mathbb{S}_x^{D_{n+1}}(\pi)$ we define $\omega \dot{+} \nabla \in \mathbb{J}_x^{D_{n+1}}(\pi)$ to be

$$(\omega \dot{+} \nabla)(\gamma) = \omega(\gamma) \dot{+} \nabla(\gamma)$$

for any $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)}$.

With these two operations, we have

Theorem 41. *The bundle $\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ is an affine bundle over the vector bundle $\mathbb{S}^{D_{n+1}}(\pi) \times_E \mathbb{J}^{D_n}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$.*

Proof. This follows simply from Theorem 35. □

3.3. The Comparison between the Second and Third Approaches

Now we are in a position to investigate the relationship between the affine bundles discussed in Theorem 25 and 41. Let us begin with

Lemma 42. *Let $\gamma_{\pm} \in (M \otimes \mathcal{W}_{D_{n+1}})_x$ with*

$$\left(\text{id}_E \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}} \right) (\gamma_+) = \left(\text{id}_E \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}} \right) (\gamma_-).$$

Then

$$\begin{aligned} & \left(\text{id}_E \otimes \mathcal{W}_{+D_{\{n+1\}_n \rightarrow D^{n+1}}} \right) \left(\left(\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}} \right) (\gamma_+) \right) \\ &= \left(\text{id}_E \otimes \mathcal{W}_{i_{D_{\{n+1\}_n \rightarrow D^{n+1}}} \right) \left(\left(\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}} \right) (\gamma_-) \right) \end{aligned}$$

obtained and we have

$$\begin{aligned} & \gamma^+ \dot{-} \gamma^- \\ &= \left(\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}} \right) (\gamma^+) \dot{-} \left(\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}} \right) (\gamma^-). \end{aligned}$$

Proof. Since the diagram

$$\begin{array}{ccccc} D\{n+1\}_n & \xrightarrow{i_{D\{n+1\}_n \rightarrow D^{n+1}}} & D^{n+1} & & \\ +D_{\{n+1\}_n \rightarrow D_n} \downarrow & \xrightarrow{\quad \quad \quad} & \downarrow +D^{n+1} \rightarrow D_{n+1} & & (1) \\ D_n & \xrightarrow{i_{D_n \rightarrow D_{n+1}}} & D_{n+1} & & \end{array}$$

is commutative, we have

$$\begin{aligned}
 & \left(\text{id}_E \otimes \mathcal{W}_{i_{D\{n+1\}_n \rightarrow D^{n+1}}} \right) \left(\left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}} \right) (\gamma^+) \right) \\
 &= \left(\text{id}_E \otimes \mathcal{W}_{+_{D\{n+1\}_n \rightarrow D_n}} \right) \left(\left(\text{id}_E \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}} \right) (\gamma^+) \right) \\
 &= \left(\text{id}_E \otimes \mathcal{W}_{+_{D\{n+1\}_n \rightarrow D_n}} \right) \left(\left(\text{id}_E \otimes \mathcal{W}_{i_{D_n \rightarrow D_{n+1}}} \right) (\gamma^-) \right) \\
 &= \left(\text{id}_E \otimes \mathcal{W}_{i_{D\{n+1\}_n \rightarrow D^{n+1}}} \right) \left(\left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}} \right) (\gamma^-) \right)
 \end{aligned}$$

which establishes the coveted first statement. The second statement follows simply from a commutative cubical diagram, which is depicted here separately as the upper square (1), the lower square and the bounding four side squares:

$$\begin{array}{ccc}
 D^{n+1} & \xrightarrow{\Phi_{D^{n+1}}} & D^{n+1} \oplus D \\
 +_{D^{n+1} \rightarrow D_{n+1}} \downarrow & \xrightarrow{\quad} \downarrow & +_{D^{n+1} \rightarrow D_{n+1}} \oplus \text{id}_D \\
 D_{n+1} & \xrightarrow{\Phi_{D_{n+1}}} & D_{n+1} \oplus D
 \end{array} \tag{2}$$

$$\begin{array}{ccccccc}
 D\{n+1\}_n & \xrightarrow{i_{D\{n+1\}_n \rightarrow D^n}} & D^{n+1} & \xrightarrow{+_{D^{n+1} \rightarrow D_{n+1}}} & D_{n+1} \\
 i_{D\{n+1\}_n \rightarrow D^{n+1}} \downarrow & \xrightarrow{\quad} \downarrow & D^{n+1} \oplus D & \xrightarrow{+_{D^{n+1} \rightarrow D_{n+1}} \oplus \text{id}_D} & D_{n+1} \oplus D \\
 D^{n+1} & & & & \Psi_{D_{n+1}} \downarrow
 \end{array} \tag{3}$$

$$\begin{array}{ccccccc}
 D\{n+1\}_n & \xrightarrow{+_{D\{n+1\}_n \rightarrow D_n}} & D_n & \xrightarrow{i_{D_n \rightarrow D_{n+1}}} & D_{n+1} \\
 i_{D\{n+1\}_n \rightarrow D^{n+1}} \downarrow & \xrightarrow{\quad} \downarrow & i_{D_n \rightarrow D_{n+1}} \downarrow & \xrightarrow{\quad} \downarrow & \Psi_{D_{n+1}} \downarrow \\
 D^{n+1} & \xrightarrow{+_{D^{n+1} \rightarrow D_{n+1}}} & D_{n+1} & \xrightarrow{\Phi_{D_{n+1}}} & D_{n+1} \oplus D
 \end{array} \tag{4}$$

□

Lemma 43. Let $t \in E \otimes \mathcal{W}_D$ and $\gamma \in E \otimes \mathcal{W}_{D_{n+1}}$ with

$$\begin{aligned}
 & \left(\text{id}_E \otimes \mathcal{W}_{i_{1 \rightarrow D}} \right) (t) \\
 &= \left(\text{id}_E \otimes \mathcal{W}_{i_{1 \rightarrow D_{n+1}}} \right) (\gamma).
 \end{aligned}$$

Then we have

$$\left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}} \right) (t \dot{+} \gamma) = t \dot{+} \left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}} \right) (\gamma)$$

Proof. This follows simply from a commutative cubical diagram, which is depicted here separately as the upper square, the lower square (2) and the

rounding four side squares:

$$\begin{array}{ccccc}
 & & 1 & \xrightarrow{i_{1 \rightarrow D}} & D \\
 & & \text{id}_1 \downarrow & \xrightarrow{\quad} & \downarrow \text{id}_D \\
 & & 1 & \xrightarrow{i_{1 \rightarrow D}} & D \\
 \\
 & 1 & \xrightarrow{i_{1 \rightarrow D}} & D & \xrightarrow{\text{id}_D} & D \\
 i_{1 \rightarrow D^{n+1}} \downarrow & \xrightarrow{\Phi_{D^{n+1}}} & \Xi_{D^{n+1}} \downarrow & \xrightarrow{+_{D^{n+1} \rightarrow D_{n+1}} \oplus \text{id}_D} & \Xi_{D_{n+1}} \downarrow & \downarrow \\
 D^{n+1} & & D^{n+1} \oplus D & & D_{n+1} & D \\
 \\
 & 1 & \xrightarrow{\text{id}_1} & 1 & \xrightarrow{i_{1 \rightarrow D_{n+1}}} & D_{n+1} \\
 i_{1 \rightarrow D^{n+1}} \downarrow & \xrightarrow{+_{D^{n+1} \rightarrow D_{n+1}}} & i_{1 \rightarrow D_{n+1}} \downarrow & \xrightarrow{\Phi_{D_{n+1}}} & \Xi_{D_{n+1}} \downarrow & \downarrow \\
 D^{n+1} & & D_{n+1} & & D_{n+1} \oplus D &
 \end{array}$$

□

Now we are ready to state the main result of this subsection.

Theorem 44. *We have the following.*

1. For any $\nabla^+, \nabla^- \in \mathbb{J}_x^{D^{n+1}}(\pi)$ and any $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)}$ with $\pi_{n+1,n}(\nabla^+) = \pi_{n+1,n}(\nabla^-)$,

we have

$$\begin{aligned}
 & \psi_{n+1}(\nabla^+)(\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\gamma) \\
 &= \psi_{n+1}^+(\left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}}\right)(\gamma)) \dot{-} \nabla^-\left(\left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}}\right)(\gamma)\right)
 \end{aligned}$$

For any $\nabla \in \mathbb{J}_x^{D^{n+1}}(\pi)$, any $t \in (M \otimes \mathcal{W}_D)_{\pi(x)}$ and any

$$\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)},$$

we have

$$\begin{aligned}
 & \left(\text{id}_E \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}}\right) \left(\left(\pi_{n+1,1}(\psi_{n+1}(\nabla))\right)(t) \dot{+} \psi_{n+1}(\nabla)(\gamma)\right) \\
 &= \left(\pi_{n+1,1}(\nabla)\right)(t) \dot{+} \nabla\left(\left(\text{id}_M \otimes \mathcal{W}_{+_{D^{n+1} \rightarrow D_{n+1}}}\right)(\gamma)\right)
 \end{aligned}$$

Proof. We deal with the two statements separately.

