International Journal of Pure and Applied Mathematics
Volume 82 No. 3 2013, 499-502
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

RATIONAL INTERPOLATION ON SMOOTH CURVES WITH POSITIVE GENUS

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: We discuss a rational interpolation problem on curves of positive genus.

AMS Subject Classification: 14H99
Key Words: rational interpolation, positive genus curve

1. Generic Rational Interpolation

See [5], [7], [1], [8] and references therein for the classical theory of rational interpolation in one variable, i.e. on $\mathbb{P}^1$. Here we propose several interpolation problems on higher genus curves which capture some of the features of the genus 0 case. We just recall that in genus 0 a solution does not always exist if we require tight bounds for the sum of the degrees of the numerator and the numerator. For the existence one needs certain genericity conditions, e.g. certain determinants to be non-zero or certain polynomials to be pairwise coprime ([2], Theorem 5.16 and Corollary 5.18). What is missing in this note are explicit formulas. We hope to come back on this topic in the genus 1 case and get formulas in terms of Weierstrass $\wp$ function.

Let C be a smooth and geometrically connected curve of genus $g > 0$ defined over a field K. We do not always assume that K is algebraically closed, but to state the problems we need to fix several points of $C(K)$. Hence we need

Received: December 3, 2012

that $C(K)$ is not too low. Let $K(C)$ be the function field of C. Fix $Q \in C(K)$. Let $H(Q,C)$ be the numerical semigroup of non-gaps of Q ([6], Theorem 1.6.8). For each divisor D of C defined over K let $L(D)$ denote the vector space union of 0 and of all $f \in K(C) \setminus \{0\}$ with $D + (f) \geq 0$. Set $\ell(D) := \dim L(D)$. Hence $\ell(mQ) = 0$ if $m < 0$, $\ell(0Q) = 1$ and $\ell(mQ) > \ell((m-1)Q)$ if and only if $m \in H(Q,C)$.

Question 1. Fix a zero-dimensional scheme $Z \subset C$ defined over K and such that $Q \notin Z_{red}$. Fix $u \in H^0(Z, \mathcal{O}_Z)$ and $a, b \in \mathbb{N}$. Under which assumptions there are $f \in L(aQ)$, $h \in L(bQ)$ such that $h \neq 0$ and $(f/h)|Z = u$? Under which assumptions on a, b and $\deg(Z)$ this is true for all $Q \in C(\overline{K})$ (resp. for some $Q \in C(\overline{K})$, resp. for a general $Q \in C(\overline{K})$? Under which assumption this is true for a general $u \in H^0(Z, \mathcal{O}_Z)$?

It is the last part (with u and Z general) that we consider in this note.