1. Since

$$\begin{aligned} & \nabla^\pm((\text{id}_M \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\gamma)) \\ &= (\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})((\psi_{n+1}(\nabla^\pm))(\gamma)) \end{aligned}$$

by the very definition of $\psi_{n+1}(\nabla^\pm)$, we have

$$\begin{aligned} & \nabla^+((\text{id}_M \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\gamma)) \dot{-} \nabla^-((\text{id}_M \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\gamma)) \\ &= (\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})((\psi_{n+1}(\nabla^+))(\gamma)) \dot{-} (\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}}) \\ & \quad ((\psi_{n+1}(\nabla^-))(\gamma)) \\ &= \psi_{n+1}(\nabla^+)(\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\gamma) \quad [\text{by Lemma 42}] \end{aligned}$$

2. Since

$$\begin{aligned} & \nabla((\text{id}_M \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\gamma)) \\ &= (\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\psi_{n+1}(\nabla)(\gamma)) \end{aligned}$$

by the very definition of $\psi_{n+1}(\nabla)$,

$$(\pi_{n+1,1}(\nabla))(\gamma) = (\pi_{n+1,1}(\psi_{n+1}(\nabla)))(t)$$

by dint of Proposition 41, we have

$$\begin{aligned} & (\pi_{n+1,1}(\nabla))(\gamma) \dot{+} \nabla((\text{id}_M \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\gamma)) \\ &= (\pi_{n+1,1}(\nabla))(\gamma) \dot{+} (\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})((\psi_{n+1}(\nabla))(\gamma)) \\ &= (\text{id}_E \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})((\pi_{n+1,1}(\psi_{n+1}(\nabla)))(t) \dot{+} \psi_{n+1}(\nabla)(\gamma)) \\ & \quad [\text{by Lemma 43}] \end{aligned}$$

□

Now we would like to discuss the relationship between $\mathbb{S}^{D^{n+1}}(\pi)$ and $\mathbb{S}^{D_{n+1}}$ (π).

Proposition 45. For any $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$, the mapping

$$\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)} \mapsto \omega \left((\text{id}_M \otimes \mathcal{W}_{+D^{n+1} \rightarrow D_{n+1}})(\gamma) \right),$$

denoted by $\phi_{n+1}(\omega)$, belongs to $\mathbb{S}_x^{D_{n+1}}(\pi)$, thereby giving rise to a function $\phi_{n+1} : \mathbb{S}^{D^{n+1}}(\pi) \rightarrow \mathbb{S}^{D_{n+1}}(\pi)$.

Proof. For $n = 0$, the statement is trivial. For any $\omega \in \mathbb{S}_x^{D^{n+1}}(\pi)$, there exist $\nabla^+, \nabla^- \in \mathbb{J}_x^{D^{n+1}}(\pi)$, by dint of Theorem 25, such that

$$\pi_{n+1,n}(\nabla^+) = \pi_{n+1,n}(\nabla^-)$$

and

$$\omega = \nabla^+ \dot{-} \nabla^-.$$

Then we have the following:

1. Let $\alpha \in \mathbb{R}$ and $\gamma \in (M \otimes \mathcal{W}_{D_{n+1}})_{\pi(x)}$. Then we have

$$\begin{aligned} & \omega \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) (\alpha\gamma) \right) \\ &= \nabla^+ \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) (\alpha\gamma) \right) \\ & \quad \dot{-} \nabla^- \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) (\alpha\gamma) \right) \\ &= \psi_{n+1}(\nabla^+)(\alpha\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\alpha\gamma) \quad [\text{by Theorem 44}] \\ &= \alpha(\psi_{n+1}(\nabla^+)(\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\gamma)) \\ &= \alpha^{n+1}(\psi_{n+1}(\nabla^+)(\gamma) \dot{-} \psi_{n+1}(\nabla^-)(\gamma)) \\ &= \alpha^{n+1}(\pi_{n+1}(\gamma_{D_{n+1}}) \dot{-} \pi_{n+1}(\gamma_{D_{n+1}})) \quad [\text{by Theorem 44}] \\ &= \alpha^{n+1} \omega \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) (\gamma) \right) \end{aligned}$$

so that $\phi_{n+1}(\omega)$ abides by the first condition in Definition 36.

2. The proof that the mapping $\phi_{n+1}(\omega)$ abides by the second condition in Definition 36, which is similar to the above, is safely left to the reader.

Let ρ be a simple polynomial of $d \in D_{n+1}$ and $\gamma \in (M \otimes \mathcal{W}_{D_l})_{\pi(x)}$ with $\text{deg}_{n+1} \rho = l < n + 1$, we have

$$\begin{aligned} & \omega \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) \left((\text{id}_M \otimes \mathcal{W}_\rho) (\gamma) \right) \right) \\ &= \nabla^+ \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) \left((\text{id}_M \otimes \mathcal{W}_\rho) (\gamma) \right) \right) \dot{-} \\ & \quad \nabla^- \left(\left(\text{id}_M \otimes \mathcal{W}_{+D_{n+1} \rightarrow D_{n+1}} \right) \left((\text{id}_M \otimes \mathcal{W}_\rho) (\gamma) \right) \right) \\ &= (\psi_{n+1}(\nabla^+)) \left((\text{id}_M \otimes \mathcal{W}_\rho) (\gamma) \right) \dot{-} (\psi_{n+1}(\nabla^-)) \left((\text{id}_M \otimes \mathcal{W}_\rho) (\gamma) \right) \\ & [\text{by Theorem 44}] \\ &= (\text{id}_E \otimes \mathcal{W}_\rho) \left((\pi_{n+1,l}(\psi_{n+1}(\nabla^+))) (\gamma) \right) \end{aligned}$$

$$\begin{aligned} & \dot{-} (\text{id}_E \otimes \mathcal{W}_\rho) ((\pi_{n+1,l}(\psi_{n+1}(\nabla^-))) (\gamma)) \\ &= (\text{id}_E \otimes \mathcal{W}_\rho) ((\psi_l(\pi_{n+1,l}(\nabla^+))) (\gamma)) \dot{-} (\text{id}_E \otimes \mathcal{W}_\rho) ((\psi_l(\pi_{n+1,l}(\nabla^-))) (\gamma)) \\ & \text{[by Proposition 4]} \\ &= 0, \end{aligned}$$

so that $\phi_{n+1}(\omega)$ abides by the third condition in Definition 36. □

Let us fix our terminology.

Definition 46. Given an affine bundle $\pi_1 : E_1 \rightarrow M_1$ over a vector bundle $\xi_1 : P_1 \rightarrow M_1$ and another affine bundle $\pi_2 : E_2 \rightarrow M_2$ over another vector bundle $\xi_2 : P_2 \rightarrow M_2$, a triple (f, g, h) of mappings $f : M_1 \rightarrow M_2$, $g : E_1 \rightarrow E_2$ and $h : P_1 \rightarrow P_2$ is called a *morphism of affine bundles* from the affine bundle $\pi_1 : E_1 \rightarrow M_1$ over the vector bundle $\xi_1 : P_1 \rightarrow M_1$ to the affine bundle $\pi_2 : E_2 \rightarrow M_2$ over the vector bundle $\xi_2 : P_2 \rightarrow M_2$ provided that they abide by the following three conditions:

1. (f, g) is a morphism of bundles from π_1 to π_2 . In other words, the following diagram is commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

2. (f, h) is a morphism of bundles from ξ_1 to ξ_2 . In other words, the following diagram is commutative:

$$\begin{array}{ccc} P_1 & \xrightarrow{h} & P_2 \\ \xi_1 \downarrow & & \downarrow \xi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

3. For any $x \in M_1$, $(g|_{E_{1,x}}, h|_{P_{1,x}})$ is a morphism of affine spaces from $(E_{1,x}, P_{1,x})$ to $(E_{2,x}, P_{2,x})$.

Using this terminology, we can summarize Theorem 44 succinctly as follows:

Theorem 47. *The triple $(\psi_n, \psi_{n+1}, \phi_{n+1} \times \psi_n)$ of mappings is a morphism of affine bundles from the affine bundle $\pi_{n+1,n} : \mathbb{J}^{D^{n+1}}(\pi) \rightarrow \mathbb{J}^{D^n}(\pi)$ over the vector bundle $\mathbb{S}^{D^{n+1}}(\pi) \times \mathbb{J}^{D^n}(\pi) \rightarrow \mathbb{J}^{D^n}(\pi)$ in Theorem 25 to the affine bundle*

$\pi_{n+1,n} : \mathbb{J}^{D_{n+1}}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ over the vector bundle $\mathbb{S}^{D_{n+1}}(\pi) \times_{\mathbb{J}^{D_n}(\pi)} \mathbb{J}^{D_n}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ in Theorem 41.

4. Taylor Representations

The following results are merely special cases of the general Taylor representation theorem such as seen in [2] (Part III, Proposition 5.2).

Proposition 48. Any $f \in \mathbb{R}^p \otimes \mathcal{W}_{D_n}$ is of a unique Taylor representation of the form

$$\delta \in \mathbb{R} \mapsto (x^i) + \delta(y_1^i) + \delta^2(y_2^i) + \dots + \delta^n(y_n^i) \in \mathbb{R}^p$$

with $(x^i), (y_1^i), (y_2^i), \dots, (y_n^i) \in \mathbb{R}^p$.

Proposition 49. Any $f \in \mathbb{R}^p \otimes \mathcal{W}_{D_n}$ is of a unique Taylor representation of the form

$$(\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mapsto (x^i) + \sum_{r=1}^n \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p$$

with $(x^i), (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p$

Proposition 50. Any $f \in \mathbb{R}^p \otimes \mathcal{W}_{D(n)_m}$ is of a unique Taylor representation of the form

$$(\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mapsto (x^i) + \sum_{r=1}^m \sum_{1 \leq k_1 \leq \dots \leq k_r \leq n} \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p$$

with $(x^i), (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p$.

5. The Basic Framework with Coordinates

This section is inspired much by [1].

5.1. The Basic Framework

Notation 51. We denote by $\mathcal{J}^n(\pi)$ the totality of

$$\gamma \in E \otimes \mathcal{W}_{D(p)_n}$$

such that

$$\left(\pi \otimes \text{id}_{\mathcal{W}_{D(p)n}}\right)(\gamma) \in M \otimes \mathcal{W}_{D(p)n}$$

is degenerate in the sense that

$$\begin{aligned} &\left(\pi \otimes \text{id}_{\mathcal{W}_{D(p)n}}\right)(\gamma) \\ &= (\text{id}_M \otimes \mathcal{W}_{D(p)n \rightarrow 1})(\gamma') \end{aligned}$$

for some $\gamma' \in M \otimes \mathcal{W}_1 = M$.

Notation 52. We denote by $\mathcal{S}^n(\pi)$ the totality of

$$t \in E \otimes \mathcal{W}_{D(p+n)C_{n+1}}$$

such that

$$\left(\pi \otimes \text{id}_{\mathcal{W}_{D(p+n)C_{n+1}}}\right)(t) \in M \otimes \mathcal{W}_{D(p+n)C_{n+1}}$$

is degenerate in the sense that

$$\begin{aligned} &\left(\pi \otimes \text{id}_{\mathcal{W}_{D(p+n)C_{n+1}}}\right)(t) \\ &= (\text{id}_M \otimes \mathcal{W}_{D(p+n)C_{n+1} \rightarrow 1})(t') \end{aligned}$$

for some $t' \in M \otimes \mathcal{W}_1 = M$.

Remark 53. 1. Each $\gamma \in E \otimes \mathcal{W}_{D(p)n}$ can be identified uniquely with a sequence

$$\left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1}^i, x_{i_1, i_2}^i, \dots, x_{i_1, i_2, \dots, i_n}^i, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j\right)_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq p}$$

of real numbers at length

$$p + q + (p + q)p + \dots + (p + q)_{p+n-1} C_n$$

in the sense that the Taylor representation of γ is

$$\begin{aligned} (\delta_1, \dots, \delta_p) \in \mathbb{R}^p &\mapsto (x^1, \dots, x^p, u^1, \dots, u^q) + \sum_{1 \leq i_1 \leq p} (x_{i_1}^1, \dots, x_{i_1}^p, u_{i_1}^1, \dots, u_{i_1}^q) \delta_{i_1} \\ &+ \sum_{1 \leq i_1 \leq i_2 \leq p} (x_{i_1, i_2}^1, \dots, x_{i_1, i_2}^p, u_{i_1, i_2}^1, \dots, u_{i_1, i_2}^q) \delta_{i_1} \delta_{i_2} + \dots \end{aligned}$$

$$\begin{aligned}
 & + \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq p} \left(x_{i_1, i_2, \dots, i_n}^1, \dots, x_{i_1, i_2, \dots, i_n}^p, u_{i_1, i_2, \dots, i_n}^1, \dots, \right. \\
 & \left. u_{i_1, i_2, \dots, i_n}^q \right) \delta_{i_1} \delta_{i_2} \dots \delta_{i_n} \\
 & \in \mathbb{R}^{p+q}
 \end{aligned}$$

2. Each $\nabla \in \mathcal{J}^n(\pi)$ can be identified uniquely with a sequence

$$\left(x^1, \dots, x^p, u^1, \dots, u^q, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j \right)_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq p}}$$

of real numbers at length

$$p + q + qp + \dots + \binom{p+q-1}{p-1} C_n$$

in the sense that the Taylor representation of ∇ is

$$\begin{aligned}
 (\delta_1, \dots, \delta_p) \in \mathbb{R}^p & \mapsto (x^1, \dots, x^p, u^1, \dots, u^q) + \sum_{1 \leq i_1 \leq p} (0, \dots, 0, u_{i_1}^1, \dots, u_{i_1}^q) \delta_{i_1} \\
 & + \sum_{1 \leq i_1 \leq i_2 \leq p} (0, \dots, u_{i_1, i_2}^1, \dots, u_{i_1, i_2}^q) \delta_{i_1} \delta_{i_2} + \dots \\
 & + \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq p \\ n \leq p+q}} (0, \dots, 0, u_{i_1, i_2, \dots, i_n}^1, \dots, u_{i_1, i_2, \dots, i_n}^q) \delta_{i_1} \delta_{i_2} \dots \delta_{i_n}
 \end{aligned}$$

3. Each $\nabla \in F \otimes \mathcal{W}_{D(p+n, C_{n+1})}$ can be identified uniquely with a sequence

$$\left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1, i_2, \dots, i_{n+1}}^1, \dots, x_{i_1, i_2, \dots, i_{n+1}}^p, u_{i_1, i_2, \dots, i_{n+1}}^1, \dots, \right. \\
 \left. u_{i_1, i_2, \dots, i_{n+1}}^q \right)_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq p}$$

of real numbers at length

$$p + q + (p + q)_{p+n} C_{n+1}$$

in the sense that the Taylor representation of t is

$$\begin{aligned}
 & (\delta_{i_1, i_2, \dots, i_{n+1}})_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq p} \\
 & \in \mathbb{R}^{(p+n)C_{n+1}} \mapsto (x^1, \dots, x^p, u^1, \dots, u^q)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq p} \left(x_{i_1, i_2, \dots, i_{n+1}}^1, \dots, x_{i_1, i_2, \dots, i_{n+1}}^p, u_{i_1, i_2, \dots, i_{n+1}}^1, \dots, \right. \\
 & \left. u_{i_1, i_2, \dots, i_{n+1}}^q \right) \delta_{i_1, i_2, \dots, i_{n+1}} \\
 & \in \mathbb{R}^{p+q}
 \end{aligned}$$

4. Each $\omega \in \mathcal{S}^{n+1}(\pi)$ can be identified uniquely with a sequence

$$\left(x^1, \dots, x^p, u^1, \dots, u^q, u_{i_1, i_2, \dots, i_{n+1}}^1, \dots, u_{i_1, i_2, \dots, i_{n+1}}^q \right)_{1 \leq i_1 \leq \dots \leq i_{n+1} \leq p}$$

of real numbers at length

$$p + q + q_{p+n} C_{n+1}$$

in the sense that the Taylor representation of ω is

$$\begin{aligned}
 & \left(\delta_{i_1, i_2, \dots, i_{n+1}} \right)_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq p} \\
 & \in \mathbb{R}^{(p+n)C_{n+1}} \mapsto \left(x^1, \dots, x^p, u^1, \dots, u^q \right) \\
 & \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_{n+1} \leq p} \left(0, \dots, 0, u_{i_1, i_2, \dots, i_{n+1}}^1, \dots, u_{i_1, i_2, \dots, i_{n+1}}^q \right) \delta_{i_1, i_2, \dots, i_{n+1}} \in \mathbb{R}^{p+q}
 \end{aligned}$$

Notation 54. Since $\mathcal{W}_{D(p)/n} = D(p)_{n+1}$, there is a canonical projection $E \otimes \mathcal{W}_{D(p)/n+1} \rightarrow E \otimes \mathcal{W}_{D(p)/n}$, which restricts itself naturally to $\mathcal{J}^{n+1}(\pi) \rightarrow \mathcal{J}^n(\pi)$. Both of them are denoted by $\pi_{n+1, n}$. We have

$$\begin{aligned}
 \pi_{n+1, n} \left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1}^i, \dots, x_{i_1, i_2, \dots, i_n}^j, x_{i_1, i_2, \dots, i_{n+1}}^i, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j, \right. \\
 \left. u_{i_1, i_2, \dots, i_{n+1}}^j \right) \\
 \left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1}^i, x_{i_1, i_2}^i, \dots, x_{i_1, i_2, \dots, i_n}^i, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j \right)
 \end{aligned}$$

and

$$\begin{aligned}
 & \pi_{n+1, n} \left(x^1, \dots, x^p, u^1, \dots, u^q, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j, u_{i_1, i_2, \dots, i_{n+1}}^j \right) \\
 & = \left(x^1, \dots, x^p, u^1, \dots, u^q, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j \right)
 \end{aligned}$$

5.2. The Affine Bundle Theorem within the Basic Framework

It is easy to see that

Lemma 55. *The diagram*

$$\begin{array}{ccccc}
 D(p)_n & & i_{D(p)_n \rightarrow D(p)_{n+1}} & & D(p)_{n+1} \\
 \downarrow i_{D(p)_n \rightarrow D(p)_{n+1}} & & \xrightarrow{\Phi_{D(p)_{n+1}}} & & \downarrow \Psi_{D(p)_{n+1}} \\
 D(p)_{n+1} & & & & D(p)_{n+1} \oplus D(p+n)C_{n+1}
 \end{array}$$

is a quasi-colimit diagram, where $\Phi_{D(p)_{n+1}}$ is the canonical injection of $D(p)_{n+1}$ into $D(p)_{n+1} \oplus D(p+n)C_{n+1}$, and $\Psi_{D(p)_{n+1}}$ is the mapping

$$(d_1, \dots, d_p) \in D(p)_{n+1} \mapsto (d_1, \dots, d_p, d_1^{n+1}, d_1^n d_2, \dots) \in D(p)_{n+1} \oplus D(p+n)C_{n+1}$$

with the sequence $d_1^{n+1}, d_1^n d_2, \dots$ being that of $d_{k_1}, \dots, d_{k_{n+1}}$'s ($1 \leq k_1 \leq k_2 \leq \dots \leq k_{n+1} \leq n+1$) in lexicographic order.