Assume that K is infinite. Fix a zero-dimension scheme $Z \subset C \setminus \{Q\}$ and an integer $b > 0$. Set $L(bQ)_Z := \{h \in L(bQ) : h(P) \neq 0$ for all $P \in Z_{red}$. Since $1 \in L(bQ)$, $L(bQ)_Z$ is a non-empty open subset of $L(bQ)$ for the Zariski topology. For each $(f,h) \in L(aQ) \oplus L(bQ)_Z$ set $f_{a,b,Z}(f,h) : (f/h)|Z \in H^0(Z, \mathcal{O}_Z)$. We define in this way a map $f_{a,b,Z} : L(aQ) \times L(bQ)_Z \to H^0(Z, \mathcal{O}_Z)$. We say that the rational interpolation problem is *generically solvable* for Q, a, b and Z if for a general $u \in H^0(Z, \mathcal{O}_Z)$ there are $f \in L(aQ)$, $h \in L(bQ)$ such that $b \neq 0$ and $(f/h)|Z = u$. Notice that $H^0(Z, \mathcal{O}_Z)$ is a finite vector space (it has dimension $\deg(Z)$) over the infinite field K and hence it is an irreducible topological space with respect to the Zariski topology. Hence the words " a general $u \in H^0(Z, \mathcal{O}_Z)$ " make sense. This condition is equivalent to say that the image of $f_{a,b,Z}$ contains a non-empty open subset of $H^0(Z, \mathcal{O}_Z)$. Since $\mathrm{Im}(f_{a,b,Z})$ is constructible, it is equivalent to assuming that $\mathrm{Im}(f_{a,b,Z})$ is Zariski dense in $H^0(Z, \mathcal{O}_Z)$. If either $K = \mathbb{R}$ or $K = \mathbb{C}$ we may take the euclidean topology for $H^0(Z, \mathcal{O}_Z)$. In this case this condition is satisfied for all u in a non-empty open subset for the euclidean topology of the vector space $H^0(Z, \mathcal{O}_Z)$, because any nonempty open subset of $H^0(Z, \mathcal{O}_Z)$ is Zariski-dense in in $H^0(Z, \mathcal{O}_Z)$. The map $f_{a,b,Z}$ has fibers of dimension at least 1, because $(cf/ch) = f/h$ for all $c \in K \setminus \{0\}$. Hence $\dim(\mathrm{Im}(f_{a,b,Z})) \leq \min\{\deg(Z), \ell(aQ) + \ell(bQ) - 1\}$. We say that the rational interpolation problem for Q, a, b and Z satisfies *generic uniqueness* if $\dim(\mathrm{Im}(f_{a,b,Z})) = \ell(aQ) + \ell(bQ) - 1$. We say that the rational interpolation problem for Q, a, b, z is *generically unique* (resp. *generically solvable*) if $\mathrm{Im}(f_{a,b,Z})$ has dimension z (resp. $\ell(aQ) + \ell(bQ) - 1$), where Z is a general union of z points of C. Fix positive integers e, z_i, $1 \leq i \leq e$ and set $z := z_1 + \cdots + z_e$. We say that the rational interpolation problem for Q, a,

b, $e, z_1, \ldots, z_e$ is *generically solvable* (resp. *generically unique*) if $\mathrm{Im}(f_{a,b,Z})$ has dimension z (resp. $\ell(aQ) + \ell(bQ) - 1$), where $Z \subset C$ is a general union of e zero-dimensional schemes of degree $z_1, \ldots, z_e$.

Extensions 1. *Instead of the family* $\{mP\}_{m\geq 0}$ *we may take two families of Cartier divisors* $\{D_m\}_{m\geq 0}$, $\{D'_m\}_{m\geq 0}$ *with* $\deg(D_m) = \deg(D'_m) = m$, $(f) + D_m \geq 0$ *and* $(g) + D'_m \geq 0$ *for all* m. *We do not impose that the divisors* D_m *and* D'_m *are effective, but we impose that* Z_{red} *does not contain any point in the support of either* D_m *or* D'_m. *Define* $f_{D_\mathsf{a},D'_\mathsf{b},Z} : H^0(C, \mathcal{O}_{D_\mathsf{a}}) \oplus H^0(C, \mathcal{O}_C(D'_b)_Z \to H^0(Z, \mathcal{O}_Z)$ *as in the case* $D_a = aQ$ *and* $D'_b = bQ$.

Theorem 1. *Assume* $K = \overline{K}$. *Fix* C, Q, *divisors* D_a *and* D_b *and a positive integer* z. *Set* $\ell(a) := h^0(C, \mathcal{O}_C(D_a))$, $\ell(b) := h^0(C, \mathcal{O}_C(D_a))$.

(a) If $z \geq \ell(a) + \ell(b) - 1$, *then the rational interpolation problem is generically unique.*

(b) If $z \leq \ell(a) + \ell(b) - 1$, *then the rational interpolation problem is generically solvable.*