This implies at once that

Proposition 56. *Given $\gamma_+, \gamma_- \in E \otimes \mathcal{W}_{D(p)_{n+1}}$ with*

$$\left(\text{id}_E \otimes \mathcal{W}_{i_{D(p)_n \rightarrow D(p)_{n+1}}} \right) (\gamma_+) = \left(\text{id}_E \otimes \mathcal{W}_{i_{D(p)_n \rightarrow D(p)_{n+1}}} \right) (\gamma_-),$$

there exists unique $\gamma \in E \otimes \mathcal{W}_{D(p)_{n+1} \oplus D(p+n)C_{n+1}}$ with

$$\left(\text{id}_E \otimes \mathcal{W}_{\Psi_{D(p)_{n+1}}} \right) (\gamma) = \gamma_+ \text{ and}$$

$$\left(\text{id}_E \otimes \mathcal{W}_{\Phi_{D(p)_{n+1}}} \right) (\gamma) = \gamma_-$$

Notation 57. Under the same notation as in the above proposition, we denote

$$\left(\text{id}_E \otimes \mathcal{W}_{\Xi_{D(p)_{n+1}}} \right) (\gamma)$$

by

$$\gamma_+ \dot{-} \gamma_-,$$

where $\Xi_{D^n} : D(p+n)C_{n+1} \rightarrow D(p)_{n+1} \oplus D(p+n)C_{n+1}$ is the canonical injection.

Remark 58. Given

$$\gamma_+ = \left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1}^i, x_{i_1, i_2}^i, \dots, x_{i_1, i_2, \dots, i_{n+1}}^i, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_{n+1}}^j \right),$$

$$\gamma_- = \left(y^1, \dots, y^p, v^1, \dots, v^q, y_{i_1}^i, y_{i_1, i_2}^i, \dots, y_{i_1, i_2, \dots, i_{n+1}}^i, v_{i_1}^j, v_{i_1, i_2}^j, \dots, v_{i_1, i_2, \dots, i_{n+1}}^j \right)$$

$$\in E \otimes \mathcal{W}_{D(p)_{n+1}},$$

we have

$$\left(\text{id}_E \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}\right) (\gamma_+) = \left(\text{id}_E \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}\right) (\gamma_-)$$

iff

$$\begin{aligned} & \left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1, i_2}^i, \dots, x_{i_1, i_2, \dots, i_n}^i, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j\right) \\ & = \left(y^1, \dots, y^p, v^1, \dots, v^q, y_{i_1, i_2}^i, \dots, y_{i_1, i_2, \dots, i_n}^i, v_{i_1, i_2}^j, \dots, v_{i_1, i_2, \dots, i_n}^j\right), \end{aligned}$$

in which we get

$$\begin{aligned} & \gamma_+ \dot{-} \gamma_- \\ & = \left(x^1, \dots, x^p, u^1, \dots, u^q, u_{i_1, i_2, \dots, i_{n+1}}^1 - v_{i_1, i_2, \dots, i_{n+1}}^1, \dots, u_{i_1, i_2, \dots, i_{n+1}}^q - v_{i_1, i_2, \dots, i_{n+1}}^q\right) \end{aligned}$$

From the very definition of $\dot{-}$, we have

Proposition 59. *Let F be a mapping of E into E' . Given $\gamma_+, \gamma_- \in E \otimes \mathcal{W}_{D(p)n+1}$ with*

$$\left(\text{id}_E \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}\right) (\gamma_+) = \left(\text{id}_E \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}\right) (\gamma_-),$$

we have

$$\begin{aligned} & \left(\text{id}_{E'} \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}\right) \left(\left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}}\right) (\gamma_+)\right) \\ & = \left(\text{id}_{E'} \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}\right) \left(\left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}}\right) (\gamma_-)\right) \end{aligned}$$

and

$$\begin{aligned} & \left(F \otimes \text{id}_{\mathcal{W}_{D(p+n)C_{n+1}}}\right) (\gamma_+ \dot{-} \gamma_-) \\ & = \left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}}\right) (\gamma_+) \dot{-} \left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}}\right) (\gamma_-) \end{aligned}$$

Lemma 60. *The diagram*

$$\begin{array}{ccc} 1 & \xrightarrow{i_1 \rightarrow D(p+n)C_{n+1}} & D(p+n)C_{n+1} \\ i_1 \rightarrow D(p)n+1 \downarrow & \xrightarrow{\Phi_{D(p)n+1}} & \downarrow \Xi_{D(p)n+1} \\ D(p)n+1 & & D(p)n+1 \oplus D(p+n)C_{n+1} \end{array}$$

is a quasi-colimit diagram.

This implies at once that

Proposition 61. Given $t \in E \otimes \mathcal{W}_{D(p+n)C_{n+1}}$ and $\gamma \in E \otimes \mathcal{W}_{D(p)n_{n+1}}$ with

$$\left(\text{id}_E \otimes \mathcal{W}_{i_1 \rightarrow D(p+n)C_{n+1}} \right) (t) = \left(\text{id}_E \otimes \mathcal{W}_{i_1 \rightarrow D(p)n_{n+1}} \right) (\gamma),$$

there exists unique $\gamma' \in E \otimes \mathcal{W}_{D(p)n_{n+1} \oplus D(p+n)C_{n+1}}$ with

$$\begin{aligned} \left(\text{id}_E \otimes \mathcal{W}_{\Xi_{D(p)n_{n+1}}} \right) (\gamma') &= t \text{ and} \\ \left(\text{id}_E \otimes \mathcal{W}_{\Phi_{D(p)n_{n+1}}} \right) (\gamma') &= \gamma. \end{aligned}$$

Notation 62. Under the same notation as in the above proposition, we denote

$$\left(\text{id}_E \otimes \mathcal{W}_{\Psi_{D(p)n_{n+1}}} \right) (\gamma')$$

by $t \dot{+} \gamma$.

Remark 63. Given

$$\begin{aligned} \gamma &= \left(x^1, \dots, x^p, u^1, \dots, u^q, x_{i_1, i_2, \dots, i_{n+1}}^1, \dots, x_{i_1, i_2, \dots, i_{n+1}}^p, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_{n+1}}^j \right) \\ &\in E \otimes \mathcal{W}_{D(p)n_{n+1}} \end{aligned}$$

and

$$\begin{aligned} t &= \left(y^1, \dots, y^p, v^1, \dots, v^q, y_{i_1, i_2, \dots, i_{n+1}}^1, \dots, y_{i_1, i_2, \dots, i_{n+1}}^p, v_{i_1, i_2, \dots, i_{n+1}}^1, \dots, v_{i_1, i_2, \dots, i_{n+1}}^q \right) \\ &\in E \otimes \mathcal{W}_{D(p+n)C_{n+1}}, \end{aligned}$$

we have

$$\left(\text{id}_E \otimes \mathcal{W}_{i_1 \rightarrow D(p+n)C_{n+1}} \right) (t) = \left(\text{id}_E \otimes \mathcal{W}_{i_1 \rightarrow D(p)n_{n+1}} \right) (\gamma)$$

iff

$$(x^1, \dots, x^p, u^1, \dots, u^q) = (y^1, \dots, y^p, v^1, \dots, v^q),$$

in which we get

$$\begin{aligned} t \dot{+} \gamma &= \left(x^i, u^j, x_{i_1}^i, \dots, x_{i_1, i_2, \dots, i_n}^i, x_{i_1, i_2, \dots, i_{n+1}}^i + y_{i_1, i_2, \dots, i_{n+1}}^i, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j, \right. \\ &\quad \left. u_{i_1, i_2, \dots, i_{n+1}}^j + v_{i_1, i_2, \dots, i_{n+1}}^j \right). \end{aligned}$$

From the very definition of $\dot{+}$, we have

Proposition 64. Let F be a mapping of E into E' . Given $t \in E \otimes \mathcal{W}_{D(p+n)C_{n+1}}$ and $\gamma \in E \otimes \mathcal{W}_{D(p)n+1}$ with

$$\left(\text{id}_E \otimes \mathcal{W}_{i_1 \rightarrow D(p+n)C_{n+1}} \right) (t) = \left(\text{id}_E \otimes \mathcal{W}_{i_1 \rightarrow D(p)n+1} \right) (\gamma),$$

we have

$$\begin{aligned} & \left(\text{id}_{E'} \otimes \mathcal{W}_{i_1 \rightarrow D(p+n)C_{n+1}} \right) \left(\left(F \otimes \text{id}_{\mathcal{W}_{D(p+n)C_{n+1}}} \right) (t) \right) \\ &= \left(\text{id}_{E'} \otimes \mathcal{W}_{i_1 \rightarrow D(p)n+1} \right) \left(\left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}} \right) (\gamma) \right) \end{aligned}$$

and

$$\left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}} \right) (t + \gamma) = \left(F \otimes \text{id}_{\mathcal{W}_{D(p+n)C_{n+1}}} \right) (t) + \left(F \otimes \text{id}_{\mathcal{W}_{D(p)n+1}} \right) (\gamma),$$

Now we have

Theorem 65. The canonical projection $\text{id}_{E \otimes \mathcal{W}_{i_{D(p)n \rightarrow D(p)n+1}}} : E \otimes \mathcal{W}_{D(p)n+1} \rightarrow E \otimes \mathcal{W}_{D(p)n}$ is an affine bundle over the vector bundle

$$\left(E \otimes \mathcal{W}_{D(p+n)C_{n+1}} \right)^\perp_{E \otimes \mathcal{W}_{D(p)n+1}} \rightarrow E \otimes \mathcal{W}_{D(p)n}.$$

Theorem 66. The map $\text{id}_{\mathcal{S}^{n+1}(\pi) \times \mathcal{J}^n(\pi)} : \mathcal{J}^{n+1}(\pi) \rightarrow \mathcal{J}^n(\pi)$ is an affine bundle over the vector bundle $\mathcal{S}^n(\pi) \times \mathcal{J}^n(\pi) \rightarrow \mathcal{J}^n(\pi)$.

2. The First Approach with Coordinates

Definition 67. We define $\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} : \mathcal{J}^1(\pi) \rightarrow \mathbf{J}^1(\pi)$ to be

$$\begin{aligned} & \theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} \left(x^1, \dots, x^p, u^1, \dots, u^q, u_i^j \right) \\ &= \left[\delta \in \mathbb{R} \mapsto (x^1, \dots, x^p) + (y^1, \dots, y^p) \delta \in \mathbb{R}^p \right] \in (M \otimes \mathcal{W}_D)_{(x^1, \dots, x^p)} \mapsto \\ & \left[\delta \in \mathbb{R} \mapsto (x^1, \dots, x^p, u^1, \dots, u^q) + \left(y^1, \dots, y^p, \sum_{i=1}^p u_i^j y^i \right) \delta \in \mathbb{R}^{p+q} \right] \\ & \in (E \otimes \mathcal{W}_D)_{(x^1, \dots, x^p, u^1, \dots, u^q)} \end{aligned}$$

Remark 68. It is easy to see that the right-hand side of the above formula belongs to $\mathbf{J}^1(\pi)$.

Theorem 69. *The mapping $\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} : \mathcal{J}^1(\pi) \rightarrow \mathbf{J}^1(\pi)$ is bijective.*

Remark 70. This gives a coordinate description of $\mathbf{J}^1(\pi)$.

Now we are going to consider $\tilde{\mathbf{J}}^2(\pi) = \mathbf{J}^1(\pi_1)$, which has a coordinate description as follows:

$$\begin{aligned} & \theta_{\mathbf{J}^1(\pi_1)}^{\mathcal{J}^1(\pi_1)} \left(x^i, u^j, u_{i_1}^j, u_{i_2}^j, u_{i_1; i_2}^j \right) \\ &= \left[\delta \in \mathbb{R} \mapsto (x^i) + (y^i) \delta \in \mathbb{R}^p \right] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\ & \left[\delta \in \mathbb{R} \mapsto (x^i, u^j, u_{i_1}^j) + \left(y^i, \sum_{i_2=1}^p u_{i_2}^j y^{i_2}, \sum_{i_2=1}^p u_{i_1}^j y^{i_2} \right) \delta \in \mathbb{R}^{q+pq} \right] \\ & \in (E \otimes \mathcal{W}_D)_{(x^i, u^j, u_{i_1}^j)}, \end{aligned}$$

for which we get

Proposition 71. *We have*

$$(x^i, u^j; u_{i_1}^j, u_{i_2}^j, u_{i_1; i_2}^j) \in \hat{\mathbf{J}}^2(\pi)$$

iff $u_{i_1}^j = u_{i_2}^j$ for all $1 \leq i \leq p$ and all $1 \leq j \leq q$.

Proof. It is easy to see that $(x^i, u^j, u_{i_1}^j, u_{i_2}^j, u_{i_1; i_2}^j) \in \mathbf{J}^1(\pi_1)$ is $\pi_{1,0}$ -related to $(x^i, u^j, u_{i_1}^j) \in \mathbf{J}^1(\pi)$ iff

$$u_{i_2}^j + \delta \sum_{i_1=1}^p u_{i_1}^j y^{i_1} = u^j + \delta \sum_{i_1=1}^p y^{i_1} u_{i_1}^j \quad (1 \leq j \leq q)$$

for all $(u^1, \dots, u^q) \in \mathbb{R}^q$ and all $\delta \in \mathbb{R}$, which is tantamount to saying that

$$u_{i_1}^j = u_{i_2}^j$$

for all $1 \leq i \leq p$ and all $1 \leq j \leq q$. This completes the proof. □

Notation 72. Thus the coordinate $(x^i, u^j; u_{i_1}^j, u_{i_2}^j, u_{i_1; i_2}^j) \in \hat{\mathbf{J}}^2(\pi)$ can be simplified to $(x^i, u^j, u_{i_1}^j, u_{i_1; i_2}^j)$.

Now we take a step forward.

Proposition 73. *Let $(x^i, u^j, u_{i_1}^j, u_{i_1; i_2}^j) \in \hat{\mathbf{J}}^2(\pi)$. Then $(x^i, u^j, u_{i_1}^j, u_{i_1; i_2}^j) \in \mathbf{J}^2(\pi)$ iff*

$$u_{i_1; i_2}^j = u_{i_2; i_1}^j$$

for all $1 \leq i_1, i_2 \leq p$ and all $1 \leq j \leq q$.

Proof. Let $\gamma \in (M \otimes \mathcal{W}_{D^2})_{(x^i)}$. Then γ is of the Taylor representation

$$\begin{aligned} (\delta_1, \delta_2) \in \mathbb{R}^2 &\mapsto (x^1, \dots, x^p) + \delta_1(y_1^1, \dots, y_1^p) + \delta_2(y_2^1, \dots, y_2^p) + \delta_1\delta_2(y_{12}^1, \dots, y_{12}^p) \\ &= (x^1 + \delta_1y_1^1, \dots, x^p + \delta_1y_1^p) + \delta_2(y_2^1 + \delta_1y_{12}^1, \dots, y_2^p + \delta_1y_{12}^p) \\ &= (x^1 + \delta_2y_2^1, \dots, x^p + \delta_2y_2^p) + \delta_1(y_1^1 + \delta_2y_{12}^1, \dots, y_1^p + \delta_2y_{12}^p) \\ &\in \mathbb{R}^p \end{aligned}$$

Let $\nabla_{(x^i, u^j, u_{i_1}^j)} = (x^i, u^j, u_{i_1}^j, u_{i_1; i_2}^j)$ and $\nabla_{(x^i, u^j)} = (x^i, u^j, u^j)$. Then we have

$$\begin{aligned} \delta_1 \in \mathbb{R} &\mapsto \nabla_{(x^i, u^j)}(\gamma(\cdot, 0))(\delta_1) \\ &= \delta_1 \in \mathbb{R} \mapsto (x^i + \delta_1y_1^i, u^j + \delta_1 \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) \in \mathbb{R}^{p+q} \\ \delta_1 \in \mathbb{R} &\mapsto \nabla_{(x^i, u^j, u_{i_1}^j)}(\gamma(\cdot, 0))(\delta_1) \in \mathbb{R}^{p+q+pq} \\ &= \delta_1 \in \mathbb{R} \mapsto (x^i + \delta_1y_1^i, u^j + \delta_1 \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j + \delta_1 \sum_{i_2=1}^p y_1^{i_2} u_{i_1; i_2}^j) \in \mathbb{R}^{p+q+pq} \end{aligned}$$

while we have

$$\begin{aligned} (\delta_1, \delta_2) \in \mathbb{R}^2 &\mapsto \nabla_{\nabla_{(x^i, u^j)}(\gamma(\cdot, 0))(\delta_1)}(\gamma(\cdot, 0))(\delta_1) \in \mathbb{R}^{p+q} \\ &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \left(x^i + \delta_1y_1^i + \delta_2y_2^i + \delta_1\delta_2y_{12}^i, u^j + \delta_1 \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j + \delta_2 \sum_{i_2=1}^p y_2^{i_2} u_{i_2}^j + \delta_1\delta_2 \sum_{i_1=1}^p y_1^{i_1} u_{i_1; i_2}^j \right) \in \mathbb{R}^{p+q} \\ &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \left(x^i + \delta_1y_1^i + \delta_2y_2^i + \delta_1\delta_2y_{12}^i, u^j + \delta_1 \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j + \delta_2 \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j + \delta_1\delta_2 \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} y_2^{i_2} u_{i_1; i_2}^j \right) \in \mathbb{R}^{p+q} \end{aligned} \tag{5}$$