Proof. We prove both parts by induction on z, starting from the case $z = 1$. Let E be the union of the support of D_a and of D_b. Assume Theorem 1 for the integer $z-1$ and fix a general $S \subset C(K)$ such that $\sharp(S) = z-1$. By the inductive assumption we have $\dim(\mathrm{Im}(f_{D_\mathsf{a},D_\mathsf{b},S})) = \min\{z-1, \ell(a)+\ell(b)-1\}$. First assume $z-1 \geq \ell(a)+\ell(b)-1$. Take $Z := S \cup \{P\}$ with any $P \in C \setminus (S \cup E)$. We have $\dim(\mathrm{Im}(f_{D_\mathsf{a},D_\mathsf{b},S})) \geq \dim(\mathrm{Im}(f_{D_\mathsf{a},D_\mathsf{b},S}))$ and hence $\dim(\mathrm{Im}(f_{D_\mathsf{a},D_\mathsf{b},Z})) = \ell(a)+\ell(b)-1$. Now assume $z \leq \ell(a)+\ell(b)-1$. Fix a general $w \in H^0(S, \mathcal{O}_S)$. Let $\{T_\gamma\}_{\gamma\in\Gamma}$ be the set of all irreducible components of $f^{-1}_{D_\mathsf{a},D_\mathsf{b},S}(w)$. For each $\gamma \in T_\gamma$ fix a general $(f_\gamma, h_\gamma) \in T_\gamma(K)$. Since u is general and $L(aQ) \oplus L(bQ)_S$ is irreducible, for each $\gamma \in \Gamma$ we have $\dim(T_\gamma) = \ell(a) + \ell(b) - z$. For each $\gamma \in \Gamma$ fix $(f_\gamma, h_\gamma) \in T_\gamma$ and $(f'_\gamma, h'_\gamma) \in T_\gamma$ such that $f_\gamma/h_\gamma \neq f'_\gamma/h'_\gamma$. Since K is algebraically closed, there is $P \in C(K) \setminus (\{Q\} \cup S)$ such that $(f_\gamma/h_\gamma)(P) \neq (f'_\gamma/h'_\gamma)(P)$ for all $\gamma \in \Gamma$. Fix $c \in K \setminus \{0\}$. Set $Z := S \cup \{P\}$ and take $u \in H^0(Z, \mathcal{O}_Z)$ such that $u|S = w$ and $u|\{P\} = c$. We find that each irreducible component of $f^{-1}_{a,b,Z}(u)$ has dimension at most $z - \ell(a) - \ell(b) + 1$. This is not enough (a priori) to prove that $\dim(\mathrm{Im}(f_{D_\mathsf{a},D_\mathsf{b},S}) = z-1$, because $f_{D_\mathsf{a},D_\mathsf{b},Z}$ is not proper. We also need to observe that this fiber is non-empty. Then we may apply [3], Ex. II.3.22. □

In the case $e, z_1, \ldots, z_e$ with $z_i \geq 2$ for some i, there are differences between the case in which either $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > 2g - 2 + z$ and the case in which $\mathrm{char}(K)$ is low ([4], Theorem 15).

Question 2. Compute the dimension of the $Z \subset C(K) \setminus \{Q\}$ of degree z for which generic uniqueness or generic solvability fails. Assume $z \leq \ell(aQ) + \ell(bQ) - 1$. For one Z with generic solvability (or for the generic Z) compute the codimension of the set of all $u \in H^0(Z, \mathcal{O}_Z)$ which have no solution.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] C. D'Andrea, T. Krick, A. Szanto, Subresultants, Sylvester sums and the rational interpolation problem, *ArXiv:* 1211.6895v1.

[2] J. von zur Gathen, J. Gerhard, *Modern Computer Algebra*, Cambridge University Press, Cambridge (2003).

[3] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Berlin (1977).

[4] D. Laksov, Wronskians and Plücker formulas for linear systems on curves, *Ann. Scient. École Norm. Sup. (4)*, **17**, No. 4 (1984), 565-579.

[5] H.E. Salzer, Note on osculatory rational interpolation, *Math. Comp.*, **16** (1962), 486-491.

[6] H. Stichtenoth, *Algebraic Function Fields and Codes*, Second Edition, Springer-Verlag (2009).

[7] J. Tan, Y. Fang, Newton-Thiele's rational interpolants, Computational methods from rational approximation theory (Wilrijk, 1999), *Numer. Algorithms*, **24**, No-s: 1, 2 (2000), 141-157.

[8] L. Wuytack, On the osculatory rational interpolation problem, *Math. Comput.*, **29** (1975), 837-843.