On the other hand we have

$$\begin{aligned} \delta_2 \in \mathbb{R} &\mapsto \nabla_{(x^i, u^j)}(\gamma(0, \cdot))(\delta_2) \in \mathbb{R}^{p+q} \\ &= \delta_2 \in \mathbb{R} \mapsto \left(x^i + \delta_2y_2^i, u^j + \delta_2 \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j \right) \in \mathbb{R}^{p+q} \\ \delta_2 \in \mathbb{R} &\mapsto \nabla_{(x^i, u^j, u_{i_1}^j)}(\gamma(0, \cdot))(\delta_2) \in \mathbb{R}^{p+q+pq} \\ &= \delta_2 \in \mathbb{R} \mapsto \left(x^i + \delta_2y_2^i, u^j + \delta_2 \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j + \delta_2 \sum_{i_2=1}^p y_2^{i_2} u_{i_1; i_2}^j \right) \in \mathbb{R}^{p+q+pq} \end{aligned}$$

while we have

$$(\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \nabla_{\nabla_{(x^i, u^j)}(\gamma(0, \cdot))(\delta_2)}(\gamma(\cdot, \delta_2))(\delta_1) \in \mathbb{R}^{p+q}$$

$$\begin{aligned}
 &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \left(\begin{array}{l} x^i + \delta_1 y_1^i + \delta_2 y_2^i + \delta_1 \delta_2 y_3^i, u^j + \delta_2 \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j + \\ \delta_1 \sum_{i_1=1}^p (y_1^{i_1} + \delta_2 y_3^{i_1}) (u_{i_1}^j + \delta_2 \sum_{i_2=1}^p y_2^{i_2} u_{i_1; i_2}^j) \end{array} \right) \in \mathbb{R}^{p+q} \\
 &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \left(\begin{array}{l} x^i + \delta_1 y_1^i + \delta_2 y_2^i + \delta_1 \delta_2 y_3^i, u^j + \\ \delta_1 \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j + \delta_2 \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j + \\ \delta_1 \delta_2 \sum_{i_1=1}^p y_3^{i_1} u_{i_1}^j + \delta_1 \delta_2 \sum_{i_2=1}^p \sum_{i_1=1}^p y_1^{i_1} y_2^{i_2} u_{i_1; i_2}^j \end{array} \right) \in \mathbb{R}^{p+q} \\
 &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \left(\begin{array}{l} x^i + \delta_1 y_1^i + \delta_2 y_2^i + \delta_1 \delta_2 y_3^i, u^j + \\ \delta_1 \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j + \delta_2 \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j + \\ \delta_1 \delta_2 \sum_{i_2=1}^p y_3^{i_2} u_{i_2}^j + \delta_1 \delta_2 \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} u_{i_1; i_2}^j \end{array} \right) \in \mathbb{R}^{p+q}
 \end{aligned} \tag{6}$$

Therefore it follows from (5) and (6) that

$$\begin{aligned}
 (\delta_1, \delta_2) \in \mathbb{R}^2 &\mapsto \nabla_{\nabla_{(x^i, u^j)}(\gamma(\cdot, 0))(\delta_1)}(\gamma(\delta_1, \delta_2))(\delta_2) \in \mathbb{R}^{p+q} \\
 &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \nabla_{\nabla_{(x^i, u^j)}(\gamma(\cdot, 0))(\delta_2)}(\gamma(\delta_1, \delta_2))(\delta_1) \in \mathbb{R}^{p+q}
 \end{aligned}$$

for all $\gamma \in (M \otimes \mathcal{W}_{D^2})_{(x^i)}$ iff

for all $1 \leq i_1, i_2 \leq p$ and all $1 \leq j \leq q$. This completes the proof. □

Definition 74. The authors have defined a bijection $\theta_{\mathbf{J}^2(\pi)}^{\mathcal{J}^2(\pi)} : \mathcal{J}^2(\pi) \rightarrow \mathbf{J}^2(\pi)$, which goes formally as follows:

$$\begin{aligned}
 &\theta_{\mathbf{J}^2(\pi)}^{\mathcal{J}^2(\pi)} \left(x^i, u^j, u_{i_1; i_2}^j \right) \\
 &= [\delta \in \mathbb{R} \mapsto (x^i) + (y^i) \delta \in \mathbb{R}^p] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\
 &\left[\begin{array}{l} \in \mathbb{R} \mapsto \theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)} \left((x^i, u^j, u_{i_1}^j) + \left(y^i, \sum_{i_1=1}^p u_{i_1}^j y^{i_1}, \sum_{i_2=1}^p u_{i_1; i_2}^j y^{i_2} \right) \delta \right) \in \mathbf{J}^1(\pi) \end{array} \right] \\
 &\in (\mathbf{J}^1(\pi) \otimes \mathcal{W}_D)_{\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)}(x^i, u^j, u_{i_1}^j)}
 \end{aligned}$$

We can go on by induction on n .

Theorem 75. The mapping $\theta_{\mathbf{J}^{n+1}(\pi)}^{\mathcal{J}^{n+1}(\pi)} : \mathcal{J}^{n+1}(\pi) \rightarrow \mathbf{J}^{n+1}(\pi)$, which is defined to be

$$\begin{aligned}
 &\theta_{\mathbf{J}^{n+1}(\pi)}^{\mathcal{J}^{n+1}(\pi)} \left(x^i, u^j, u_{i_1}^j, u_{i_1; i_2}^j, \dots, u_{i_1; i_2, \dots, i_{n+1}}^j \right) \\
 &= [\delta \in \mathbb{R} \mapsto (x^i) + (y^i) \delta \in \mathbb{R}^p] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto
 \end{aligned}$$

$$\left[\begin{array}{c} \delta \in \mathbb{R} \mapsto \\ \theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)} \left(\begin{array}{c} (x^i, u^j, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) + \\ (y^i, \sum_{i_1=1}^p u_{i_1}^j y^{i_1}, \dots, \sum_{i_{n+1}=1}^p u_{i_1, i_2, \dots, i_{n+1}}^j y^{i_{n+1}}) \delta \end{array} \right) \in \mathbf{J}^n(\pi) \end{array} \right] \\ \in (\mathbf{J}^n(\pi) \otimes \mathcal{W}_D)_{\theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}(x^i, u^j, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j)}$$

by induction on n , is bijective.

7. The Second Approach with Coordinates

Definition 76. We define mappings $\theta_{\mathbb{J}^D(\pi)}^{\mathcal{J}^D(\pi)} : \mathcal{J}^D(\pi) \rightarrow \mathbb{J}^D(\pi)$ as $\varphi_n \circ \theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}$.

Remark 77. Since $\mathbb{J}^D(\pi) = \mathbf{J}^1(\pi)$ and φ_1 is the identity transformation, we have

$$\begin{aligned} & \theta_{\mathbb{J}^D(\pi)}^{\mathcal{J}^D(\pi)}(x^i, u^j, u_{i_1}^j) \\ &= [\delta \in \mathbb{R} \mapsto (x^i) + (y^i, \dots, y^p)] \in (M \otimes \mathcal{W}_D)_{(x^i)} \mapsto \\ & \left[\delta \in \mathbb{R} \mapsto (x^i, u^j) + \delta \left(y^i, \sum_{i_1=1}^p y^{i_1} u_{i_1}^j \right) \in \mathbb{R}^{p+q} \right] \\ & \in (M \otimes \mathcal{W}_D)_{(x^i)} \end{aligned}$$

With due regard to Theorem 75, it is easy to see that

Lemma 78. Given $(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_{n+1}}^j) \in \mathcal{J}^{n+1}(\pi)$, we have

$$\begin{aligned} & \theta_{\mathbb{J}^{n+1}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_{n+1}}^j) \\ &= \left[(\delta_1, \dots, \delta_{n+1}) \in \mathbb{R}^{n+1} \mapsto (x^i) + \sum_{r=1}^{n+1} \sum_{1 \leq k_1 < \dots < k_r \leq n+1} \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p \right] \\ & \in (M \otimes \mathcal{W}_{D^{n+1}})_{(x^i)} \mapsto \end{aligned}$$

$$\left[\begin{array}{c} (\delta_1, \dots, \delta_{n+1}) \in \mathbb{R}^{n+1} \mapsto \\ \theta_{\mathbb{J}^D(\pi)}^{\mathcal{J}^D(\pi)} \left(\begin{array}{c} (x^i, u^j, u_{i_1}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) + \\ (y_{n+1}^i, \sum_{i_1=1}^p u_{i_1}^j y_{n+1}^{i_1}, \dots, \sum_{i_{n+1}=1}^p u_{i_1, i_2, \dots, i_{n+1}}^j y_{n+1}^{i_{n+1}}) \delta_{n+1} \end{array} \right) \\ \left(\begin{array}{c} (\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mapsto (x^i) + \\ \left(\sum_{r=1}^n \sum_{1 \leq k_1 < \dots < k_r \leq n} \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i + y_{k_1, \dots, k_r, n+1}^i \delta_{n+1}) \in \mathbb{R}^p \right) \end{array} \right) \end{array} \right]$$

$$\in (E \otimes \mathcal{W}_{D^{n+1}})_{(x^i, u^j)}$$

Now we are going to determine $\theta_{\mathbb{J}D^2}^{\mathcal{J}^2(\pi)}$.

Theorem 79. Given $(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j) \in \mathcal{J}^2(\pi)$, we have

$$\begin{aligned} &\theta_{\mathbb{J}D^2}^{\mathcal{J}^2(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j) \\ &= [(\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i) + (y_1^i) \delta_1 + (y_2^i) \delta_2 + (y_{12}^i) \delta_1 \delta_2 \in \mathbb{R}^p] \\ &\in (M \otimes \mathcal{W}_{D^2})_{(x^i)} \mapsto \\ &\left[(\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i, u^j) + (y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) \delta_1 + (y_2^i, \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j) \delta_2 + \right. \\ &\quad \left. (y_{12}^i, \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} y_2^{i_2} u_{i_1, i_2}^j + \sum_{i_1=1}^p y_{12}^{i_1} u_{i_1}^j) \delta_1 \delta_2 \in \mathbb{R}^{p+q} \right] \\ &\in (E \otimes \mathcal{W}_{D^2})_{(x^i, u^j)} \end{aligned}$$

Proof. The Taylor representation of

$$\begin{aligned} &\theta_{\mathbb{J}D^2}^{\mathcal{J}^2(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j) \\ &= ((\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i) + (y_1^i) \delta_1 + (y_2^i) \delta_2 + (y_{12}^i) \delta_1 \delta_2 \in \mathbb{R}^p) \\ &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto \\ &\theta_{\mathbb{J}D}^{\mathcal{J}^1(\pi)}\left(\left(x^i, u^j\right) + \left(y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j, \sum_{i_2=1}^p y_2^{i_2} u_{i_1, i_2}^j\right) \delta_2\right) \\ &= (\delta_1 \in \mathbb{R}^p \mapsto (x^i + y_2^i \delta_2) + (y_1^i + y_{12}^i \delta_2) \delta_1) \\ &\in \mathbb{R}^{p+q} \end{aligned}$$

goes as follows

$$\begin{aligned} &\left(x^i + y_2^i \delta_2, u^j + \left(\sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j\right) \delta_2\right) + \\ &\left(y_1^i + y_{12}^i \delta_2, \sum_{i_1=1}^p \left(u_{i_1}^j + \left(\sum_{i_2=1}^p y_2^{i_2} u_{i_1, i_2}^j\right) \delta_2\right) (y_1^{i_1} + y_{12}^{i_1} \delta_2)\right) \delta_1 \\ &= (x^i, u^j) + (y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) \delta_1 + (y_2^i, \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j) \delta_2 + \\ &\quad (y_{12}^i, \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} y_2^{i_2} u_{i_1, i_2}^j + \sum_{i_1=1}^p y_{12}^{i_1} u_{i_1}^j) \delta_1 \delta_2 \end{aligned}$$

so that we have the coveted result. □

Generally, by the same token, we have

Theorem 80. Given $(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) \in \mathcal{J}^n(\pi)$, we have

$$\begin{aligned} & \theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) \\ &= \left[(\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mapsto (x^i) + \sum_{r=1}^n \sum_{1 \leq k_1 < \dots < k_r \leq n} \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p \right] \\ & \in (M \otimes \mathcal{W}_{D^n})(x^i) \mapsto \\ & \left[(\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mapsto (x^i, u^j) + \left(\sum_{r=1}^n \sum_{1 \leq k_1 < \dots < k_r \leq n} \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i), \sum_{i_1=1}^p \dots \sum_{i_s=1}^p y_{i_1, \dots, i_s}^{i_s} \dots y_{i_1}^{i_s} u_{i_1}^j \right) \in \mathbb{R}^{p+q} \right] \\ & \in (E \otimes \mathcal{W}_{D^n})(x^i, u^j) \end{aligned}$$

where the completely undecorated \sum is taken over all partitions of the set $\{k_1, \dots, k_r\}$ into nonempty subsets $\{\mathbf{J}_1, \dots, \mathbf{J}_s\}$, and $\mathbf{J} = \{k_1, \dots, k_t\}$ is a set of natural numbers with $k_1 < \dots < k_t$, then y_{k_1, \dots, k_t}^i denotes $y_{k_1, \dots, k_t}^{i_1, \dots, i_t}$.

Proof. By using Lemma 78, we can proceed by induction on n . The details are safely left to the reader. □

Definition 81. We define mappings $\theta_{\mathbb{S}^{D^n}(\pi)}^{\mathcal{S}^n(\pi)} : \mathcal{S}^n(\pi) \rightarrow \mathbb{S}^{D^n}(\pi)$ to be

$$\begin{aligned} & \theta_{\mathbb{S}^{D^n}(\pi)}^{\mathcal{S}^n(\pi)}(x^i, u^j, u_{i_1, i_2, \dots}^j) \\ &= \left[(\delta_1, \dots, \delta_n) \in \mathbb{R}^n \mapsto (x^i) + \sum_{r=1}^n \sum_{1 \leq k_1 < \dots < k_r \leq n} \delta_{k_1} \dots \delta_{k_r} (y_{k_1, \dots, k_r}^i) \in \mathbb{R}^p \right] \\ & \in (M \otimes \mathcal{W}_{D^n})(x^i) \mapsto \\ & \left[\delta \in \mathbb{R} \mapsto (x^i, u^j + \delta \sum_{1 \leq i_1 \leq \dots \leq i_n \leq p} y_1^{i_1} \dots y_n^{i_n} u_{i_1, i_2, \dots, i_n}^j) \in \mathbb{R}^{p+q} \right] \\ & \in (E \otimes \mathcal{W}_D)^\perp(x^i, u^j) \end{aligned}$$

It is easy to see that

Proposition 82. The mappings $\theta_{\mathbb{S}^{D^n}(\pi)}^{\mathcal{S}^n(\pi)} : \mathcal{S}^n(\pi) \rightarrow \mathbb{S}^{D^n}(\pi)$ are bijective.

It is also easy to see that

Proposition 83. Given

$$\nabla = (x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_{n+1}}^j),$$

$$\nabla' = (y^i, v^j, v_{i_1}^j, v_{i_1, i_2}^j, \dots, v_{i_1, i_2, \dots, i_{n+1}}^j) \mathcal{J}^{n+1}(\pi),$$

we have

$$\pi_n^{n+1}(\nabla) = \pi_n^{n+1}(\nabla')$$

iff

$$\pi_n^{n+1} \left(\theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla) \right) = \pi_n^{n+1} \left(\theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla') \right),$$

in which we get

$$\theta_{\mathbb{S}^{D^{n+1}}(\pi)}^{\mathcal{S}^{n+1}(\pi)}(\nabla - \nabla') = \theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla) - \theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla')$$

Theorem 84. *The mappings $\theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \rightarrow \mathbb{J}^{D^n}(\pi)$ are bijective.*

Proof. We proceed by induction on n . The mapping $\theta_{\mathbb{J}^D(\pi)}^{\mathcal{J}^1(\pi)} : \mathcal{J}^1(\pi) \rightarrow \mathbb{J}^D(\pi)$ is obviously bijective. By Proposition 83 and the induction hypothesis, $\left(\theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}, \theta_{\mathbb{S}^{D^{n+1}}(\pi)}^{\mathcal{S}^{n+1}(\pi)} \times_E \theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)} \right)$ gives a morphism of affine bundles from the affine bundle $\mathcal{J}^{n+1}(\pi) \rightarrow \mathcal{J}^n(\pi)$ over the vector bundle $\mathbb{S}^{n+1}(\pi) \times \mathcal{J}^n(\pi) \rightarrow \mathcal{J}^n(\pi)$ to the affine bundle $\pi_{n+1, n} : \mathbb{J}^{D^{n+1}}(\pi) \rightarrow \mathbb{J}^{D^n}(\pi)$ over the vector bundle $\mathbb{S}^{D^{n+1}}(\pi) \times_E \mathbb{J}^{D^n}(\pi) \rightarrow \mathbb{J}^{D^n}(\pi)$ is an isomorphism of affine bundles, so that the mapping $\theta_{\mathbb{J}^{D^{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)} : \mathcal{J}^{n+1}(\pi) \rightarrow \mathbb{J}^{D^{n+1}}(\pi)$ is bijective. \square

Corollary 85. *The mappings $\varphi_n : \mathbf{J}^n(\pi) \rightarrow \mathbb{J}^{D^n}(\pi)$ are bijective.*

Proof. This follows simply from Theorems 75 and 84 and the commutativity of the following diagram:

$$\begin{array}{ccc} & \mathcal{J}^n(\pi) & \\ \theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)} \swarrow & & \searrow \theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)} \\ \mathbf{J}^n(\pi) & \xrightarrow{\varphi_n} & \mathbb{J}^{D^n}(\pi) \end{array}$$

\square

8. The Third Approach with Coordinates

Definition 86. We define mappings $\theta_{\mathbb{J}D_n(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \rightarrow \mathbb{J}D_n(\pi)$ as $\psi_n \circ \theta_{\mathbb{J}D^n(\pi)}^{\mathcal{J}^n(\pi)}$.

Now we are going to determine $\theta_{\mathbb{J}D^2(\pi)}^{\mathcal{J}^2(\pi)}$.

Theorem 87. Given $(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j) \in \mathcal{J}^2(\pi)$, we have

$$\begin{aligned} & \theta_{\mathbb{J}D^2(\pi)}^{\mathcal{J}^2(\pi)} \left(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j \right) \\ &= \left[\delta \in \mathbb{R} \mapsto (x^i) + (y_1^i) \delta + \frac{1}{2} (y_2^i) \delta^2 \in \mathbb{R}^p \right] \\ &\in (M \otimes \mathcal{W}_{D^2})_{(x^i)} \mapsto \\ &\left[\begin{aligned} & \delta \in \mathbb{R} \mapsto (x^i, u^j) + (y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) \delta + \\ & \frac{1}{2} (y_2^i, \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} y_1^{i_2} u_{i_1, i_2}^j + \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j) \delta^2 \in \mathbb{R}^{p+q} \end{aligned} \right] \\ &\in (E \otimes \mathcal{W}_{D^2})_{(x^i, u^j)} \end{aligned}$$

Proof. The Taylor representation of $\theta_{\mathbb{J}D^2(\pi)}^{\mathcal{J}^2(\pi)}$ is

$$(\text{id}_M \otimes \mathcal{W}_{(d_1, d_2) \in D^2 \rightarrow d_1, d_2}) \left(\delta \in \mathbb{R} \mapsto (x^i) + (y_1^i) \delta + \frac{1}{2} (y_2^i) \delta^2 \in \mathbb{R}^p \right)$$

is

$$\begin{aligned} & (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i) + (y_1^i) (\delta_1 + \delta_2) + \frac{1}{2} (y_2^i) (\delta_1 + \delta_2)^2 \in \mathbb{R}^p \\ &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i) + (y_1^i) \delta_1 + (y_1^i) \delta_2 + (y_2^i) \delta_1 \delta_2 \in \mathbb{R}^p \end{aligned}$$

so that the transformation under the mapping $\theta_{\mathbb{J}D^2(\pi)}^{\mathcal{J}^2(\pi)} \left(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j \right)$ is

$$\begin{aligned} & (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i, u^j) + (y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) \delta_1 + (y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) \delta_2 + \\ & (y_2^i, \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} y_1^{i_2} u_{i_1, i_2}^j + \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j) \delta_1 \delta_2 \\ & \in \mathbb{R}^{p+q} \\ &= (\delta_1, \delta_2) \in \mathbb{R}^2 \mapsto (x^i, u^j) + (y_1^i, \sum_{i_1=1}^p y_1^{i_1} u_{i_1}^j) (\delta_1 + \delta_2) + \end{aligned}$$

$$\frac{1}{2} \left(y_2^i, \sum_{i_1=1}^p \sum_{i_2=1}^p y_1^{i_1} y_1^{i_2} u_{i_1, i_2}^j + \sum_{i_1=1}^p y_2^{i_1} u_{i_1}^j \right) (\delta_1 + \delta_2)^2 \in \mathbb{R}^{p+q}$$

Therefore we have the coveted result. □

Generally, by the same token, we have

Theorem 88. Given $(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) \in \mathcal{T}_{\mathbb{J}D_n}(\pi)$, we have

$$\begin{aligned} & \theta_{\mathbb{J}D_n(\pi)}^{\mathcal{J}^n(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) \\ &= \left[\delta \in \mathbb{R} \mapsto (x^i) + \sum_{k=1}^n \frac{\delta^k}{k!} (y_k^i) \in \mathbb{R}^p \right] \in (M \otimes \mathcal{W}_{D_n})_{(x^i)} \mapsto \\ & \left[\delta \in \mathbb{R} \mapsto (x^i, u^j) + \sum_{k=1}^n \frac{\delta^k}{k!} \sum_{i_1=1}^p \dots \sum_{i_r=1}^p (y_k^i, u_{i_1, \dots, i_r}^j, y_{k_1}^{i_1} \dots y_{k_r}^{i_r}) \in \mathbb{R}^{p+q} \right] \\ & \in (E \otimes \mathcal{W}_{D_n})_{(x^i, u^j)} \end{aligned}$$

where the undecorated \sum is taken over all partitions of the positive integer k into positive integers k_1, \dots, k_r (such that $k = k_1 + \dots + k_r$) with $1 \leq k_1 \leq \dots \leq k_r \leq n$.

Definition 89. We define mappings $\theta_{\mathbb{S}D_n(\pi)}^{\mathcal{S}^n(\pi)} : \mathcal{S}^n(\pi) \rightarrow \mathbb{S}^{D_n}(\pi)$ to be

$$\begin{aligned} & \theta_{\mathbb{S}D_n(\pi)}^{\mathcal{S}^n(\pi)}(x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_n}^j) \\ &= \left[\delta \in \mathbb{R} \mapsto (x^i) + \sum_{k=1}^n \frac{\delta^k}{k!} (y_k^i) \in \mathbb{R}^p \right] \in (M \otimes \mathcal{W}_{D_n})_{(x^i)} \mapsto \\ & \left[\delta \in \mathbb{R} \mapsto \left(x^i, u^j + \frac{\delta}{n!} \sum_{1 \leq i_1 \leq \dots \leq i_n \leq p} y_1^{i_1} \dots y_1^{i_n} u_{i_1, i_2, \dots, i_n}^j \right) \in \mathbb{R}^{p+q} \right] \\ & \in (E \otimes \mathcal{W}_D)_{(x^i, u^j)}^\perp \end{aligned}$$

It is easy to see that

Proposition 90. The mappings $\theta_{\mathbb{S}D_n(\pi)}^{\mathcal{S}^n(\pi)} : \mathcal{S}^n(\pi) \rightarrow \mathbb{S}^{D_n}(\pi)$ are bijective.

It is also easy to see that

Proposition 91. Given

$$\nabla = (x^i, u^j, u_{i_1}^j, u_{i_1, i_2}^j, \dots, u_{i_1, i_2, \dots, i_{n+1}}^j),$$

$$\nabla' = (y^i, v^j, v_{i_1}^j, v_{i_1, i_2}^j, \dots, v_{i_1, i_2, \dots, i_{n+1}}^j) \mathcal{J}^{n+1}(\pi),$$

we have

$$\pi_n^{n+1}(\nabla) = \pi_n^{n+1}(\nabla')$$

iff

$$\pi_n^{n+1} \left(\theta_{\mathbb{J}^{D_{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla) \right) = \pi_n^{n+1} \left(\theta_{\mathbb{J}^{D_{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla') \right),$$

in which we get

$$\theta_{\mathbb{S}^{D_{n+1}}(\pi)}^{\mathcal{S}^{n+1}(\pi)}(\nabla - \nabla') = \theta_{\mathbb{J}^{D_{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla) - \theta_{\mathbb{J}^{D_{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}(\nabla')$$

Now we have

Theorem 92. *The mappings $\theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ are bijective.*

Proof. The mapping $\theta_{\mathbb{J}^{D^1}(\pi)}^{\mathcal{J}^1(\pi)} : \mathcal{J}^1(\pi) \rightarrow \mathbb{J}^{D^1}(\pi)$ is obviously bijective. We proceed by induction on n . By Proposition 91 and the induction hypothesis, $\left(\theta_{\mathbb{J}^{D_{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)}, \theta_{\mathbb{S}^{D_{n+1}}(\pi)}^{\mathcal{S}^{n+1}(\pi)} \times \theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)}, \theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)} \right)$ gives a morphism of affine bundles from the affine bundle $\mathcal{J}^{n+1}(\pi) \rightarrow \mathcal{J}^n(\pi)$ over the vector bundle $\mathcal{S}^{n+1}(\pi) \times_E \mathcal{J}^n(\pi) \rightarrow \mathcal{J}^n(\pi)$ to the affine bundle $\pi_{n+1, n} : \mathbb{J}^{D_{n+1}}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ over the vector bundle $\mathbb{S}^{D_{n+1}}(\pi) \times_E \mathbb{J}^{D_n}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ is an isomorphism of affine bundles, so that the mapping $\theta_{\mathbb{J}^{D_{n+1}}(\pi)}^{\mathcal{J}^{n+1}(\pi)} : \mathcal{J}^{n+1}(\pi) \rightarrow \mathbb{J}^{D_{n+1}}(\pi)$ is bijective. \square

Corollary 93. *The mappings $\psi_n : \mathbb{J}^{D^n}(\pi) \rightarrow \mathbb{J}^{D_n}(\pi)$ are bijective.*

Proof. This follows simply from Theorems 84 and 92 and the commutativity of the following diagram:

$$\begin{array}{ccc} & \mathcal{J}^n(\pi) & \\ \theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)} \swarrow & & \searrow \theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)} \\ \mathbb{J}^{D^n}(\pi) & \xrightarrow{\psi_n} & \mathbb{J}^{D_n}(\pi) \end{array}$$

\square

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